The logarithmic Brunn-Minkowski inequality and Minkowski problem

Gaoyong Zhang
New York University

Conference on Geometric and Analytic Inequalities

BIRS, Banff, Canada

July 10 – 15, 2016

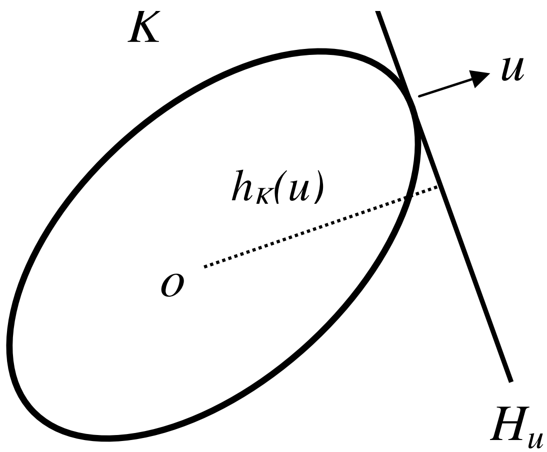
Basic concepts in convex geometry

- Convex body K. A compact convex set with nonempty interior in $\mathbb{R}^n$. A complete space in Hausdorff metric. Ovaloids, polytopes.

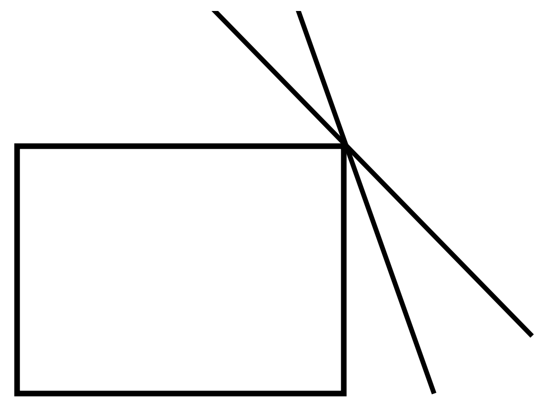
Basic concepts in convex geometry

- Convex body K. A compact convex set with nonempty interior in $\mathbb{R}^n$. A complete space in Hausdorff metric. Ovaloids, polytopes.

- Support function h_K. It is the distance from the origin to the support hyperplane H_u with outer normal $u \in \mathbb{S}^{n-1}$.



Convex



Non-smooth

The Brunn-Minkowski Theory

- It studies geometric invariants and geometric measures of convex bodies from both geometric and analytic viewpoints.

The Brunn-Minkowski Theory

- It studies geometric invariants and geometric measures of convex bodies from both geometric and analytic viewpoints.
- Geometric invariants.

  Volume V, surface area S, quermassintegrals W_i, ...

The Brunn-Minkowski Theory

- It studies geometric invariants and geometric measures of convex bodies from both geometric and analytic viewpoints.

- Geometric invariants.

  Volume V, surface area S, quermassintegrals W_i, ...

- Geometric measures. (global analogs of curvatures)

  Surface area measure S_K, cone volume measure V_K, integral Gauss curvature J_K, ...

Two fundamental theorems

- The Brunn-Minkowski inequality. For convex bodies K, L in $\mathbb{R}^n$,

$$V((1-t)K+tL) \geq V(K)^{1-t}V(L)^t,$$

where $(1-t)K+tL = \{(1-t)x+ty : x \in K,\ y \in L\}$, $0 \leq t \leq 1$, is the vector sum, and $V(\cdot)$ is the volume functional (Lebesgue measure). (log concave)

(Brunn, Minkowski, Blaschke)

Two fundamental theorems

- The Brunn-Minkowski inequality. For convex bodies K, L in $\mathbb{R}^n$,

$$V((1-t)K + tL) \geq V(K)^{1-t}V(L)^t,$$

where $(1-t)K+tL = \{(1-t)x+ty : x \in K,\ y \in L\}$, $0 \leq t \leq 1$, is the vector sum, and $V(\cdot)$ is the volume functional (Lebesgue measure). (log concave)

(Brunn, Minkowski, Blaschke)

- Solution to the Minkowski problem. For each finite Borel measure μ on $\mathbb{S}^{n-1}$ not concentrated in a closed hemisphere, there exists a unique (up to translation) convex body K so that μ equals the surface area measure S_K of K if and only if

$$\int_{\mathbb{S}^{n-1}} u\, d\mu(u) = 0.$$

(Minkowski, Aleksandrov, Fenchel-Jessen)

Surface area measure

- Surface area measure S_K.

$$\frac{dV(K+tL)}{dt}\Big|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(u)\,dS_K(u).$$

Surface area measure

- Surface area measure S_K.

$$\frac{dV(K+tL)}{dt}\Big|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(u)\, dS_K(u).$$

- The differential of volume functional. First mixed volume.
  The global concept of reciprocal Gauss curvature,

$$dS_K(u) = \frac{1}{G_K(x)} du,$$

where $G_K(x)$ is the Gauss curvature at $x \in \partial K$ with outer unit normal u.

The Minkowski inequality of mixed volume

- The Minkowski inequality. For convex bodies K, L in $\mathbb{R}^n$,

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)\, dS_K(u) \geq V(L)^{\frac{1}{n}} V(K)^{\frac{n-1}{n}}.$$

The Minkowski inequality of mixed volume

- The Minkowski inequality. For convex bodies K, L in $\mathbb{R}^n$,

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)\, dS_K(u) \geq V(L)^{\frac{1}{n}} V(K)^{\frac{n-1}{n}}.$$

- The Minkowski inequality of mixed volume and the Brunn-Minkowski inequality are equivalent.

The Minkowski inequality of mixed volume

- The Minkowski inequality. For convex bodies K, L in $\mathbb{R}^n$,

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)\, dS_K(u) \geq V(L)^{\frac{1}{n}} V(K)^{\frac{n-1}{n}}.$$

- The Minkowski inequality of mixed volume and the Brunn-Minkowski inequality are equivalent.

- The isoperimetric inequality. The Minkowski inequality implies

$$S(K) \geq nV(B)^{\frac{1}{n}} V(K)^{\frac{n-1}{n}}.$$

Development of the Brunn-Minkowski theory

- Replace volume by quermassintegrals. (1930s, Aleksandrov, Fenchel)
  - General Brunn-Minkowski inequalities for quermassintegrals.
    The Aleksandrov-Fenchel inequality.

Development of the Brunn-Minkowski theory

- Replace volume by quermassintegrals. (1930s, Aleksandrov, Fenchel)
  - General Brunn-Minkowski inequalities for quermassintegrals.
    The Aleksandrov-Fenchel inequality.
  - The Christoffel-Minkowski problem for area measures.
    Sufficient conditions (Guan-Ma, 2003).

Development of the Brunn-Minkowski theory

- Replace volume by quermassintegrals. (1930s, Aleksandrov, Fenchel)
  - General Brunn-Minkowski inequalities for quermassintegrals.
    The Aleksandrov-Fenchel inequality.
  - The Christoffel-Minkowski problem for area measures.
    Sufficient conditions (Guan-Ma, 2003).

- Replace vector sum by L_p sum. (1950s, Firey)
  - L_p Brunn-Minkowski theory. (1990s, Lutwak)
  - L_p affine isoperimetric and Sobolev inequalities. (Lutwak, LYZ, Cianchi-LYZ, Haberl-Schuster, ...)
  - L_p Minkowski problem. (Lutwak, Lutwak-Oliker, Chou-Wang, Guan-Lin, Böröczky-LYZ, Zhu, ...)
  - The case of $p < 1$ is of great interest.

Geometric mean of convex bodies

Geometric mean $K^{1-t} \cdot L^t$. The largest convex body whose support function is smaller than $h_K^{1-t} h_L^t$,

$$K^{1-t} \cdot L^t = \{x \in \mathbb{R}^n : x \cdot u \leq h_K^{1-t}(u) h_L^t(u), u \in \mathbb{S}^{n-1}\}.$$

Geometric mean of convex bodies

Geometric mean $K^{1-t} \cdot L^t$. The largest convex body whose support function is smaller than $h_K^{1-t} h_L^t$,

$$K^{1-t} \cdot L^t = \{x \in \mathbb{R}^n : x \cdot u \leq h_K^{1-t}(u) h_L^t(u), u \in \mathbb{S}^{n-1}\}.$$

- Arithmetic mean $(1-t)K + tL$. It is the convex body whose support function is $(1-t)h_K + th_L$.

Geometric mean of convex bodies

Geometric mean $K^{1-t} \cdot L^t$. The largest convex body whose support function is smaller than $h_K^{1-t} h_L^t$,

$$K^{1-t} \cdot L^t = \{x \in \mathbb{R}^n : x \cdot u \leq h_K^{1-t}(u) h_L^t(u), u \in \mathbb{S}^{n-1}\}.$$

- Arithmetic mean $(1-t)K + tL$. It is the convex body whose support function is $(1-t)h_K + th_L$.
- Inclusion,
$$K^{1-t} \cdot L^t \subset (1-t)K + tL.$$

The logarithmic Brunn-Minkowski inequality

Conjecture 1. For origin-symmetric convex bodies K, L, there is the inequality,

$$V(K^{1-t} \cdot L^t) \geq V(K)^{1-t} V(L)^t, \quad 0 < t < 1.$$

The logarithmic Brunn-Minkowski inequality

Conjecture 1. For origin-symmetric convex bodies K, L, there is the inequality,

$$V(K^{1-t} \cdot L^t) \geq V(K)^{1-t} V(L)^t, \quad 0 < t < 1.$$

- Stronger than the classical Brunn-Minkowski inequality.

$$V((1-t)K + tL) \geq V(K^{1-t} \cdot L^t) \geq V(K)^{1-t} V(L)^t.$$

Cone volume measure

- Let K, L be convex bodies in $\mathbb{R}^n$ that contain the origin in their interior. Then

$$\frac{dV(K_t)}{dt}\Big|_{t=0} = n\int_{\mathbb{S}^{n-1}} \log h_L(u)\, dV_K(u),$$

where $K_t = K \cdot L^t$ is the geometric mean that is the maximal convex body so that $\log h_{K_t} \leq \log h_K + t\log h_L$.

Cone volume measure

- Let K, L be convex bodies in $\mathbb{R}^n$ that contain the origin in their interior. Then

$$\left.\frac{dV(K_t)}{dt}\right|_{t=0} = n \int_{\mathbb{S}^{n-1}} \log h_L(u)\, dV_K(u),$$

  where $K_t = K \cdot L^t$ is the geometric mean that is the maximal convex body so that $\log h_{K_t} \leq \log h_K + t \log h_L$.

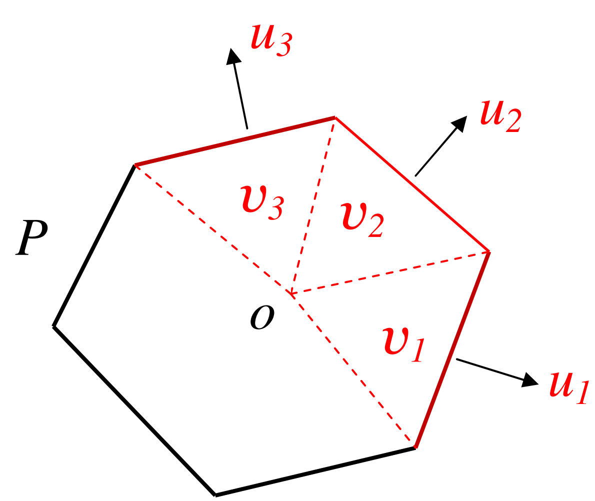
- Cone volume measure V_K is the log differential of the volume functional.

Cone volume measure of convex polytopes

- If P is a convex polytope containing the origin in its interior with unit normals $u_1, \ldots, u_m$ and cone volumes $v_1, \ldots, v_m$, then the discrete measure on $\mathbb{S}^{n-1}$,

$$V_P = \sum_{i=1}^{m} v_i \, \delta_{u_i},$$

  is the cone volume measure of P



The logarithmic Minkowski inequality

Conjecture 2. For origin-symmetric convex bodies K, L in $\mathbb{R}^n$, there is the inequality,

$$\frac{1}{V(K)} \int_{\mathbb{S}^{n-1}} \log \frac{h_L}{h_K} \, dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}.$$

The logarithmic Minkowski inequality

Conjecture 2. For origin-symmetric convex bodies K, L in $\mathbb{R}^n$, there is the inequality,

$$\frac{1}{V(K)} \int_{\mathbb{S}^{n-1}} \log \frac{h_L}{h_K} \, dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}.$$

- It is stronger than the classical Minkowski inequality of mixed volumes, and thus it is stronger than the classical isoperimetric inequality for symmetric bodies.

The logarithmic Minkowski inequality

Conjecture 2. For origin-symmetric convex bodies K, L in $\mathbb{R}^n$, there is the inequality,

$$\frac{1}{V(K)}\int_{\mathbb{S}^{n-1}} \log\frac{h_L}{h_K}\,dV_K \geq \frac{1}{n}\log\frac{V(L)}{V(K)}.$$

- It is stronger than the classical Minkowski inequality of mixed volumes, and thus it is stronger than the classical isoperimetric inequality for symmetric bodies.
- Answers are affirmative in $\mathbb{R}^2$ (Böröczky-Lutwak-Yang-Z., 2012), and in $\mathbb{R}^n$ under the condition of coordinates symmetry (Saroglou, 2014).

The logarithmic Minkowski problem

The logarithmic Minkowski problem. What are the necessary and sufficient conditions for a finite Borel measure μ on $\mathbb{S}^{n-1}$ so that it is the cone volume measure V_K of a convex body in $\mathbb{R}^n$?

The logarithmic Minkowski problem

The logarithmic Minkowski problem. What are the necessary and sufficient conditions for a finite Borel measure μ on $\mathbb{S}^{n-1}$ so that it is the cone volume measure V_K of a convex body in $\mathbb{R}^n$?

- The geometric equation,

$$V_K = \mu.$$

The logarithmic Minkowski problem

The logarithmic Minkowski problem. What are the necessary and sufficient conditions for a finite Borel measure μ on $\mathbb{S}^{n-1}$ so that it is the cone volume measure V_K of a convex body in $\mathbb{R}^n$?

- The geometric equation,

$$V_K = \mu.$$

- The PDE on $\mathbb{S}^{n-1}$,

$$\det(h_{ij} + h\delta_{ij}) = \frac{f}{h}.$$

The logarithmic Minkowski problem

The logarithmic Minkowski problem. What are the necessary and sufficient conditions for a finite Borel measure μ on $\mathbb{S}^{n-1}$ so that it is the cone volume measure V_K of a convex body in $\mathbb{R}^n$?

- The geometric equation,

$$V_K = \mu.$$

- The PDE on $\mathbb{S}^{n-1}$,

$$\det(h_{ij} + h\delta_{ij}) = \frac{f}{h}.$$

- Affine problem. The general case for measures is much harder than the case for functions. Existence of solutions involves measure concentration.

Measure concentration

The subspace concentration condition. A finite Borel measure μ on $\mathbb{S}^{n-1}$ satisfies the condition:

For any m-dimensional subspace $\xi \subset \mathbb{R}^n$, $0 < m < n$, there is

$$\frac{\mu(\xi \cap \mathbb{S}^{n-1})}{\mu(\mathbb{S}^{n-1})} \leq \frac{m}{n},$$

with equality only if μ is concentrated on complementary subspaces $\xi_m \cup \xi_{n-m}$.

Measure concentration

The subspace concentration condition. A finite Borel measure μ on $\mathbb{S}^{n-1}$ satisfies the condition:

For any m-dimensional subspace $\xi \subset \mathbb{R}^n$, $0 < m < n$, there is

$$\frac{\mu(\xi \cap \mathbb{S}^{n-1})}{\mu(\mathbb{S}^{n-1})} \leq \frac{m}{n},$$

with equality only if μ is concentrated on complementary subspaces $\xi_m \cup \xi_{n-m}$.

- A measure that has a positive continuous density satisfies the subspace concentration condition. A measure that concentrates most of its mass on the equator does not satisfies the subspace concentration condition.

Solution to the symmetric log Minkowski problem

Theorem. A non-zero finite even Borel measure on $\mathbb{S}^{n-1}$ is the cone volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.

(Böröczky-Lutwak-Yang-Z., 2012)

Solution to the symmetric log Minkowski problem

Theorem. A non-zero finite even Borel measure on $\mathbb{S}^{n-1}$ is the cone volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.

(Böröczky-Lutwak-Yang-Z., 2012)

- Solving singular PDE with measure concentration.

Solution to the symmetric log Minkowski problem

Theorem. A non-zero finite even Borel measure on $\mathbb{S}^{n-1}$ is the cone volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.

(Böröczky-Lutwak-Yang-Z., 2012)

- Solving singular PDE with measure concentration.
- Asymmetric case. Partial results. (Zhu, 2014; Böröczky-Zhu, 2015)

Solving a log minimization problem

- The functional $\Phi : C_e^+(\mathbb{S}^{n-1}) \to \mathbb{R}$ is defined by

$$\Phi(f) = \frac{1}{|\mu|}\int_{\mathbb{S}^{n-1}} \log f(u)\, d\mu(u) - \frac{1}{n}\log V([f]),$$

where $[f] = \{x \in \mathbb{R}^n : x \cdot u \leq f(u),\ u \in \mathbb{S}^{n-1}\}$ is the Wulff shape, $f \in C_e^+(\mathbb{S}^{n-1})$ is positive.

Solving a log minimization problem

- The functional $\Phi : C_e^+(\mathbb{S}^{n-1}) \to \mathbb{R}$ is defined by

$$\Phi(f) = \frac{1}{|\mu|} \int_{\mathbb{S}^{n-1}} \log f(u)\, d\mu(u) - \frac{1}{n} \log V([f]),$$

where $[f] = \{x \in \mathbb{R}^n : x \cdot u \leq f(u),\ u \in \mathbb{S}^{n-1}\}$ is the Wulff shape, $f \in C_e^+(\mathbb{S}^{n-1})$ is positive.

- The minimization problem,

$$\inf \left\{ \Phi(f) : f \in C_e^+(\mathbb{S}^{n-1}) \right\}.$$

Solving a log minimization problem

- The functional $\Phi : C_e^+(\mathbb{S}^{n-1}) \to \mathbb{R}$ is defined by

$$\Phi(f) = \frac{1}{|\mu|}\int_{\mathbb{S}^{n-1}} \log f(u)\, d\mu(u) - \frac{1}{n}\log V([f]),$$

  where $[f] = \{x \in \mathbb{R}^n : x \cdot u \le f(u),\ u \in \mathbb{S}^{n-1}\}$ is the Wulff shape, $f \in C_e^+(\mathbb{S}^{n-1})$ is positive.

- The minimization problem,

$$\inf \left\{\Phi(f) : f \in C_e^+(\mathbb{S}^{n-1})\right\}.$$

- Solving the minimization problem. Convergence (compactness) and non-degeneracy (positivity). (Delicate estimates of integrals)

Uniqueness of the log Minkowski problem

Conjecture 3. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. If $V_K = V_L$, then K and L have dilated vector summands. (Firey, BLYZ)

Uniqueness of the log Minkowski problem

Conjecture 3. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. If $V_K = V_L$, then K and L have dilated vector summands. (Firey, BLYZ)

- Affirmative in $\mathbb{R}^2$. (Gage, 1993; Stancu, 2003; Böröczky-LYZ, 2012)

Uniqueness of the log Minkowski problem

Conjecture 3. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. If $V_K = V_L$, then K and L have dilated vector summands. (Firey, BLYZ)

- Affirmative in $\mathbb{R}^2$. (Gage, 1993; Stancu, 2003; Böröczky-LYZ, 2012)
- If L is a ball in $\mathbb{R}^n$, then the solution is affirmative. (Firey, 1974, *worn stone problem*. He asked if the symmetry assumption could be removed.)

Uniqueness of the log Minkowski problem

Conjecture 3. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. If $V_K = V_L$, then K and L have dilated vector summands. (Firey, BLYZ)

- Affirmative in $\mathbb{R}^2$. (Gage, 1993; Stancu, 2003; Böröczky-LYZ, 2012)
- If L is a ball in $\mathbb{R}^n$, then the solution is affirmative. (Firey, 1974, *worn stone problem*. He asked if the symmetry assumption could be removed.)
- Let K be a smooth convex body in $\mathbb{R}^3$ that contains the origin and B a ball in $\mathbb{R}^3$ centered at the origin. If $V_K = V_B$ then $K = B$. (Andrews, 1999, curvature flow. Open in higher dimensions. Partial results (Guan-Ni, 2013))

Uniqueness of the log Minkowski problem

Conjecture 3. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. If $V_K = V_L$, then K and L have dilated vector summands. (Firey, BLYZ)

- Affirmative in $\mathbb{R}^2$. (Gage, 1993; Stancu, 2003; Böröczky-LYZ, 2012)
- If L is a ball in $\mathbb{R}^n$, then the solution is affirmative. (Firey, 1974, *worn stone problem*. He asked if the symmetry assumption could be removed.)
- Let K be a smooth convex body in $\mathbb{R}^3$ that contains the origin and B a ball in $\mathbb{R}^3$ centered at the origin. If $V_K = V_B$ then $K = B$. (Andrews, 1999, curvature flow. Open in higher dimensions. Partial results (Guan-Ni, 2013))
- Conjecture 3 and solution to the log minimization problem implies Conjecture 2.

The B-conjecture of the log concave measures

Conjecture 4. If γ is a log concave measure in $\mathbb{R}^n$ and L is an origin-symmetric convex body in $\mathbb{R}^n$, then the function,

$$f(t) = \gamma(e^t L),$$

is log concave in $(0, \infty)$.

The B-conjecture of the log concave measures

Conjecture 4. If γ is a log concave measure in $\mathbb{R}^n$ and L is an origin-symmetric convex body in $\mathbb{R}^n$, then the function,

$$f(t) = \gamma(e^t L),$$

is log concave in $(0, \infty)$.

- It is true for the Gaussian measure. (Banaszczyk, Latala (2002), Cordero-Fradelizi-Maurey (2004))

Equivalence

Conjectures $1-4$ are equivalent.

Thank you!