On Transverse Stability of Discrete Line Solitons

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada
URL: http://dmpeli.math.mcmaster.ca

Joint work with **Jianke Yang** (University of Vermont, USA)

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- In many Hamiltonian PDEs, one-dimensional solitons are unstable with respect to transverse perturbations:
  - Two-dimensional nonlinear Schrödinger equation

$$iu_t + u_{xx} \pm u_{yy} + |u|^2 u = 0.$$

  - Dark solitons and KP-I equation

$$(u_t + uu_x + u_{xxx})_x = u_{yy}$$

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- Old works: Kadomtsev–Petviashvili (1970), Zakharov–Rubenchik (1971), Zakharov (1975), Pelinovsky–Stepanyants (1993), Bridges (2000).
- Recent works: Rousset–Tzvetkov (2008), Johnson–Zumbrun (2010), Stefanov–Stanislavova (2011), Haragus (2012), ...

Mathematical techniques

- Direct perturbation theory for eigenvalues
- Multi-symplectic geometric perturbation theory
- Evans function and algebraic perturbation theory
- Functional analysis framework and negative index theory (*)

Lattice NLS equation

The discrete NLS (dNLS) equation

$$i\dot{u}_{m,n}+\epsilon(u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n})+|u_{m,n}|^2u_{m,n}=0,$$

where $(m,n)\in\mathbb{Z}^2$, $u_{m,n}\in\mathbb{C}$, and $\epsilon\in\mathbb{R}$.

The Gross–Pitaevskii equation with a periodic potential:

$$iu_t+u_{xx}+u_{yy}-V_0\sin^2(x)\sin^2(y)u+|u|^2u=0,$$

where $(x,y)\in\mathbb{R}^2$, $u\in\mathbb{C}$, and $V_0\in\mathbb{R}$.

Yang [PRA **84**, 033840 (2011)] found that line solitons can become stable with respect to transverse perturbations.

One-dimensional (stripe) dNLS lattice

$$i\frac{\partial u_m}{\partial t} + \epsilon(u_{m+1} + u_{m-1} - 2u_m) + \kappa\frac{\partial^2 u_m}{\partial y^2} + |u_m|^2 u_m = 0,$$

where $m \in \mathbb{Z}$, $y \in \mathbb{R}$, $u_m \in \mathbb{C}$, and $\epsilon, \kappa \in \mathbb{R}$.

Yang *et al.* [Opt. Lett. **37**, 1571 (2012)] found again numerically that line solitons can become transversely stable.

Our objective is to study this phenomenon analytically by using the negative index theory .

Stability of nonlinear waves in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J\nabla H(u), \quad u(t) \in X$$

where $X \subset L^2$ is a phase space, $J^+ = -J$ is the symplectic operator, and $H : X \to \mathbb{R}$ is the Hamiltonian function.

- Assume existence of the stationary state (nonlinear wave) $u_0 \in X$ such that $\nabla H(u_0) = 0$.
- Perform linearization at the stationary solution

  $$u(t) = u_0 + ve^{\lambda t},$$

  where $(\lambda, v) \in \mathbb{C} \times X$ satisfies the spectral problem

  $$JD^2 H(u_0)v = \lambda v.$$

Main Questions

Consider the spectral stability problem:

$$JD^2H(u_0)v = \lambda v, \quad v \in X.$$

- Let stationary solutions u_0 decay exponentially as $|x| \to \infty$ (solitary waves, vortices, etc).
- Let the skew-symmetric operator J be invertible
- Let the self-adjoint operator $D^2H(u_0)$ have a positive essential spectrum and finitely many negative eigenvalues.

Question: Is there a relation between unstable eigenvalues of $JD^2H(u_0)$ and negative eigenvalues of $D^2H(u_0)$?

State of the art

Consider the spectral stability problem:

$$JD^2H(u_0)v = \lambda v, \quad v \in X.$$

For simplicity, assume a zero-dimensional kernel of $D^2H(u_0)$.
If λ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

- **Grillakis, Shatah, Strauss, 1990** Orbital Stability Theory:
  - If $D^2H(u_0)$ has no negative eigenvalue, then $JD^2H(u_0)$ has no unstable eigenvalues.
  - If $D^2H(u_0)$ has an odd number of negative eigenvalues, then $JD^2H(u_0)$ has at least one real unstable eigenvalue.

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Negative Index Theory

- **Kapitula, Kevrekidis, Sandstede, 2004**:

$$N_{\rm re}(JD^2H(u_0))+2N_{\rm c}(JD^2H(u_0))+2N^-_{\rm im}(JD^2H(u_0)) = N_{\rm neg}(D^2H),$$

where $N_{\rm re}$ is the number of positive real eigenvalues, $N_{\rm c}$ is the number of complex eigenvalues in the first quadrant, and $N^-_{\rm im}$ is the number of positive imaginary eigenvalues of negative Krein signature.

- Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JD^2H(u_0)$ with the eigenvector v. Then, the sign of

$$E''_\omega(v) = \langle D^2H(u_0)v, v\rangle_{L^2}$$

is called the Krein signature of the eigenvalue λ.

Sharp Negative Index Theory

Consider the spectral stability problem:

$$L_+ u = -\lambda w, \quad L_- w = \lambda u, \quad u, w \in X,$$

and assume again zero-dimensional kernels of L_+ and L_-.

- **Pelinovsky, 2005** Sharp Negative Index Theory:

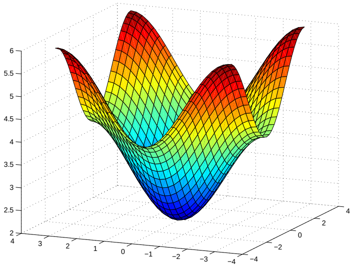
$$\left\{ \begin{array}{l} N_{\rm re}^-(JD^2H(u_0)) + N_{\rm c}(JD^2H(u_0)) + N_{\rm im}^-(JD^2H(u_0)) = N_{\rm neg}(L_+), \\ N_{\rm re}^+(JD^2H(u_0)) + N_{\rm c}(JD^2H(u_0)) + N_{\rm im}^-(JD^2H(u_0)) = N_{\rm neg}(L_-), \end{array} \right.$$

where $N_{\rm re}^+$ ($N_{\rm re}^-$) is the number of positive eigenvalues with positive (negative) quadratic form $\langle L_+ u, u \rangle_{L^2}$.

Linearized dNLS equation:

$$i\dot{u}_{m,n} + \epsilon(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n}) = 0.$$

Bifurcations of stationary solitons occur from critical points of the dispersion surface, where $\nabla\omega = 0$.



Linear waves $e^{ikm+ipn-i\omega t}$ with $(k,p) \in [-\pi,\pi] \times [-\pi,\pi]$ satisfies the dispersion relation

$$\omega(k,p) = \epsilon(4 - 2\cos(k) - 2\cos(p))$$

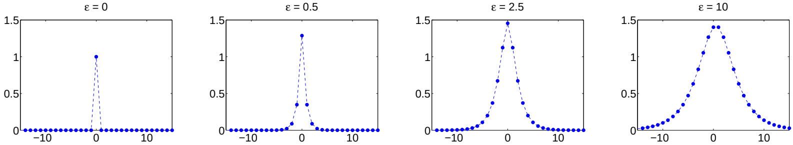
Critical points at $(0,0)$, $(\pi,0)$, $(0,\pi)$, and (π,π).

Minimum point Γ : $k = p = 0$, $\omega(0,0) = 0$

Line solitons $u_{m,n}(t) = e^{i\mu^2 t}\psi_m$ satisfy the 1D dNLS equation

$$-\mu^2\psi_m + \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2\psi_m = 0,$$

A fundamental soliton exists for any $\epsilon > 0$ (Hermann, 2011)



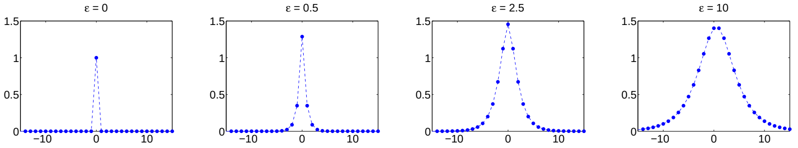
Continuous approximation $\psi_m \sim \sqrt{2}\mu\,\mathrm{sech}\left(\frac{\mu m}{\sqrt{\epsilon}}\right)$ as $\mu \to 0$ (Bambusi and Penati, 2010).

Saddle point X : $k = 0$, $p = \pi$, $\omega(0, \pi) = 4\epsilon$

Line solitons $u_{m,n}(t) = (-1)^n e^{i(\mu^2 - 4\epsilon)t} \psi_m$ satisfy the same 1D dNLS equation

$$-\mu^2 \psi_m + \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2 \psi_m = 0,$$

Another family of line solitons exist.



Saddle point X' : $k = \pi$, $p = 0$, $\omega(\pi, 0) = 4\epsilon$

Line solitons $u_{m,n}(t) = (-1)^m e^{i(-\mu^2 - 4\epsilon)t} \psi_m$ satisfy the 1D dNLS equation

$$\mu^2 \psi_m - \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2 \psi_m = 0.$$

No line solitons exist because

$$\mu^2 \|\psi\|_{l^2}^2 + \epsilon \langle \psi, (-\Delta)\psi \rangle + \|\psi\|_{l^4}^4 = 0$$

yields a contradiction.

Maximum point M : $k = \pi$, $p = \pi$, $\omega(\pi, \pi) = 8\epsilon$

Line solitons $u_{m,n}(t) = (-1)^{m+n} e^{i(-\mu^2 - 8\epsilon)t} \psi_m$ satisfy the same 1D dNLS equation

$$\mu^2 \psi_m - \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2 \psi_m = 0.$$

No line solitons exist.

Minimum point Γ : $k = p = 0$, $\omega(0,0) = 0$

At the minimum point Γ, we can substitute

$$u_{m,n}(t) = U(X,Y,t)e^{i\mu^2 t}, \quad X = \frac{m}{\sqrt{\epsilon}}, \quad Y = \frac{n}{\sqrt{\epsilon}}$$

and obtain an elliptic 2D NLS equation as $\epsilon \to \infty$:

$$i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + (|U|^2 - \mu^2)U = 0.$$

Line solitons are unstable as $\epsilon \to \infty$.

Would the same be true for all $\epsilon > 0$?

Saddle point X : $k = 0$, $p = \pi$, $\omega(0,\pi) = 4\epsilon$

At the saddle point X, we can substitute

$$u_{m,n}(t) = (-1)^n U(X,Y,T) e^{i(\mu^2 - 4\epsilon)t}, \quad X = \frac{m}{\sqrt{\epsilon}}, \quad Y = \frac{n}{\sqrt{\epsilon}}$$

and obtain a hyperbolic 2D NLS equation as $\epsilon \to \infty$:

$$i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U}{\partial Y^2} + (|U|^2 - \mu^2)U = 0.$$

Line solitons are unstable as $\epsilon \to \infty$.

Would the same be true for all $\epsilon > 0$?

Instability Theorem

Linearizing at the discrete line soliton,

$$u_{m,n}(t) = e^{i\mu^2 t}\left[\psi_m + v_{m,n}(t)\right], \quad v_{m,n}(t) = e^{\lambda t + ipn}\left(U_m + iW_m\right),$$

we obtain the linear stability problem

$$L_+(p)U = -\lambda W, \quad L_-(p)W = \lambda U,$$

where

$$(L_+U)_m = -\epsilon\left[U_{m+1} + U_{m-1} + (2\cos(p) - 4)U_m\right] + (\mu^2 - 3\psi_m^2)U_m,$$
$$(L_-W)_m = -\epsilon\left[W_{m+1} + W_{m-1} + (2\cos(p) - 4)W_m\right] + (\mu^2 - \psi_m^2)W_m.$$

Fix $\mu = 1$ and consider a fundamental (positive, 1-humped) soliton:

$$\psi_m = \delta_{m,0} + \epsilon(\delta_{m,1} + \delta_{m,0} + \delta_{m,-1}) + \mathcal{O}(\epsilon^2).$$

Theorem

Consider the fundamental soliton bifurcating from the Γ *point. For any* $\epsilon > 0$*, there is* $p_0(\epsilon) \in (0, \pi]$ *such that for any* p *with* $0 < |p| < p_0(\epsilon)$ *the linear-stability problem admits a pair of real eigenvalues* $\pm\lambda(\epsilon, p)$ *with* $\lambda(\epsilon, p) > 0$.

In addition, $p_0(\epsilon) = \pi$ *if* $0 < \epsilon < \frac{1}{2}$. *Furthermore, for any* $p \in [-\pi, \pi]$*, the eigenvalue* $\lambda(\epsilon, p)$ *has the following asymptotic expansion in the anti-continuum limit,*

$$\lambda^2(\epsilon, p) = 8\epsilon \sin^2\left(\frac{p}{2}\right) + \mathcal{O}(\epsilon^2) \text{ as } \epsilon \to 0.$$

- We have

$$L_{\pm}(p) = L_{\pm}(0) + 2\epsilon\left[1 - \cos(p)\right] \geq L_{\pm}(0).$$

- $L_-(0)\psi = 0$ with $\psi > 0$. Hence $L_-(0) \geq 0$ and 0 is at the bottom of $L_-(0)$.
- By the perturbation theory, $L_-(p) > 0$ for all $p \neq 0$.
- $L_+(0)$ has at least one negative eigenvalue

$$\langle L_+(0)\psi, \psi\rangle = -2\|\psi\|_{l^4}^4 < 0,$$

moreover, there is only one negative eigenvalue for any $\epsilon > 0$.

- $L_+(p)$ has exactly one negative and no zero eigenvalues for small $p \neq 0$.

Negative Index Theory:

$$\begin{aligned} N_{\rm real}^- + N_{\rm imag}^- + N_{\rm comp} &= n(L_+(p)) = 1, \\ N_{\rm real}^+ + N_{\rm imag}^- + N_{\rm comp} &= n(L_-(p)) = 0, \end{aligned} \quad p \neq 0,$$

where

- $N_{\rm real}^+$ ($N_{\rm real}^-$) are the numbers of real positive eigenvalues λ with positive (negative) quadratic form $\langle L_+(p) U, U \rangle$ at the eigenvector (U, W) of the linear stability problem;
- $N_{\rm imag}^-$ is the number of purely imaginary eigenvalues λ with $\mathrm{Im}(\lambda) > 0$ and negative quadratic form $\langle L_+(p) U, U \rangle$;
- $N_{\rm comp}$ is the number of complex eigenvalues λ with $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Im}(\lambda) > 0$.

Hence

$$N_{\rm real}^- = 1, \quad N_{\rm real}^+ = N_{\rm imag}^- = N_{\rm comp} = 0.$$

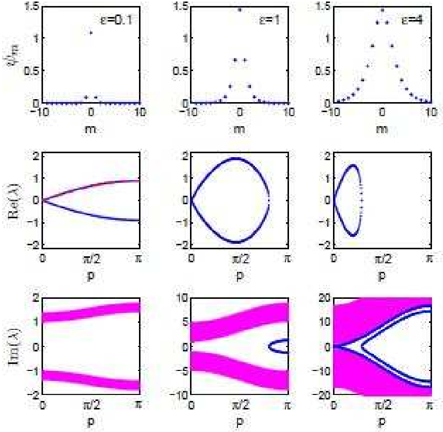


Figure : Left: $\epsilon = 0.1$; middle: $\epsilon = 1$; right: $\epsilon = 4$.

Stability Theorem

Linearizing at the discrete line soliton,

$$u_{m,n}(t) = (-1)^n e^{i(\mu^2 - 4\epsilon)t} \left[\psi_m + v_{m,n}(t)\right], \; v_{m,n}(t) = e^{\lambda t + ipn} \left(U_m + iW_m\right)$$

we obtain the linear stability problem

$$L_+(p)U = -\lambda W, \quad L_-(p)W = \lambda U,$$

where

$$(L_+U)_m = -\epsilon \left[U_{m+1} + U_{m-1} - 2\cos(p)U_m\right] + (\mu^2 - 3\psi_m^2)U_m,$$
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Theorem

Consider the fundamental soliton bifurcating from the X point. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and $p \in [\pi, \pi]$, the linear-stability problem does not admit any unstable eigenvalues but admits a pair of purely imaginary eigenvalues $\pm i\omega(\epsilon, p)$ of negative Krein signature.

For any $p \in [-\pi, \pi]$ and small ϵ, this eigenvalue $\omega(\epsilon, p)$ has the following asymptotic expression,

$$\omega^2(\epsilon, p) = 8\epsilon \sin^2\left(\frac{p}{2}\right) + \mathcal{O}(\epsilon^2) \text{ as } \epsilon \to 0.$$

- We have
$$L_{\pm}(p) = L_{\pm}(0) - 2\epsilon\left[1 - \cos(p)\right].$$

- $L_{-}(0)\psi = 0$ with $\psi > 0$. Hence $L_{-}(0) \geq 0$ and 0 is at the bottom of $L_{-}(0)$.

- By the perturbation theory, $L_{-}(p)$ has exactly one negative eigenvalue for small $\epsilon > 0$ and $p \neq 0$.

- $L_{+}(0)$ has exactly one negative eigenvalue and no zero eigenvalue for any $\epsilon > 0$.

- $L_{+}(p)$ has exactly one negative and no zero eigenvalues for small $\epsilon > 0$ and $p \neq 0$.

Negative Index Theory:

$$\begin{array}{l} N^-_{\rm real} + N^-_{\rm imag} + N_{\rm comp} = n(L_+(p)) = 1, \\ N^+_{\rm real} + N^-_{\rm imag} + N_{\rm comp} = n(L_-(p)) = 1, \end{array} \quad p \neq 0,$$

At $p = 0$, a double zero eigenvalue exists, which splits for $p \neq 0$ outside the continuous spectrum. Hence,

$$N^-_{\rm imag} = 1, \quad N^+_{\rm real} = N^-_{\rm real} = N_{\rm comp} = 0,$$

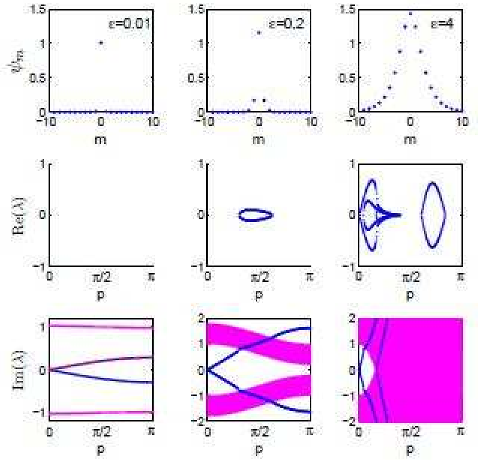

Figure : Left: $\epsilon = 0.01$; middle: $\epsilon = 0.2$; right: $\epsilon = 4$.

Consider the 1D Stripe dNLS lattice:

$$i\frac{\partial u_m}{\partial t} + \epsilon(u_{m+1} + u_{m-1} - 2u_m) + \kappa\frac{\partial^2 u_m}{\partial y^2} + |u_m|^2 u_m = 0, \quad m \in \mathbb{Z},$$

where $\epsilon > 0$ is small and $\kappa = \pm 1$.

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where $\epsilon > 0$ is small and $\kappa = \pm 1$.

Linearizing at the discrete line soliton,

$$u_m(y,t) = e^{i\mu^2 t}\left[\psi_m + v_m(y,t)\right], \; v_m(y,t) = e^{\lambda t + ipy}\left(U_m + iW_m\right),$$

we obtain the linear stability problem

$$L_+(p)U = -\lambda W, \quad L_-(p)W = \lambda U,$$

where

$$(L_+(p)U)_m = -\epsilon(U_{m+1} + U_{m-1} - 2U_m) + (\mu^2 + \kappa p^2 - 3\psi_m^2)U_m,$$
$$(L_-(p)W)_m = -\epsilon(W_{m+1} + W_{m-1} - 2W_m) + (\mu^2 + \kappa p^2 - \psi_m^2)W_m.$$

- At $\epsilon = 0$, the linear system has two semi-simple eigenvalue of infinite multiplicity at $\lambda = \pm i(1+\kappa p^2)$ and two simple eigenvalues at $\lambda = \pm\sqrt{\kappa p^2(2-\kappa p^2)}$.
- We also have

$$L_{\pm}(p) = L_{\pm}(0) + \kappa p^2.$$

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$$L_\pm(p) = L_\pm(0) + \kappa p^2.$$

- For $\kappa = 1$ and $\epsilon = 0$, simple eigenvalues $\lambda = \pm p\sqrt{2 - p^2}$ are real for $p \in (0, \sqrt{2})$ and purely imaginary eigenvalues for $p > \sqrt{2}$ bounded away from the continuum spectrum.
- For small $\epsilon > 0$, the negative index count gives

$$N^-_{\text{real}} = 1, \quad p \in (0, p_0(\epsilon))$$

and

$$n(L_+(p)) = n(L_-(p)) = 0, \quad p > p_0(\epsilon),$$

where $p_0(\epsilon) = \sqrt{2} + \mathcal{O}(\epsilon)$.

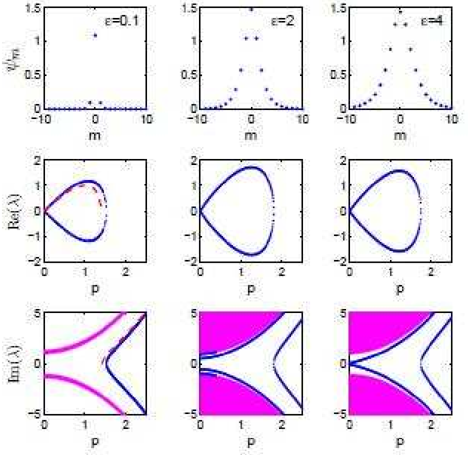

Figure : Left: $\epsilon = 0.1$; middle: $\epsilon = 2$; right: $\epsilon = 4$.

- For $\kappa = -1$ and $\epsilon = 0$, simple eigenvalues $\lambda = \pm i p \sqrt{2 + p^2}$ are in resonance with the essential spectrum $\lambda = \pm i(1 - p^2)$ at $p = p_c = \frac{1}{2}$.

- The simple eigenvalues have negative Krein signature and the essential spectrum has positive Krein signature for $p \in (-1, 1)$. For small $\epsilon > 0$, the resonance gives rise to complex instabilities with $N_{\rm comp} = 1$ for p near p_c.

- Asymptotic theory gives

$$\lambda(\epsilon, p) = \frac{3}{4} i + \frac{i\epsilon}{15}(14 + 17\delta) + \frac{2\epsilon}{15}\sqrt{15 - 4(1 - 2\delta)^2} + \mathcal{O}(\epsilon^2),$$

where $\delta = (p^2 - p_c^2)/\epsilon$.

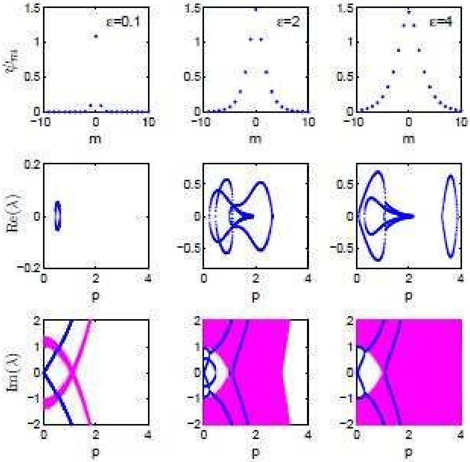

Figure : Left: $\epsilon = 0.1$; middle: $\epsilon = 2$; right: $\epsilon = 4$.

Summary

- Transverse stability problems are much easier than regular stability problems because symmetry-breaking perturbations remove kernels of the linearized operators.

- Applications of the negative index theory are developed in regular l^2 spaces, there is no necessity of constrained spaces.

- Lattice problems have additional simplifications near the anti-continuum limit, where asymptotic methods can be used in conjugation with the negative stability theory.

- Discretization may induce transverse stability of continuously unstable solitons. The role of discretization may be taken by the periodic potentials in the continuous NLS equations.