Frequency Downshift in a Viscous Fluid

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Major Collaborators

- Alex Govan (Seattle University)
- Diane Henderson (Penn State University)
- Harvey Segur (University of Colorado at Boulder)

Background

History

Some (very) select historical results:

1. Experimental
   - Benjamin & Feir (1967)
   - Lake & Yuen (1977) and Lake *et al.* (1977)
   - Segur *et al.* (2005)

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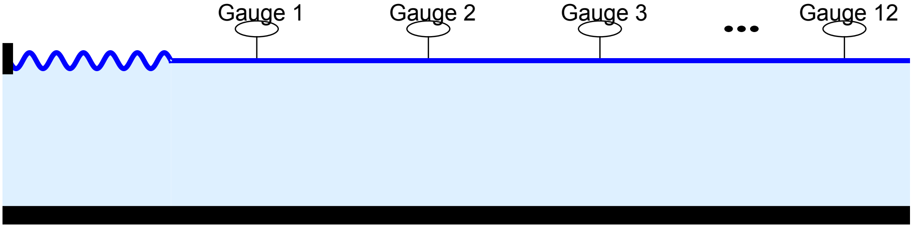
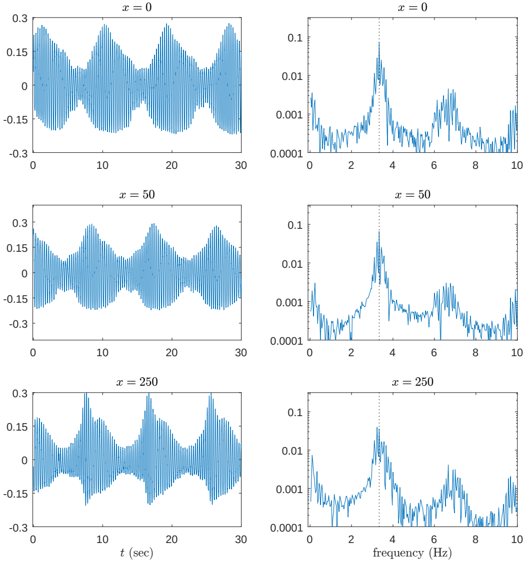
2. Modeling
   - Zakharov (1966)
   - Benjamin & Feir (1967)
   - Dysthe (1979)
   - Segur *et al.* (2005)
   - Dias *et al.* (2008)

Basic Experimental Setup



Experiments conducted by Diane Henderson (Penn State University).

Experimental Measurements



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A wave train is said to exhibit frequency downshifting (FD) if ω_m or ω_p decreases monotonically as it travels down the tank.

More Experimental Background

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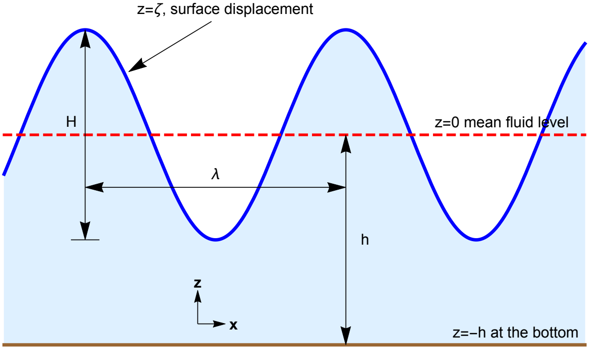
Our goal is to provide a mathematical justification for these observations without relying on wind or wave breaking effects.

Theoretical Background

Physical System



- $\zeta = \zeta(x, y, t)$ represents the surface displacement
- $\phi = \phi(x, y, z, t)$ represents the velocity potential

Governing Equations

The equations for an infinitely deep, inviscid, irrotational, incompressible fluid are

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad \text{for} \quad -\infty < z < \zeta(x,y,t)$$

$$\phi_z \to 0, \quad \text{as} \quad z \to -\infty$$

$$\zeta_t + \phi_x\zeta_x + \phi_y\zeta_y - \phi_z = 0, \quad \text{for} \quad z = \zeta(x,y,t)$$

$$\phi_t + g\zeta + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) = 0, \quad \text{for} \quad z = \zeta(x,y,t)$$

Approximate Models

In 1966, Zakharov assumed

$$\zeta(x,y,t) = \epsilon B e^{ik_0x - i\omega_0 t} + \epsilon^2 B_2 e^{2(ik_0x - i\omega_0 t)} + \epsilon^3 B_3 e^{3(ik_0x - i\omega_0 t)} + \cdots + c.c.$$

$$\phi(x,y,z,t) = \epsilon A_1 e^{k_0z + ik_0x - i\omega_0 t} + \epsilon^2 A_2 e^{2(k_0z + ik_0x - i\omega_0 t)} + \epsilon^3 A_3 e^{3(k_0z + ik_0x - i\omega_0 t)} + \cdots + c.c.$$

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in order to study the evolution of modulated wave trains. Here

- $\epsilon = 2|a_0|k_0 \ll 1$ is the dimensionless wave steepness
- a_0 represents a typical amplitude
- $k_0 > 0$ represents the wave number of the carrier wave
- $\omega_0 > 0$ represents the frequency of the carrier wave
- The A's depend on $X = \epsilon x$, $Y = \epsilon Y$, $Z = \epsilon z$, and $T = \epsilon t$
- The B's depend on X, Y, and T
- $c.c.$ stands for complex conjugate

NLS Equation

This led to the nonlinear Schrödinger (NLS) equation

$$2i\omega_0\Big(B_T + \frac{g}{2\omega_0}B_X\Big) + \epsilon\Big(\frac{g}{4k_0}B_{XX} - \frac{g}{2k_0}B_{YY} + 4gk_0^3|B|^2B\Big) = 0$$

where

$$\omega_0^2 = gk_0$$

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where

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- NLS preserves mass, $\mathcal{M}$
- NLS preserves linear momentum, $\mathcal{P}$
- NLS preserves the spectral mean, ω_m

Dysthe System

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$$2i\omega_0\left(B_T+\frac{g}{2\omega_0}B_X\right)+\epsilon\left(\frac{g}{4k_0}B_{XX}-\frac{g}{2k_0}B_{YY}+4gk_0^3|B|^2B\right)$$

$$+\epsilon^2\left(-i\frac{g}{8k_0^2}B_{XXX}+i\frac{3g}{4k_0^2}B_{XYY}+2igk_0^2B^2B_X^*+12igk_0^2|B|^2B_X+2k_0\omega_0B\Phi_X\right)=0, \text{ at } Z=0$$

$$\Phi_Z=2\omega_0\left(|B|^2\right)_X, \qquad \text{at } Z=0$$

$$\Phi_{XX}+\Phi_{YY}+\Phi_{ZZ}=0, \qquad \text{for } -\infty<Z<0$$

$$\Phi_Z\to 0, \qquad \text{as } Z\to-\infty$$

Dysthe System

- The Dysthe system preserves $\mathcal{M}$
- The Dysthe system does not preserve $\mathcal{P}$
- The Dysthe system does not preserve ω_m

Derivation of the Viscous Dysthe System

Governing Equations with Weak Viscosity

Dias *et al.* (2008) derived a weakly viscous generalization of the Euler equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad \text{for} \quad -\infty < z < \zeta(x,y,t)$$

$$\phi_z \to 0, \quad \text{as} \quad z \to -\infty$$

$$\zeta_t + \phi_x\zeta_x + \phi_y\zeta_y - \phi_z = 2\bar{\nu}\left(\zeta_{xx} + \zeta_{yy}\right), \quad \text{for} \quad z = \zeta(x,y,t)$$

$$\phi_t + g\zeta + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) = -2\bar{\nu}\phi_{zz}, \quad \text{for} \quad z = \zeta(x,y,t)$$

Where $\bar{\nu}$ is the kinematic viscosity.

Solution Ansatz

Generalizing the work of Dysthe, assume

$$\zeta(x,y,t)=\epsilon^3\bar{\eta}+\epsilon B e^{i\omega_0 t-ik_0x}+\epsilon^2 B_2 e^{2(i\omega_0 t-ik_0x)}+\epsilon^3 B_3 e^{3(i\omega_0 t-ik_0x)}+\cdots+c.c.$$

$$\phi(x,y,z,t)=\epsilon^2\bar{\phi}+\epsilon A_1 e^{k_0z+i\omega_0 t-ik_0x}+\epsilon^2 A_2 e^{2(k_0z+i\omega_0 t-ik_0x)}+\epsilon^3 A_3 e^{3(k_0z+i\omega_0 t-ik_0x)}+\cdots+c.c.$$

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Here

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- a_0 represents a typical amplitude
- $\omega_0 > 0$ represents the frequency of the carrier wave
- $k_0 > 0$ represents the wave number of the carrier wave
- The A_j's and $\bar{\phi}$ depend on $X = \epsilon x$, $Y = \epsilon Y$, $Z = \epsilon x$, $T = \epsilon t$
- The B_j's and $\bar{\eta}$ depend on X, Y, and T

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$$\phi(x,y,z,t) = \epsilon^2\bar{\phi} + \epsilon A_1 e^{k_0 z + i\omega_0 t - ik_0 x} + \epsilon^2 A_2 e^{2(k_0 z + i\omega_0 t - ik_0 x)} + \epsilon^3 A_3 e^{3(k_0 z + i\omega_0 t - ik_0 x)} + \cdots + c.c.$$

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Solution Ansatz

Further, assume

$$A_j = A_{j0} + \epsilon A_{j1} + \epsilon^2 A_{j2} + \epsilon^3 A_{j3} + \dots, \qquad \text{for } j = 1, 2, 3, \dots,$$

$$B_j = B_{j0} + \epsilon B_{j1} + \epsilon^2 B_{j2} + \epsilon^3 B_{j3} + \dots, \qquad \text{for } j = 2, 3, 4, \dots,$$

$$\bar{\eta} = \bar{\eta}_0 + \epsilon \bar{\eta}_1 + \epsilon^2 \bar{\eta}_2 + \dots,$$

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$$\bar{\phi} = \bar{\phi}_0 + \epsilon \bar{\phi}_1 + \epsilon^2 \bar{\phi}_2 + \dots.$$

$$\boxed{\bar{\nu} = \epsilon^2 \nu}$$

Dissipative NLS Equation

At $\mathcal{O}(\epsilon^3)$, this leads to the dissipative NLS (dNLS) equation

$$2i\omega_0\big(B_T+\frac{g}{2\omega_0}B_X\big)+\epsilon\Big(-\frac{g}{4k_0}B_{XX}+\frac{g}{2k_0}B_{YY}-4gk_0^3|B|^2B+4ik_0^2\omega_0\nu B\Big)=0$$

Viscous Dysthe System

At $\mathcal{O}(\epsilon^4)$, this leads to the viscous Dysthe (vDysthe) system

$$2i\omega_0\left(B_T+\frac{g}{2\omega_0}B_X\right)+\epsilon\left(\frac{g}{4k_0}B_{XX}-\frac{g}{2k_0}B_{YY}+4gk_0^3|B|^2B+4ik_0^2\omega_0\nu B\right)$$

$$+\epsilon^2\left(-i\frac{g}{8k_0^2}B_{XXX}+i\frac{3g}{4k_0^2}B_{XYY}+2igk_0^2B^2B_X^*+12igk_0^2|B|^2B_X+2k_0\omega_0B\Phi_X-8k_0\omega_0\nu B_X\right)=0,\text{ at } Z=0$$

$$\Phi_Z=2\omega_0\left(|B|^2\right)_X, \qquad \text{at } Z=0$$

$$\Phi_{XX}+\Phi_{YY}+\Phi_{ZZ}=0, \qquad \text{for } -\infty<Z<0$$

$$\Phi_Z\to 0, \qquad \text{as } Z\to-\infty$$

Change Variables

$$k_0 B(X, Y, T) = \tilde{B}(\xi, \chi)$$

$$\frac{k_0^2}{\omega_0} A(X, Y, Z, T) = \tilde{A}(\xi, \chi, \zeta)$$

$$\frac{k_0^2}{4\omega_0} \bar{\phi}_0(X, Y, Z, T) = \tilde{\Phi}(\xi, \chi, \zeta)$$

$$\frac{4k_0^2}{\omega_0} \nu = \delta$$

$$\chi = \epsilon k_0 X$$

$$\xi = \omega_0 T - 2k_0 X$$

$$\zeta = k_0 Z$$

The Dimensionless Viscous Dysthe System

$$iB_\chi + B_{\xi\xi} + 4|B|^2 B + i\delta B + \epsilon\Big(-8iB^2 B_\xi^* - 32i|B|^2 B_\xi - 16B\Phi_\xi + 5\delta B_\xi\Big) = 0, \qquad \text{at } \zeta = 0$$

$$\Phi_\zeta = -\Big(|B|^2\Big)_\xi, \qquad \text{at } \zeta = 0$$

$$4\Phi_{\xi\xi} + \Phi_{\zeta\zeta} = 0, \qquad \text{for} -\infty < \zeta < 0$$

$$\Phi_\zeta \to 0, \qquad \text{as } \zeta \to -\infty$$

There is only one free parameter, δ, in this system.

Properties of the Viscous Dysthe System

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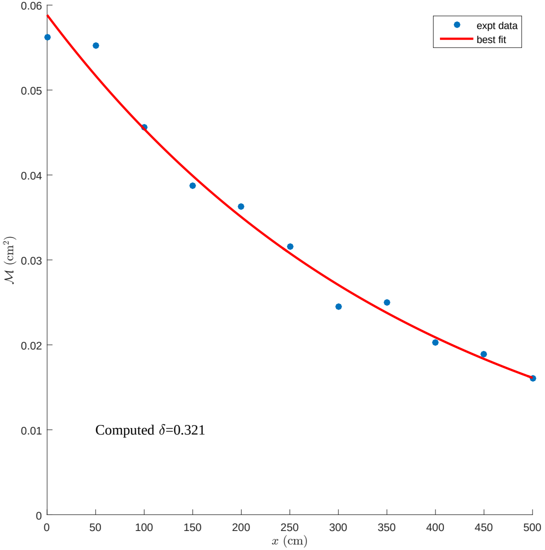
The χ dependency of $\mathcal{M}$ is given by

$$\mathcal{M}_\chi = -2\delta\mathcal{M} - 10\frac{\delta}{\omega_0}\mathcal{P}$$

At leading order in ϵ, this relationship determines δ.

Determining δ



Properties of the Viscous Dysthe System

The viscous Dysthe system does not preserve the spectral mean

$$\left(\omega_m\right)_\chi = \left(\frac{\mathcal{P}}{\mathcal{M}}\right)_\chi = -\frac{10\delta}{\omega_0\mathcal{M}^2}\Big(\mathcal{M}\mathcal{Q} - \mathcal{P}^2\Big) - \frac{16}{\omega_0}\frac{\mathcal{R}}{\mathcal{M}}$$

where

$$\mathcal{Q} = \frac{\epsilon^4\omega_0^2}{k_0^2}\frac{1}{\epsilon\omega_0 L}\int_0^{\epsilon\omega_0 L}|B_\xi|^2 d\xi$$

$$\mathcal{R} = \frac{\epsilon^4\omega_0^2}{k_0^2}\frac{1}{\epsilon\omega_0 L}\,\mathrm{Im}\left(\int_0^{\epsilon\omega_0 L}|B|^2 B^* B_{\xi\xi} d\xi\right)$$

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The Cauchy-Schwarz inequality establishes that
$(\mathcal{M}\mathcal{Q} - \mathcal{P}^2) \geq 0$.

Plane-Wave Solutions of the Viscous Dysthe System

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The viscous Dysthe system admits plane-wave solutions given by

$$B(\xi,\chi) = B_0 \exp\left(w_r(\chi) + iw_i(\chi)\right)$$

$$\Phi(\xi,\chi) = 0$$

where

$$w_r(\chi) = -\delta\chi$$

$$w_i(\chi) = \frac{2B_0^2 k_0^2}{\delta}\left(\mathrm{e}^{-2\delta\chi} - 1\right)$$

and B_0 is a real parameter.

Stability of Plane-Wave Solutions

Consider perturbed solutions of the form

$$B_{\text{pert}}(\xi,\chi) = \Big(B_0 + \mu u(\xi,\chi) + i\mu v(\xi,\chi) + \mathcal{O}(\mu^2)\Big)\exp\Big(w_r(\chi) + iw_i(\chi)\Big)$$

$$\Phi_{\text{pert}}(\xi,\chi,\zeta) = 0 + \mu p(\xi,\chi,\zeta) + \mathcal{O}(\mu^2)$$

where

- μ is a small real parameter
- u, v, and p are real-valued functions

Plane-Wave Stability **Observations**

The non-transient linear stability problem gives (in physical coordinates)

$$\eta(x,t) = d_0 \exp\left(i\omega_0 t + if_0(x) - 4\bar{\nu}\frac{k_0^3}{\omega_0}x\right)$$
$$+ d_1 \exp\left(i\omega_0(1-\epsilon q)t + if_1(x) - 4\bar{\nu}\frac{k_0^3}{\omega_0}(1-5\epsilon q)x\right)$$
$$+ d_2 \exp\left(i\omega_0(1+\epsilon q)t + if_2(x) - 4\bar{\nu}\frac{k_0^3}{\omega_0}(1+5\epsilon q)x\right) + c.c.$$

where d_j are complex constants and f_j are real-valued functions.

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- The amplitude of the carrier wave (the mode with wave number $k_0 > 0$) decays exponentially.
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- The amplitude of the upper sideband (the mode with wave number $k_0 + \epsilon|q|$) decays more rapidly than the amplitude of the carrier wave.
- The amplitude of the lower sideband ($k_0 - \epsilon|q|$) decays more slowly than does the amplitude of the carrier wave.

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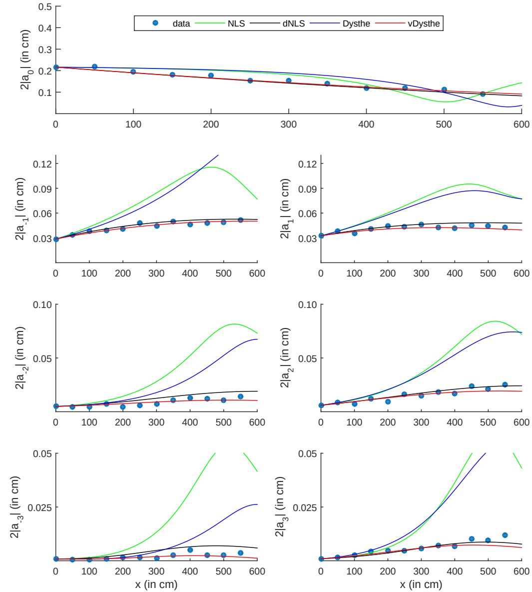
$$B_2 = k_0 B^2$$

$$B_3 = \frac{3}{2} k_0^2 B^3$$

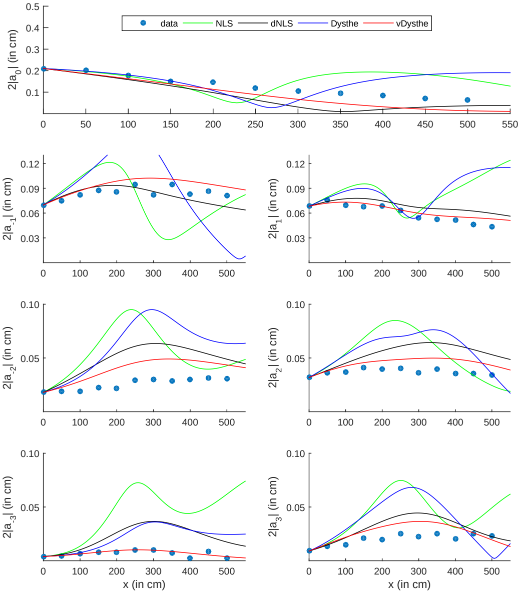
- This suggests that FD will be observed in the higher harmonics before it is observed in the fundamental.

Comparisons with Experiments

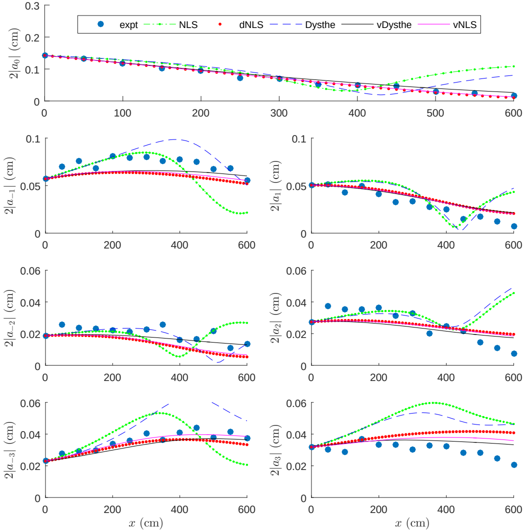
Moderate Amplitude Experiment



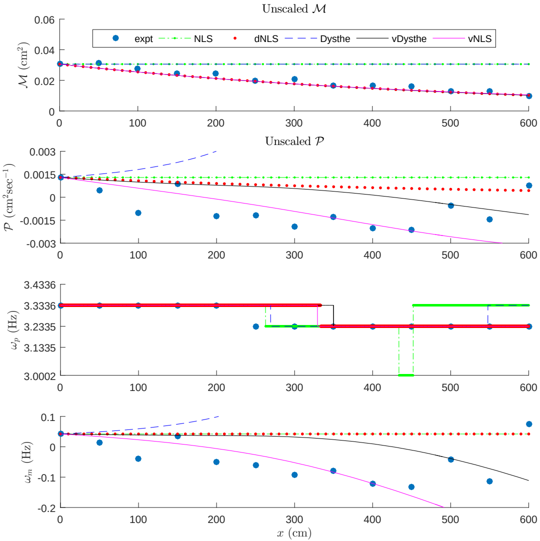
Large Amplitude Experiment



Feb 11 Experiment Fourier Amplitudes



Feb 11 Experiment Quantities



Summary

The viscous Dysthe system

- Accurately models experiments of “moderate” amplitude
- Accurately models experiments of “large” amplitude
- Admits plane-wave instabilities that yield FD
- ω_m, ω_p and $\mathcal{P}$ usually decrease