Alberta Number Theory Days 2017

THE SIZE FUNCTION FOR A NUMBER FIELD

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Content

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- Let Δ be the discriminant of F.
- Let O_F be the ring of integers of F.
- Let $\sigma_1, \ldots, \sigma_n$ be n real embeddings of F.
- Denote by $\Phi = (\sigma_1, \ldots, \sigma_n)$. Then

$$\Phi : F \hookrightarrow \mathbb{R}^n \text{ takes } x \in F \text{ to } (\sigma_i(x))_i \in \mathbb{R}^n.$$

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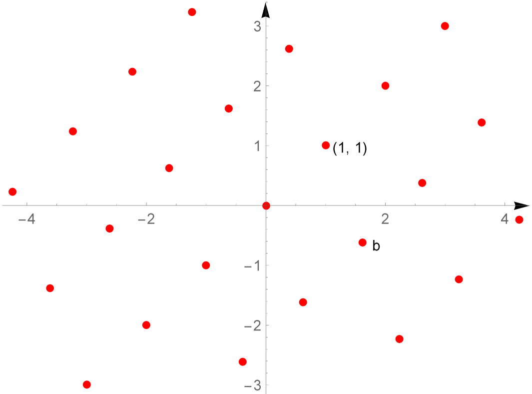

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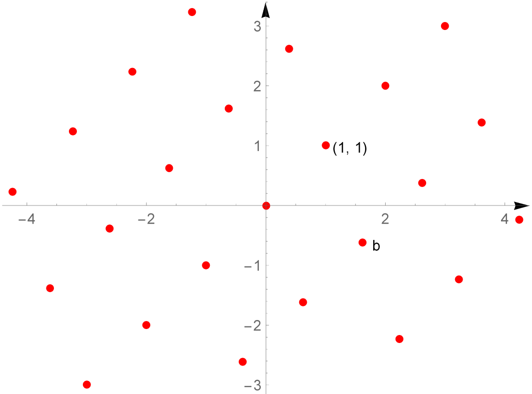
Ex: Let $F = \mathbb{Q}(\sqrt{5})$. Then $\Phi(O_F)$ is a lattice in $\mathbb{R}^2$.



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Proposition

Let I be a factional ideal of F. Then $\Phi(I)$ is a lattice in $\mathbb{R}^n$.

Ideal lattices

Definition (Ideal lattices)

An ideal lattice is a lattice (I, q), where

- I is a (fractional) O_F-ideal and
- $q : I \times I \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form st
  $q(\lambda x, y) = q(x, \bar{\lambda} y)$ (Hermitian property)
  for all $x, y \in I$ and for all $\lambda \in O_F$.

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Define $q_u(x, y) = \langle u\Phi(x), u\Phi(y)\rangle$ for any $x, y \in I$.

$$\|x\|_u^2 = q_u(x, x) = \|u\Phi(x)\|^2 = \sum_{i=1}^{n} u_i^2[\sigma_i(x)]^2.$$

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Then (I, q_u) is an ideal lattice.

The size function for lattices

Let L be a lattice of $\mathbb{R}^n$.

$$h^0(L) := \log \sum_{x \in L} e^{-\pi \|x\|^2}.$$

The size function for a number field

Similarly, h^0 is defined for the ideal lattice (I, q_u).

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Definition

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- (I, q_u) is also called the ideal lattice associated to D.
- $h^0(D) := h^0(I, q_u)$.

Analogies

Algebraic curve

- Divisor.

Number field F

- Arakelov divisor.

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- Divisor.
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The Riemann-Roch Theorem

For an algebraic curve

$$h^0(D) - h^0(\kappa - D) = deg(D) - (g - 1).$$

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We define the canonical Arakelov divisor κ to be the Arakelov divisor $(\partial, 1)$ whose ideal part is the inverse of the different ∂ of F.

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van der Geer and Schoof (1999)

Let F be a number field with discriminant Δ and let D be an Arakelov divisor. Then

$$h^0(D) - h^0(\kappa - D) = \deg(D) - \frac{1}{2}\log|\Delta|.$$

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- A principal Arakelov divisor has the form (I, u) where $I = x^{-1}O_F$ and $u = |\Phi(x)| = (|\sigma_i(x)|)_i$ and $x \in F^\times$.

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$$\mathrm{Pic}_F^0 \longrightarrow \{\text{isometry classes of ideal lattices of covolume } \sqrt{\Delta}\}$$
$$\text{the class of } D = (I, u) \longmapsto \text{ the isometry class of } (I, q_u)$$
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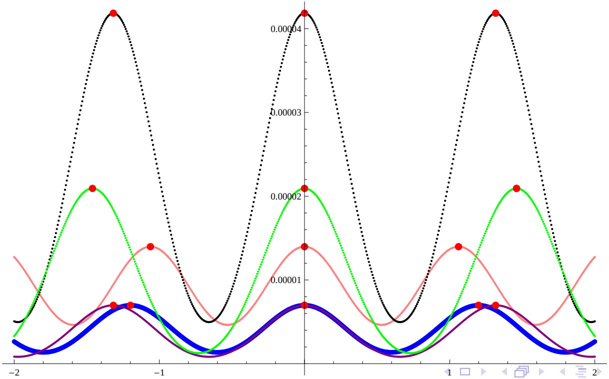
Note: h^0 is well defined on Pic_F^0.

The conjecture of van der Geer and Schoof

Let F be a real quadratic field (Galois over $\mathbb{Q}$) or quadratic extension of a complex quadratic field K (Galois over K). The origin is the divisor $(O_F, 1)$.



The conjecture of van der Geer and Schoof

A cyclic cubic field (Galois over $\mathbb{Q}$).
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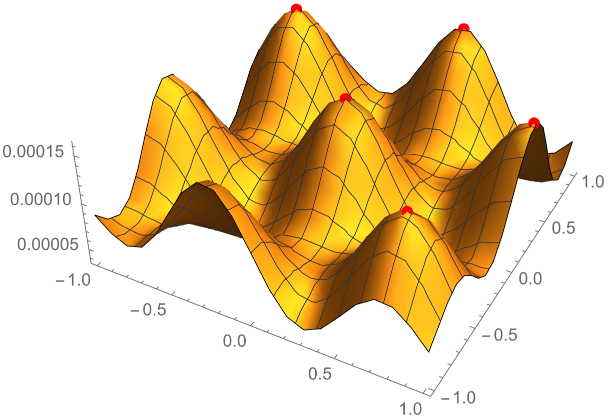

The conjecture of van der Geer and Schoof

Conjecture. Let F be a number field that is Galois over $\mathbb{Q}$ or over an imaginary quadratic field. Then the function h^0 on Pic_F^0 assumes its maximum on the trivial class $(O_F, 1)$.

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Results. The conjecture is proved for number fields of degree:

- $n = 2$: Francini (2001).
- $n = 3$: Francini (2004) - For some certain pure cubic fields.
- $n = 4$: (2014) For quadratic extensions of imaginary quadratic fields.
- $n = 3$: (2016) For cyclic cubic fields.

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