Optimal designs in regression models with correlated errors

Anatoly Zhigljavsky, Cardiff

Collaborators: Holger Dette and Andrey Pepelyshev

Banff, August 10, 2017

Regression with correlated errors

Linear regression model:

$$
\begin{aligned}
y(\mathbf{x}) &= \theta_1 f_1(\mathbf{x}) + \ldots + \theta_m f_m(\mathbf{x}) + \varepsilon(\mathbf{x}) \\
&= \boldsymbol{\theta}^T f(\mathbf{x}) + \varepsilon(\mathbf{x}) \ ,
\end{aligned}
$$

where $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^d$,
$f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T$,
$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)^T$,
$E[\varepsilon(\mathbf{x})] = 0$,
$K(\mathbf{x}, \mathbf{x}') = \mathbb{E}[\varepsilon(\mathbf{x})\varepsilon(\mathbf{x}')]$.
Here $K(\mathbf{x}, \mathbf{x}')$ is a covariance kernel (a positive definite function).
For stationary processes, $K(\mathbf{x}, \mathbf{x}') = \rho(\mathbf{x} - \mathbf{x}')$.

Standard Estimators

For observations at $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$:

$$\text{WLSE}: \qquad \widehat{\boldsymbol{\theta}}_{WLSE} = (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}\mathbf{Y},$$

$$\mathrm{Var}(\widehat{\boldsymbol{\theta}}_{WLSE}) = (\mathbf{X}^T\mathbf{W}\,\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}\,\boldsymbol{\Sigma}\mathbf{W}\,\mathbf{X}(\mathbf{X}^T\mathbf{W}\,\mathbf{X})^{-1},$$

where $\mathbf{X} = (f_i(\mathbf{x}_j))_{j=1,\ldots,N}^{i=1,\ldots,m}$ and $\boldsymbol{\Sigma} = (K(\mathbf{x}_i, \mathbf{x}_j))_{i,j=1,\ldots,N}$.

$$\begin{aligned}
\text{OLSE}: &\qquad \widehat{\boldsymbol{\theta}}_{OLSE} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}, \\
\text{BLUE}: &\qquad \widehat{\boldsymbol{\theta}}_{BLUE} = (\mathbf{X}^T\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\boldsymbol{\Sigma}^{-1}\mathbf{Y}, \\
\text{SLSE}: &\qquad \widehat{\boldsymbol{\theta}}_{SLSE} = (\mathbf{X}^T\mathbf{S}\,\mathbf{X})^{-1}\mathbf{X}^T\mathbf{S}\,\mathbf{Y}.
\end{aligned}$$

Here $\mathbf{S}$ is an $N \times N$ diagonal matrix with entries $+1$ and -1 on the diagonal; note that if $\mathbf{S} \neq \mathbf{I}_N$ then SLSE is not a standard OLSE.

Continuous version

General estimator:

$$\hat{\boldsymbol{\theta}}_\zeta = \int y(\mathbf{x})\zeta(d\mathbf{x})\,,$$

where $\zeta(d\mathbf{x})$ is a signed vector-measure.

$$\widehat{\boldsymbol{\theta}}_{OLSE} = \int y(\mathbf{x}) M^{-1}(\xi) f(\mathbf{x}) \xi(d\mathbf{x}),$$

where

$$M(\xi) = \int f(\mathbf{x}) f^T(\mathbf{x}) \xi(d\mathbf{x}),$$

and $\xi(d\mathbf{x})$ is a design (probability measure for OLSE; a signed measure for SLSE). The covariance matrix of $\widehat{\boldsymbol{\theta}}_{OLSE}$ is

$$\mathrm{Var}(\widehat{\boldsymbol{\theta}}_{OLSE}) = \mathbf{M}(\xi)^{-1}\left[\int\int K(\mathbf{x},\mathbf{z})\mathbf{f}(\mathbf{x})\mathbf{f}^T(\mathbf{z})\xi(d\mathbf{x})\xi(d\mathbf{z})\right]\mathbf{M}(\xi)^{-1}\,.$$

Plan

- Continuous BLUE
  - Characterizations of the BLUE
  - Structure of the BLUE, examples
  - BLUE with gradient-enhanced observations
  - Discretization of the continuous BLUE
- OLSE/SLSE versus BLUE
  - One-parameter case
  - Multi-parameter case

BLUE

Let ν be a vector-measure such that

$$\int K(\mathbf{x}, \mathbf{x}')\nu(d\mathbf{x}') = f(\mathbf{x})$$

and the matrix $\int \nu(dt) f^T(t)$ is non-degenerate. Then

$$\zeta(d\mathbf{x}) = D\nu(d\mathbf{x}) \text{ with } \; D = \left[\int \nu(d\mathbf{x}) f^T(\mathbf{x})\right]^{-1}$$

determines the BLUE

$$\widehat{\boldsymbol{\theta}}_{BLUE} = \int y(\mathbf{x})\zeta(d\mathbf{x})\,;$$

$$\mathrm{Var}(\widehat{\boldsymbol{\theta}}_{BLUE}) = D.$$

BLUE, an example (Markovian noise)

$\mathcal{X} = [a, b]$. $K(t, s) = u(t)v(s)$ for $t \le s$ and $K(t, s) = v(t)u(s)$ for $t > s$, where $u(\cdot)$ and $v(\cdot)$ are positive functions such that $q(t) = u(t)/v(t)$ is monotonically increasing. Define the signed vector-measure

$$\nu(dt) = z_A \delta_A(dt) + z_B \delta_B(dt) + z(t)dt$$

with

$$\begin{aligned} z_A &= \frac{1}{v^2(A)q'(A)}\left[\frac{f(A)u'(A)}{u(A)} - f'(A)\right], \\ z(t) &= -\frac{1}{v(t)}\left[\frac{h'(t)}{q'(t)}\right]', \quad z_B = \frac{h'(B)}{v(B)q'(B)}, \end{aligned}$$

where $h(t) = f(t)/v(t)$. Assume that the matrix $C = \int f(t)\zeta^T(dt)$ is non-degenerate. Then the estimate $\hat{\boldsymbol{\theta}}_\zeta$ with $\zeta(dt) = C^{-1}\nu(dt)$ is a BLUE with covariance matrix C^{-1}.

BLUE, an example (triangular kernel)

$$K(t,s) = \max(1 - \lambda|t-s|, 0)\,, \lambda \leq 1, \quad t,s \in [0,1]\,.$$

Exact optimal designs for this covariance kernel (with $\lambda = 1$) have been considered in WM & Pazman (2003); WM & VF (2007).

$$\nu(dt) = \left[-\frac{f'(0)}{2\lambda} + f_\lambda \right] \delta_0(dt) + \left[\frac{f'(1)}{2\lambda} + f_\lambda \right] \delta_1(dt) - \frac{f''(t)}{2\lambda} dt\,,$$

where $f_\lambda = (f(0) + f(1))/(4 - 2\lambda)$. The estimator $\hat{\boldsymbol{\theta}}_\zeta$ with $\zeta(dt) = C^{-1}\nu(dt)$ with $C = \int f(t)\zeta^T(dt)$ is the BLUE.

BLUE for processes with trajectories in $C^1[A, B]$: Gradient-enhanced estimation

Assume that the error process is exactly once continuously differentiable (in the mean-square sense). General estimator:

$$\hat{\boldsymbol{\theta}}_{\zeta_0,\zeta_1} = \int y(t)\zeta_0(dt) + \int y'(t)\zeta_1(dt),$$

where $\zeta_0(dt)$ and $\zeta_1(dt)$ are signed vector-measures.
Assume ν_0 and ν_1 are vector-measures such that

$$\int K(t,s)\nu_0(dt) + \int \frac{\partial K(t,s)}{\partial t}\nu_1(dt) = f(s), \;\; \forall s \in [A, B]$$

$$C = \int f(t)\nu_0^T(dt) + \int f'(t)\nu_1^T(dt)$$

is a non-degenerate matrix. Then the estimator $\hat{\boldsymbol{\theta}}_{\zeta_0,\zeta_1}$ with $\zeta_i = C^{-1}\nu_i$ $(i = 0, 1)$ is a BLUE with covariance matrix C^{-1}.

BLUE, integrated error processes

$$K(t,s) = \int_a^t \int_a^s K_0(u,v) du dv.$$

where $0 \leq a \leq A$; $t, s \in [A, B]$. This is a more general class of kernels than that considered in S-Y.
Two examples:

$$K(t,s) = \int_a^t \int_a^s \min(t', s') dt' ds'$$

$$= \frac{\max(t,s)(\min(t,s)^2 - a^2)}{2} - \frac{a^2(\min(t,s) - a)}{2} - \frac{\min(t,s)^3 - a^3}{6},$$

$$\begin{aligned} K(t,s) &= \int_0^t \int_0^s \max\{0, 1 - \lambda |t' - s'|\} dt' ds' \\ &= ts - \lambda \min(t,s)\Big(3\max(t,s)^2 - 3ts + 2\min(t,s)^2\Big)/6. \end{aligned}$$

$CAR(2)$ and $AR(2)$ noise

$t \in [A, B]$, $\varepsilon(t)$ is a continuous autoregressive (CAR) process of order 2. Formally, it is a solution of the linear stochastic differential equation

$$d\varepsilon^{(1)}(t) = a_1\varepsilon^{(1)}(t) + a_2\varepsilon(t) + \sigma_0^2 dW(t),$$

where $W(t)$ is a standard Wiener process.
There are three different forms of the autocorrelation function $\rho(t)$ of CAR(2) processes:

$$\rho_1(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1|t|} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2|t|}\,,\ (\lambda_1 \neq \lambda_2,\ \lambda_1 > 0,\ \lambda_2 > 0)$$

$$\rho_2(t) = e^{-\lambda|t|}\left\{\cos(q|t|) + \frac{\lambda}{q}\sin(q|t|)\right\}, \quad \lambda > 0,\ q > 0,$$

$$\rho_3(t) = e^{-\lambda|t|}(1 + \lambda|t|)\,, \quad \lambda > 0,$$

The kernel associated with ρ_3 is widely known as Matérn kernel with parameter $3/2$.
Discretised $CAR(2)$ process is not $AR(2)$; it is $ARMA(2;1)$.

BLUE for processes with exactly q derivatives

Let $\mathcal{X} \subseteq [A, B]$, $K(\cdot, \cdot) \in C^q([A, B] \times [A, B])$ and $f(\cdot) \in C^q([A, B])$ for some $q \geq 0$. Suppose that the process $y(t)$ along with its q derivatives can be observed at all $t \in \mathcal{X}$, $Y = \left(y^{(0)}(t), \ldots, y^{(q)}(t)\right)^T$. Let $\nu_0, \ldots, \nu_q$ be signed vector-measures such that the matrix

$$C = \sum_{i=0}^{q} \int \nu_i(dt) \left(f^{(i)}\right)^T (t)$$

is non-degenerate. Define $\zeta = (\zeta_0, \ldots, \zeta_q)$, $\zeta_i(dt) = C^{-1}\zeta_i(dt)$ for $i = 0, \ldots, q$. The estimator $\hat{\boldsymbol{\theta}}_\zeta = \int \zeta(dt) Y(t)$ is the BLUE if and only if

$$\sum_{i=0}^{q} \int K^{(i)}(t, s)\nu_i(dt) = f(s)$$

for all s. The covariance matrix of $\hat{\boldsymbol{\theta}}_\zeta$ is $\mathrm{Var}(\hat{\boldsymbol{\theta}}_\zeta) = C^{-1}$.

Non-uniqueness of the BLUE measures

If $\mathcal{X} = [A, B]$ and f has sufficient number of derivatives, then for a given set of signed vector-measures $G = (G_0, G_1, \ldots, G_q)$ on $\mathcal{X}$ we can always find another set of measures $H = (H_0, H_1, \ldots, H_q)$ such that the signed vector-measures $H_1, \ldots, H_q$ have no continuous parts but the expectations and covariance matrices of the estimators $\hat{\boldsymbol{\theta}}_G$ and $\hat{\boldsymbol{\theta}}_H$ coincide.

Discretization of the continuous BLUE

Assume $\mathcal{X} = [A, B]$ and $m = 1$. No derivatives.
BLUE is

$$\widehat{\boldsymbol{\theta}}_{BLUE} = \int y(\mathbf{x})\zeta(d\mathbf{x})$$

where

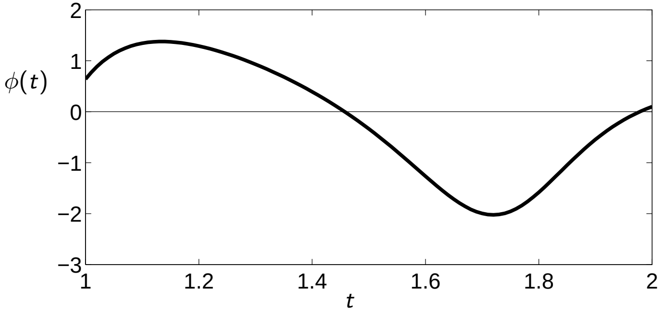
$$\zeta(d\mathbf{x}) = c_A\delta_A(d\mathbf{x}) + c_B\delta_B(d\mathbf{x}) + \phi(\mathbf{x})d\mathbf{x}\,,$$

$$D = \mathrm{Var}(\widehat{\boldsymbol{\theta}}_{BLUE}) = \left[\int \nu(d\mathbf{x})f(\mathbf{x})\right]^{-1}$$

with $\int K(\mathbf{x}, \mathbf{x}')\nu(d\mathbf{x}') = f(\mathbf{x})$, $\zeta(d\mathbf{x}) = D\nu(d\mathbf{x})$.
Most natural discretization is: take A, B and the quantiles of the density const$|\phi(\mathbf{x})|$.
S-Y advice: take A, B and the quantiles of the density const$|\phi(\mathbf{x})|^{2/3}$.
Similar with derivatives where we really have to use the derivatives (only at A and B).

The density of the optimal design

$f(t) = 1 + 0.5\sin(2\pi t)$, $t \in [1, 2]$, $K(t, t') = u(t)v(t')$ with $u(t) = t^2$ and $v(t) = t$.



Variances of the N-point designs

$f(t) = 1 + 0.5\sin(2\pi t)$, $t \in [1, 2]$, covariance kernel with $u(t) = t^2$ and $v(t) = t$.

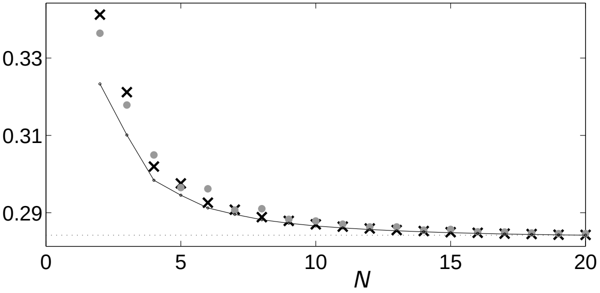


Figure: *The variance of BLUE for the proposed $(N+2)$-point designs (grey circles), the $(N+2)$-point designs from [S-Y, 1966] (crosses) and the BLUE with corresponding optimal $(N+2)$-point designs (line); $N = 2, \ldots, 20$.*

OLSE versus BLUE

OLSE vs BLUE:
Bloomfield P., and Watson G. S., "The inefficiency of least squares." Biometrika 62 (1975): 121-128.
Knott, M.. "On the minimum efficiency of least squares." Biometrika (1975): 129-132.

OLSE versus BLUE, plan:

- One-parameter case
  - SLSE vs BLUE (almost the same)
  - Location-scale model (convex, easy)
  - General f: non-convex problem but still often solvable
- Multi-parameter case: emulation of the BLUE

OLSE, $m = 1$

Model: $y(\mathbf{x}) = \theta f(\mathbf{x}) + \varepsilon(\mathbf{x}),\ m = 1.$
The variance of the OLSE is the design optimality criterion:

$$D(\xi) = \left[\int f^2(\mathbf{x})\xi(d\mathbf{x})\right]^{-2} \int\int K(\mathbf{x},\mathbf{z}) f(\mathbf{x}) f(\mathbf{z}) \xi(d\mathbf{x})\xi(d\mathbf{z})$$

as the design optimality functional. $\xi(d\mathbf{x})$ is a design (probability measure for OLSE, a signed measure with total mass 1 for SLSE).

In general, this functional is not convex.

Location-scale model: $f(\mathbf{x}) = 1$

The design optimality functional becomes

$$D(\xi) = \int_{\mathcal{X}} \int_{\mathcal{X}} K(\mathbf{x}, \mathbf{z}) \xi(d\mathbf{x}) \xi(d\mathbf{z}) \,.$$

This functional is convex:

$$D((1-\alpha)\xi + \alpha\xi_0) < (1-\alpha)D(\xi) + \alpha D(\xi_0)$$

If K is strictly positive definite, then D is strictly convex.
Optimality condition: ξ^* is optimal if and only if

$$\min_{x \in \mathcal{X}} \phi(\mathbf{x}, \xi^*) \geq D(\xi^*), \;\; \phi(\mathbf{x}, \xi) = \int K(\mathbf{x}, \mathbf{z}) \xi(d\mathbf{z}).$$

In potential theory, $1/D(\xi^*)$ is called (Wiener) capacity of the set $\mathcal{X}$.

Some examples, $f(\mathbf{x}) = 1$, $\mathcal{X} = [-1, 1]$

- $\rho(t) = e^{-\lambda|t|}$: ξ^* is a mixture of the continuous uniform measure and a two-point discrete measure supported on $\{-1, 1\}$:

$$p^*(x) = \omega^* \left(\frac{1}{2}\delta_1(x) + \frac{1}{2}\delta_{-1}(x) \right) + (1 - \omega^*)\frac{1}{2}\mathbf{1}_{[-1,1]}(x),$$

  where $\omega^* = 1/(1+\lambda)$, $b(\cdot, \xi^*) = D(\xi^*) = 1/(1+\lambda)$.
- triangular correlation function $\rho(t) = \max\{0, 1 - \lambda|t|\}$: discrete design
- $\rho(t) = 1/|t|^\alpha$, $0 < \alpha < 1$: optimal design has Beta-density

$$p^*(x) = \frac{2^{-\alpha}}{B(\frac{1+\alpha}{2}, \frac{1+\alpha}{2})}(1+x)^{\frac{\alpha-1}{2}}(1-x)^{\frac{\alpha-1}{2}}.$$

- $\rho(t) = -\ln(t^2)$ (functional is not convex): optimal design has the arcsine density

$$p^*(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

Optimal design for SLSE in one-parameter models

Assume the design space is finite: $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$. In this case, the optimal design for the SLSE can be found explicitly.

A generic approximate design on this design space is an arbitrary discrete signed measure $\xi = \{\mathbf{x}_1, \ldots, \mathbf{x}_N; w_1, \ldots, w_N\}$, where $w_i = s_i p_i$, $s_i \in \{-1, 1\}$, $p_i \geq 0$ ($i = 1, \ldots, N$) and $\sum_{i=1}^N p_i = 1$.

The variance of the SLSE:

$$D = \sum_{i=1}^{N} \sum_{j=1}^{N} K(\mathbf{x}_i, \mathbf{x}_j) w_i w_j f(\mathbf{x}_i) f(\mathbf{x}_j) \Big/ \Big(\sum_{i=1}^{N} w_i f^2(\mathbf{x}_i) \Big)^2 .$$

Optimal weights:

$$w_i^* = \mathbf{e}_i^T \mathbf{\Sigma}^{-1} \mathbf{f} / f(\mathbf{x}_i); \qquad i = 1, \ldots, N,$$

where $\mathbf{f} = (f(\mathbf{x}_1), \ldots, f(\mathbf{x}_N))^T$, $\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T$.

The resulting weighted SLSE coincides with BLUE (except that repetition of observations does not make sense)

SLSE: an explicit formula for optimal weights

Assume $K(\mathbf{x}_i, \mathbf{x}_j) = u_i v_j$ for $i \le j$ and denote $f_k = f(\mathbf{x}_k)$, $q_k = u_k/v_k$. Then If $f_i \neq 0$ $(i = 1, \ldots, N)$, then the optimal weights can be represented explicitly as follows:

$$
\begin{aligned}
w_1^* &= \frac{c}{f_1}\left(\tilde{\sigma}_{11} f_1 + \tilde{\sigma}_{12} f_2\right) = \frac{c\, u_2}{f_1 v_1 v_2 (q_2 - q_1)}\left(\frac{f_1}{u_1} - \frac{f_2}{u_2}\right), \\
w_N^* &= \frac{c}{f_N}\left(\tilde{\sigma}_{N,N} f_N + \tilde{\sigma}_{N-1,N} f_{N-1}\right) = \frac{c}{f_N v_N (q_N - q_{N-1})}\left(\frac{f_N}{v_N} - \frac{f_{N-1}}{v_{N-1}}\right), \\
w_i^* &= \frac{c}{f_i}\left(\tilde{\sigma}_{i,i} f_i + \tilde{\sigma}_{i-1,i} f_{i-1} + \tilde{\sigma}_{i,i+1} f_{i+1}\right) \\
&= \frac{c}{f_i v_i}\left(\frac{(q_{i+1} - q_{i-1}) f_i}{v_i (q_{i+1} - q_i)(q_i - q_{i-1})} - \frac{f_{i-1}}{v_{i-1}(q_i - q_{i-1})} - \frac{f_{i+1}}{v_{i+1}(q_{i+1} - q_i)}\right),
\end{aligned}
$$

for $i = 2, \ldots, N-1$. Here $\tilde{\sigma}_{ij}$ denotes the element in the position (i, j) of the matrix $\mathbf{\Sigma}^{-1} = (\tilde{\sigma}_{ij})_{i,j=1,\ldots,N}$.

Some references: Harman, R. and Stulajter, F. (2011) JSPI, 141(8), 2750–2758. AZ & Kondratovich (1984), AZ (1985).

Optimal designs, one-parameter case, Markovian noise

Assume $\mathcal{X} = [a, b]$, $K(t, t') = u(t)v(t')$, $t \leq t'$. Criterion:

$$D(\xi) = \int\int K(s,t)f(s)f(t)d\xi(s)d\xi(t) \Big/ \Big(\int f^2(t)d\xi(t)\Big)^2.$$

Optimal design: masses

$$P_a = \frac{c}{f(a)v^2(a)q'(a)}\Big[\frac{f(a)u'(a)}{u(a)} - f'(a)\Big],\ P_b = c \cdot \frac{h'(b)}{f(b)v(b)q'(b)}$$

at the points a and b, respectively, and the (signed) density

$$p(t) = -\frac{c}{f(t)v(t)}\Big[\frac{h'(t)}{q'(t)}\Big]'$$

where $h(t) = f(t)/v(t)$.

Optimality of a design ξ^* can be verified directly by checking that $D(\xi^*)$ coincides with the variance of the continuous BLUE.

OLSE/BLUE

General estimator:

$$\hat{\boldsymbol{\theta}}_\zeta = \int y(\mathbf{x})\zeta(d\mathbf{x}),$$

where $\zeta(d\mathbf{x})$ is a signed vector-measure.

$$\widehat{\boldsymbol{\theta}}_{OLSE} = \int y(\mathbf{x})M^{-1}(\xi)f(\mathbf{x})\xi(d\mathbf{x}),$$

where

$$M(\xi)=\int f(\mathbf{x})f^T(\mathbf{x})\xi(d\mathbf{x}),$$

and $\xi(d\mathbf{x})$ is a design (probability measure for OLSE; a signed measure for SLSE).

If $m = 1$ then any signed measure $\zeta(d\mathbf{x})$ can be represented in the form $M^{-1}(\xi)f(\mathbf{x})\xi(d\mathbf{x})$ and so optimal continuous SLSE is equal to continuous BLUE. Discretization is another issue.

OLSE/BLUE, $m > 1$

General estimator:

$$\hat{\boldsymbol{\theta}}_\zeta = \int y(\mathbf{x})\zeta(d\mathbf{x})\,,$$

where $\zeta(d\mathbf{x})$ is a signed vector-measure.

Continuous Matrix-Weighted estimator (MWLSE)

$$\widehat{\boldsymbol{\theta}}_{MWLSE} = \int y(\mathbf{x})M^{-1}(\xi)O(\mathbf{x})f(\mathbf{x})\xi(d\mathbf{x}),$$

where $O(\mathbf{x})$ is a matrix weight assigned to a point $\mathbf{x}$ and

$$M(\xi) = \int O(\mathbf{x})f(\mathbf{x})f^T(\mathbf{x})\xi(d\mathbf{x}),$$

and $\xi(d\mathbf{x})$ is a design.

Any signed vector-measure $\zeta(d\mathbf{x})$ can be represented in the form $M^{-1}(\xi)O(\mathbf{x})f(\mathbf{x})\xi(d\mathbf{x})$ and so optimal continuous MWLSE coincides with continuous BLUE.

In making a discretization, we only need to keep weights at A and B; the rest can be achieved by assigning $\pm$ and thinning.

All is similar for the gradient-enhanced estimation.

Results of Bickel and Herzberg and extensions

$y(t) = \theta^T f(t) + \varepsilon(t)$ with stationary error process and $\mathcal{X} = [-T, T]$. Suppose that for N observations, the correlation function is given by $\rho_N(t) = \rho_o(Nt)$, where $\rho_o(t) = \gamma\rho(t) + (1-\gamma)\delta_t$ and $\rho(t) \to 0$ as $t \to \infty$, $\gamma \in (0, 1]$. $Q(t) = \sum_{j=1}^{\infty} \rho(jt)$, $\mathbf{x}_{iN} = a\left(\frac{i-1}{N-1}\right), i = 1, \ldots, N$.

$$\mathbf{R}(a) = \left(\int_0^1 f_i(a(u))f_j(a(u))Q(a'(u))\,du\right)_{i,j=1}^m$$

B-H: the covariance matrix of the OLSE

$$\lim_{N\to\infty} \sigma^{-2} N \mathrm{Var}(\widehat{\boldsymbol{\theta}}_{OLSE}) = \mathbf{W}^{-1}(a) + 2\gamma \mathbf{W}^{-1}(a)\mathbf{R}(a)\mathbf{W}^{-1}(a),$$

where $\mathbf{W}(a) = \left(\int_0^1 f_i(a(u))f_j(a(u))\,du\right)_{i,j=1}^m$.

Two our generalizations of the B-H results: (a) LRD errors (joint with N.Leonenko), (b) different rate of expansion of the interval: $\rho_N(t) = \rho_o(N^\alpha t)$ with $0 < \alpha \leq 1$.

Thank you for listening