Permutation groups and cartesian decompositions

Csaba Schneider
with Cheryl Praeger and Robert Baddeley

UNIVERSIDADE FEDERAL
DE MINAS GERAIS

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Why cartesian decompositions?

Standard reductions in permutation group theory:

1. If $G \leq \mathsf{Sym}\,\Omega$ is intransitive, then $G \leq G^{\Omega_1} \times \cdots \times G^{\Omega_k}$ (the Ω_i are the G-orbits).
2. If $G \leq \mathsf{Sym}\,\Omega$ is imprimitive, then $G \leq (G_\Delta)^\Delta \wr \mathsf{S}_k$ (where Δ is a block).

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There may be a further reduction if $G \leq \Omega$ is primitive and $\Omega = \Gamma^\ell$ (product imprimitive).

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$W = \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ has a faithful action (the product action) on Ω:

$$(\gamma_1, \ldots, \gamma_\ell)(g_1, \ldots, g_\ell; \pi) = (\gamma_{1\pi^{-1}} g_{1\pi^{-1}}, \ldots, \gamma_{\ell\pi^{-1}} g_{\ell\pi^{-1}}).$$

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Thus $W \leq \mathsf{Sym}\,\Omega$ (maximal subgroup if Ω is finite and $|\Gamma| \geq 5$).

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The inclusion problem: Given a group $G \leq \mathsf{Sym}\,\Omega$, decide if $G \leq \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ (if $\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is an overgroup of G).

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Then we obtain the partitions $\Gamma_1, \ldots, \Gamma_\ell$ of Ω which satisfy

1. $|\Gamma_i| = |\Gamma_j|$ (homogeneous);
2. $|\delta_1 \cap \cdots \cap \delta_\ell| = 1$ for all $\delta_1 \in \Gamma_1, \ldots, \delta_\ell \in \Gamma_\ell$ (intersection property).

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THE DEFINITION OF CARTESIAN DECOMPOSITIONS

Definition

Ω is a set and $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ is a set of partitions of Ω such that 2. holds. Then $\mathcal{E}$ is said to be a cartesian decomposition of Ω.

If $\mathcal{E}$ satisfies 1., then $\mathcal{E}$ is said to be homogeneous.

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The map $\vartheta : \Gamma_1 \times \cdots \times \Gamma_\ell \to \Omega$

$$(\gamma_1, \ldots, \gamma_\ell) \mapsto \omega \quad \text{where} \quad \{\omega\} = \gamma_1 \cap \cdots \cap \gamma_\ell$$

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Laci Kovács ('89): system of product imprimitivity.

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Example (Normal (natural) inclusions)

Suppose that $M = M_1 \times \cdots \times M_\ell$ such that

1. $\{M_1, \ldots, M_\ell\}$ is a G_ω-conjugacy class;
2. $M_\omega = (M_\omega \cap M_1) \times \cdots \times (M_\omega \cap M_\ell)$.

Setting $\Gamma = [M_1 : M_\omega \cap M_1]$ (right coset space) we can embed $G \leq \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$.

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$$K_i = M_1 \times \cdots \times M_{i-1} \times (M_\omega \cap M_i) \times M_{i+1} \times \cdots \times M_\ell.$$

Lemma

Suppose that $M \underset{\text{trmin}}{\lhd} G \leq \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$. *Then* $M \leq B = (\mathsf{Sym}\,\Gamma)^\ell$, *and so* M *stabilises every partition* Γ_i.

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Consider the permutation representations $\pi_i : B \to \mathsf{Sym}\,\Gamma$:

$$g = (g_1, \ldots, g_\ell) \mapsto g_i.$$

Fix $\omega = (\gamma, \ldots, \gamma) \in \Gamma^\ell$.

$M\pi_i$ is transitive on Γ, and let K_i denote the stabiliser in M of γ under π_i.

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Another way to look at the K_i: choose $\delta_{i,\gamma} \in \Gamma_i$ ($\omega \in \delta_{i,\gamma}$). Then $K_i = M_{\delta_{i,\gamma}}$.

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Cartesian factorisations

The set $\{K_1, \ldots, K_\ell\}$ of subgroups of M satisfies:

1. $$\bigcap K_i = M_\omega; \qquad (1)$$
2. $$K_i \left(\bigcap_{j \neq i} K_j \right) = M \quad \text{for all } i; \qquad (2)$$
3. $\{K_1, \ldots, K_\ell\}$ is invariant under conjugation by G_ω;
4. homogeneous; that is, $|M : K_i| = |\Gamma|$ for all i.

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If M is a group and $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ is a family of proper subgroups of M such that (2) holds then $\mathcal{K}$ is said to be a cartesian factorisation of M.

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Theorem

Assuming $M \underset{\text{trmin}}{\lhd} G$, the group G can embedded into a wreath product $\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ iff M admits a G_ω-invariant homogeneous cartesian factorisation that satisfies (1) *and* (2).

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General hypothesis: Suppose that

1. $\Omega = \Gamma^\ell$ (not necessarily finite) with $\ell \geq 2$;
2. $W = \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ acting on Ω in product action;
3. $\pi : W \to \mathsf{S}_\ell$ is the natural projection;
4. $G \leq W$;
5. M is a transitive minimal normal subgroup of G;
6. $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ is the corresponding cartesian factorisation of M.

For instance: G is a finite primitive group of type PA, HC, TW, CD; or quasiprimitive group of type TW, CD.

Theorem (Baddeley & Praeger 1998)

If M is a finite simple group and $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ is a cartesian factorisation of M, then $\ell \leq 3$. Further,

1. *if $\ell = 3$, then $M \in \{Sp(4a, 2), P\Omega^+(8, 3), Sp(6, 2)\}$.*
2. *if $\mathcal{K}$ is homogeneous, then $\ell = 2$ and $M \in \{A_6, M_{12}, Sp(4, 2^d), P\Omega^+(8, q)\}$.*

Inclusions of groups with non-simple minimal normal subgroups

Theorem

Suppose that $M \underset{\text{trmin}}{\lhd} G \leq \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ *and* M *is transitive, non-abelian finite simple. Then*

1. $\ell = 2$;
2. $M \in \{A_6, M_{12}, Sp(4, 2^d), P\Omega^+(8, q)\}$;
3. M *is the unique minimal normal subgroup of* G, $G \leq Aut(M)$, *and the action of* M *is known up to permutational equivalence.*

INTRANSITIVE INCLUSIONS

Theorem

Under the general hypothesis, suppose that $T \underset{\text{min}}{\lhd} M \underset{\text{trmin}}{\lhd} G \leq W$ *and that* G *is finite.*

1. $G\pi$ *can have at most two orbits in* $\{1, \ldots, \ell\}$.
2. *If* $G\pi$ *has two orbits then* $T \in \{A_6, M_{12}, Sp(4, 2^d), P\Omega^+(8, q)\}$.

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where T is a simple group;
Then the cartesian factorisation $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ is a single conjugacy class.

Transitive inclusions

Let $\sigma_i : M \to T_i$ denote the coordinate projection.

Lemma (Generalisation of Scott)

If $K_j\sigma_i = T_i$ then $K_j = X \times \mathsf{C}_{K_j}(X)$ where $X \cong T$ is a diagonal subgroup that "covers" T_i.

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For instance $M = T^6$ and

$$\begin{aligned} K_1 &= A_1 \times B_2 \times \{(t, t\alpha_3) \mid t \in T_3\} \times T_5 \times T_6; \\ K_1 &= T_1 \times T_2 \times A_3 \times B_4 \times \{(t, t\alpha_5) \mid t \in T_5\}; \\ K_3 &= \{(t, t\alpha_1) \mid T \in T_1\} \times T_3 \times T_4 \times A_5 \times B_6 \end{aligned}$$

where $A_i,\ B_i < T_i$ and $\alpha_i : T_i \to T_{i+1}$ are isomorphisms.

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where $A_i,\ B_i < T_i$ and $\alpha_i : T_i \to T_{i+1}$ are isomorphisms.

The diagonal subgroup X in the lemma is called a strip involved in K_j. X is a non-trivial strip if $X \neq T_i$.

Uniform automorphisms

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Lemma

Let Y be a group and let $\alpha \in \mathrm{Aut}\,Y$. Then

$$Y \times Y = \{(y, y) \mid y \in Y\} \cdot \{(y, y\alpha) \mid y \in Y\} \tag{3}$$

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Lemma (CFSG)

Finite non-solvable groups do not admit uniform (fixed-point-free) automorphisms.

Theorem

Suppose that T does not admit a uniform automorphism and X, Y are direct products of non-trivial strips in T^k. Then $T^k \neq XY$.

The theorem applies if T is finite simple (Baddeley & Praeger 2003).

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There are infinite simple groups that admit uniform automorphisms, for instance $T = \mathrm{PSL}(d, \mathbb{F})$ where $\mathbb{F} = \overline{\mathbb{F}_p}$.

THE PROJECTIONS OF THE CARTESIAN FACTORISATIONS

Under the general hypothesis, let $\sigma_i : M \to T_i$ denote the i-th coordinate projection. Then

$$\mathcal{F}_i = \{K_j\sigma_i \mid j = 1, \ldots, \ell,\ K_j\sigma_i \neq T_i\}$$

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is a cartesian factorisation for the simple group T_i.

$\mathcal{F}_i$ is independent of i.
If G is finite, then $|\mathcal{F}_i| \leq 3$.

Theorem (6-Class Theorem)

If G is finite there are 6 different possibilities for the structure of K_i.

- CD_S $|\mathcal{F}_i| = 0$, *the K_i are subdirect subgroups of $M = T^k$ (direct products of strips)*
- CD_1 $|\mathcal{F}_i| = 1$, *the K_i do not involve strips;*
- CD_{1S} $|\mathcal{F}_i| = 1$, *the K_i involve strips;*
- $\mathrm{CD}_{2\sim}$ $|\mathcal{F}_i| = 2$, *$\mathcal{F}_i$ contain isomorphic subgroups, the K_i do not involve strips;*
- $\mathrm{CD}_{2\not\sim}$ $|\mathcal{F}_i| = 2$, *$\mathcal{F}_i$ contain non-isomorphic subgroups, the K_i do not involve strips;*
- CD_3 $|\mathcal{F}_i| = 3$, *the K_i do not involve strips.*

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Transitive inclusions

Theorem

Assume that $T^k = M \trianglelefteq G \leq W$ are as above and that $G\pi$ is transitive:

1. *The inclusions of type* CD_1 *and* CD_S *are normal.*
2. *Case* CD_S *holds iff G is quasiprimitive of type* CD.
3. *In the cases of* CD_{1S} *and* $\mathsf{CD}_{2\sim}$, *T admits a factorisation $T = AB$ with isomorphic subgroups. If G is finite, then $T \in \{A_6, M_{12}, Sp(4, 2^d), P\Omega^+(8, q)\}$.*
4. *In case of* CD_3, *T admits a cartesian factorisation with 3 subgroups. In particular, $T \in \{Sp(4a, 2), P\Omega^+(8, 3), Sp(6, 2)\}$.*
5. *G is not quasiprimitive of type* SD.

SPECIAL CASES: $\mathrm{PSL}(2,q)$

Knowing the factorisations of T, we may obtain more detailed information.

Theorem

Suppose that $T \underset{\text{min}}{\lhd} M \underset{\text{trmin}}{\lhd} G \leq W = \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ *and* $T \cong PSL(2,q)$*.*

1. *If* $q \neq 9$*, then the inclusion* $G \leq W$ *is of type* CD_1*,* CD_S *or* $\mathsf{CD}_{2\not\sim}$*.*
2. *If* $q \equiv 1 \pmod 4$ *and* $q \notin \{5, 9, 29\}$*, then the inclusion* $G \leq W$ *is of type* CD_S *or* CD_1*.*
3. *If* $q \equiv 3 \pmod 4$ *and* $q \notin \{7, 11, 19\}$ *and the inclusion* $G \leq W$ *is of type* $\mathsf{CD}_{2\not\sim}$*, then* G *admits an inclusion* $G \leq W_1$ *of type* CD_1*.*

Theorem (Li, Praeger, Sch, 2016)

Suppose that $T \underset{\text{min}}{\lhd} M \underset{\text{trmin}}{\lhd} G \leq W = \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$. *If* $\mathfrak{G}$ *is a finite* $(G, 2)$*-arc-transitive graph on the vertex set* Γ^ℓ*, then one of the following must hold:*

1. $\Gamma^\ell = 6^2$, $M = A_6$, *and* $\mathfrak{G}$ *is Sylvester's Double Six Graph;*
2. $\Gamma^\ell = 120^2$, $M = Sp(4, 4)$, *and* $\mathfrak{G}$ *is a graph of valency 17;*
3. *the inclusion* $G \leq W$ *is of type* $\mathsf{CD}_{2\not\sim}$.