Elliptic rook and file numbers

Meesue Yoo

Sungkyunkwan University

joint work with Michael Schlosser

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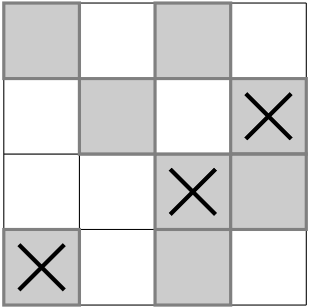
Introduction to rook theory

Let $[n] = \{1, 2, \ldots, n\}$.

- A board B is a finite subset of $[n] \times [n]$.
- We say that we *place k non-attacking rooks in B* for choosing a k-subset of B such that no two rooks lie in the same row or column.



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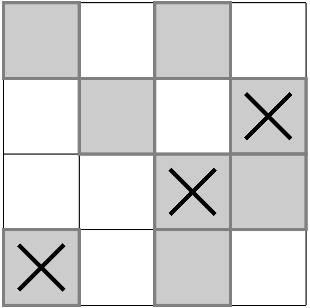
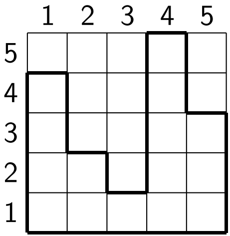


- $\mathcal{N}_k(B)$ = the set of all non-attacking k-rook placements in B.
- $r_k(B) = |\mathcal{N}_k(B)|$, the k-th rook number of B

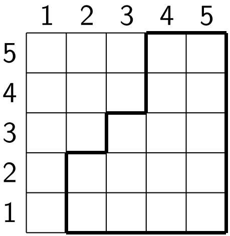
Given $b_1, \ldots, b_n \in \mathbb{N}$, $B(b_1, \ldots, b_n)$ denote the set of cells

$$B(b_1, \ldots, b_n) = \{(i,j) \mid 1 \leq i \leq n,\ 0 \leq j \leq b_i\}.$$

- If a board B equals to $B(b_1, \ldots, b_n)$ for some b_i's, then B is called a *skyline board*.
- If, in addition, $b_1 \leq b_2 \leq \cdots \leq b_n$, then a skyline board B is called a *Ferrers board*.



$B(4, 2, 1, 5, 3)$



$B(0, 2, 3, 5, 5)$

Proposition

Let B be a Ferrers board of height at most m and let $B \cup m$ denote the board obtained by adding a column of length m to B. Then we have

$$r_k(B \cup m) = r_k(B) + (m - k + 1)r_{k-1}(B).$$

Theorem (Goldman-Joichi-White, '75)

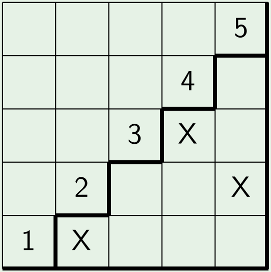
Given a Ferrers board $B = B(b_1, \ldots, b_n)$,

$$\prod_{i=1}^{n}(z + b_i - i + 1) = \sum_{k=0}^{n} r_{n-k}(B) \cdot (z) \downarrow_k,$$

where $(z) \downarrow_k = z(z-1)\cdots(z-k+1)$.

Example

- Consider a staircase board $St_n = B(0, 1, \ldots, n-1)$.



- GJW Theorem $\Rightarrow$

$$z^n = \sum_{k=0}^{n} r_{n-k}(St_n) \cdot (z)\downarrow_k,$$

$r_{n-k}(St_n) = \mathcal{S}_{n,k}$, the Stirling number of the second kind

- k-rook placement $\Leftrightarrow$ $n-k$ set partition of $[n]$
- $\mathcal{S}_{n,k+1} = \mathcal{S}_{n,k-1} + k\mathcal{S}_{n,k}$

File numbers

Let $\mathcal{C}_k(B)$ denote the set of all placements (*file placements*) of k rooks in B such that there is at most one rook in each column.

$$f_k(B) = |\mathcal{C}_k(B)|; \text{ the } k\text{-th file number of } B$$

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Theorem (Garsia-Remmel, '86)

Given a skyline board $B = B(b_1, \ldots, b_n)$,

$$\prod_{i=1}^{n}(z + b_i) = \sum_{k=0}^{n} f_{n-k}(B) z^k.$$

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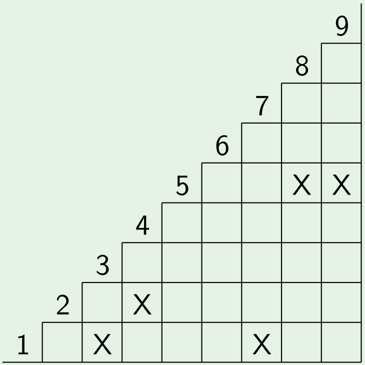


- Garsia-Remmel Theorem $\Rightarrow$

$$(z)\uparrow_n = \sum_{k=0}^{n} f_{n-k}(St_n) \cdot z^k,$$

$f_{n-k}(St_n) = c_{n,k}$, the signless Stirling number of the first kind

- $n-k$-rook placement $\Leftrightarrow$ permutations of S_n with k-cycles
- $c_{n+1,k} = c_{n,k-1} + nc_{n,k}$
- $\leftrightarrow$ $(1\ 7\ 3)(2\ 4)(5\ 9\ 8)(6)$

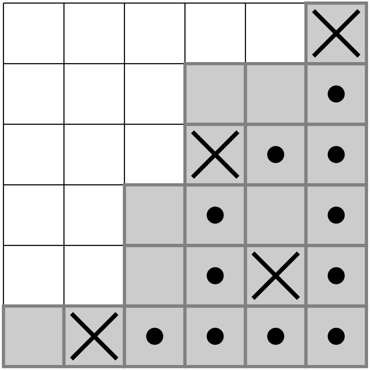
q-analogues

q-analogues

- Garsia and Remmel ('86) defined a q-analogue of the rook numbers for Ferrers boards.
  - Given a placement $P \in \mathcal{N}_k(B)$, B a Ferrers board, let each rook in P cancels all squares to its right and below it.

$$r_k(q;B) = \sum_{P \in \mathcal{N}_k(B)} q^{u_B(P)},$$

where $u_B(P)$ is the number of uncancelled squares in $B - P$.



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Let B be a Ferrers board of height at most m and let $B \cup m$ denote the board obtained by adding a column of length m to B. Then we have

$$r_k(q; B \cup m) = q^{m-k} r_k(q; B) + [m-k+1]_q r_{k-1}(q; B).$$

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Theorem (Garsia-Remmel, '86)

For a Ferrers board $B = B(b_1, \ldots, b_n)$, $b_1 \leq \cdots \leq b_n$,

$$\prod_{i=1}^{n} [z + b_i + i - 1]_q = \sum_{k=0}^{n} r_{n-k}(q; B) [z]_q \downarrow_k,$$

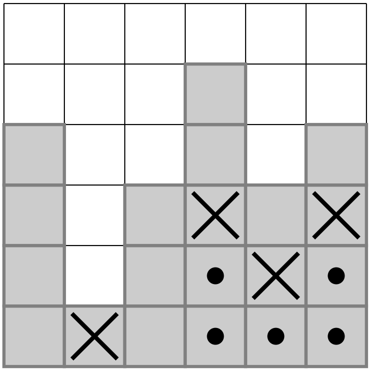
where $[n]_q = \frac{1-q^n}{1-q}$, $[z]_q \downarrow_k = [z]_q [z-1]_q \cdots [z-k+1]_q$.

q-file numbers

- Given a file placement $P \in \mathcal{C}_k(B)$, B a skyline board, let each rook in P cancels all squares below it only.

$$f_k(q;B) = \sum_{P \in \mathcal{C}_k(B)} q^{u_B(P)},$$

where $u_B(P)$ is the number of uncancelled squares in $B - P$.



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Let B be a skyline board and $B \cup m$ denote the board obtained by adding a column of length m to B. Then for any k, we have

$$f_k(q; B \cup m) = q^m f_k(q; B) + [m]_q f_{k-1}(q; B).$$

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Theorem

For a skyline board $B = B(b_1, b_2, \ldots, b_n)$,

$$\prod_{i=1}^{n} [z + b_i]_q = \sum_{k=0}^{n} f_{n-k}(q; B)[z]_q^k.$$

Elliptic analogues

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- A function is *elliptic* if it is meromorphic and doubly periodic.
- Elliptic functions can be built from quotients of theta functions.

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- Elliptic functions can be built from quotients of theta functions.

Define a *modified Jacobi theta* function with argument x and nome p by

$$\theta(x;p) := \prod_{j\geq 0}((1-p^j x)(1-p^{j+1}/x)),$$

$$\theta(x_1,\ldots,x_m;p) = \prod_{k=1}^{m}\theta(x_k;p),$$

where $x, x_1, \ldots, x_m \neq 0$, $|p| < 1$.

- $\theta(x;p) = -x\theta(1/x;p), \quad \theta(px;p) = -\frac{1}{x}\theta(x;p)$
- addition formula: $\theta(xy, x/y, uv, u/v;p)$
  $= \theta(xv, x/v, uy, u/y;p) + \frac{u}{y}\,\theta(yv, y/v, xu, x/u;p)$

Define the *theta shifted factorial* (or q, p-*shifted factorial*) by

$$(a;q,p)_n = \begin{cases} \prod_{k=0}^{n-1} \theta(aq^k;p), & n = 1,2,\ldots, \\ 1, & n = 0, \\ 1/\prod_{k=0}^{-n-1} \theta(aq^{n+k};p), & n = -1,-2,\ldots, \end{cases}$$

and

$$(a_1,a_2,\ldots,a_m;q,p)_n = \prod_{k=1}^{m} (a_k;q,p)_n.$$

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and

$$(a_1,a_2,\dots,a_m;q,p)_n = \prod_{k=1}^{m} (a_k;q,p)_n.$$

Remark

For $p=0$,

$$\theta(x;0) = 1-x, \quad \text{hence} \quad (a;q,0)_n = (a;q)_n.$$

Elliptic weights

Define the *elliptic weights* depending on two independent parameters a, b, base q, nome p by

$$w_{a,b;q,p}(k) = \frac{\theta(aq^{2k+1}, bq^k, aq^{k-2}/b; p)}{\theta(aq^{2k-1}, bq^{k+2}, aq^k/b; p)} q,$$

$$W_{a,b;q,p}(k) = \frac{\theta(aq^{1+2k}, bq, bq^2, aq^{-1}/b, a/b; p)}{\theta(aq, bq^{k+1}, bq^{k+2}, aq^{k-1}/b, aq^k/b; p)} q^k.$$

For positive integer k,

$$W_{a,b;q,p}(k) = \prod_{j=1}^{k} w_{a,b;q,p}(j).$$

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For positive integer k,

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Remark

If we let $p \to 0$, $a \to 0$ and then $b \to 0$, then

$$w_{0,0;q,0}(k) = q, \qquad W_{0,0;q,0}(k) = q^k$$

Elliptic analogue of q-integer

Define an elliptic analogue of the (complex) number n by

$$[n]_{a,b;q,p} = \frac{\theta(q^n, aq^n, bq^2, a/b; p)}{\theta(q, aq, bq^{n+1}, aq^{n-1}/b; p)}.$$

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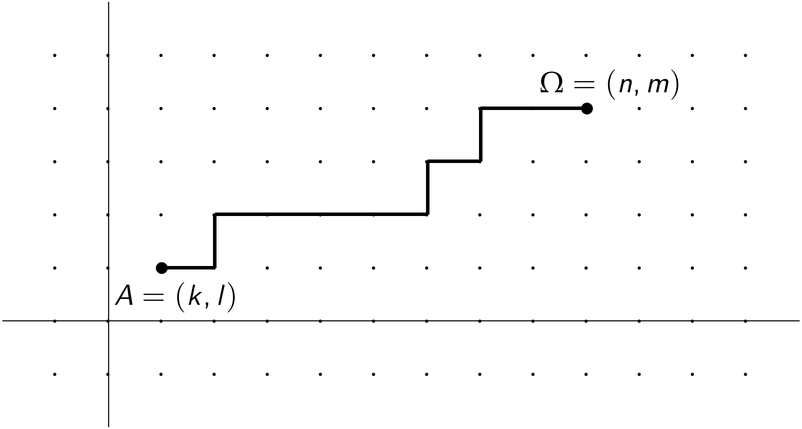
- $[n]_{a,b;q,p} = [n-1]_{a,b;q,p} + W_{a,b;q,p}(n-1)$
- For a positive integer n, the above recursion defines $[n]_{a,b;q,p}$ uniquely, with $W_{a,b;q,p}(0) = 1$.

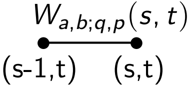
Elliptic enumeration of lattice paths

Lattice paths in $\mathbb{Z}^2$:



Given $A, \Omega \in \mathbb{Z}^2$, let $\mathcal{P}(A \to \Omega)$ denote the set of all lattice paths from A to Ω.

We weight the steps in a lattice path: to each horizontal step from $(s-1,t)$ to (s,t), assign the weight $W_{a,b;q,p}(s,t)$



where

$$W_{a,b;q,p}(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s}, bq^{2s-1}, aq^{1-s}/b, aq^{-s}/b)}{\theta(aq^{s}, bq^{2s+t}, bq^{2s+t-1}, aq^{1+t-s}/b, aq^{t-s}/b)} q^{t}$$

and 1 to each horizontal step $(s,t-1) \to (s,t)$.

Let $w(P) :=$ product of the weights of all its steps for a lattice path P .

Theorem (Schlosser, '07)

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = \sum_{P \in \mathcal{P}((0,0) \to (k,n-k))} w(P)$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}}.$$

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The elliptic number $[n]_{a,b;q,p}$ is

$$[n]_{a,b;q,p} = \begin{bmatrix} n \\ 1 \end{bmatrix}_{a,b;q,p}$$

: it can be interpreted as an area generating function of lattice paths from $(0,0)$ to $(1, n-1)$.

Addition of numbers

Ordinary case: $n + (m - n) = m$

q-analogue: $[n]_q + q^n[m - n]_q = [m]_q$

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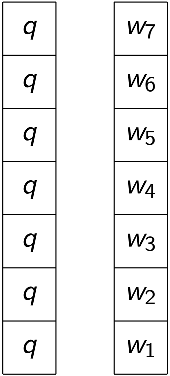
Elliptic case:

$$[n]_{a,b;q,p} + W_{a,b;q,p}(n)[m-n]_{aq^{2n},bq^n;q,p} \\ = [m]_{a,b;q,p}$$



Elliptic rook numbers

Given a Ferrers board B and a placement $P \in \mathcal{N}_k(B)$, let $U_B(P)$ denote the set of cells in $B - P$ which are not cancelled by any rooks in P.

Definition

We define an elliptic analogue of the k-th rook number by

$$r_k(a, b; q, p; B) = \sum_{P \in \mathcal{N}_k(B)} wt(P),$$

where

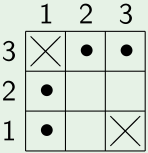
$$wt(P) = \prod_{(i,j) \in U_B(P)} w_{a,b;q,p}(i - j - r_{(i,j)}(P)),$$

and $r_{(i,j)}(P)$ is the number of rooks in P which are in the north-west region of (i, j).

Example

Consider a Ferrers board $B = B(3,3,3)$ and let P be a placement of two rooks in $(1,3)$ and $(3,1)$ in B.

$$U_B(P) = \{(2,1),(2,2),(3,2)\}$$



$$\begin{aligned}
\Rightarrow & wt(P) \\
&= w_{a,b;q,p}(2-1-1) \times w_{a,b;q,p}(2-2-1) \\
&\quad \times w_{a,b;q,p}(3-2-1) \\
&= \frac{\theta(aq^{-1}, bq^{-1}, aq^{-3}/b; p)}{\theta(aq^{-3}, bq, aq^{-1}/b; p)} \cdot \frac{\theta(aq, b, aq^{-2}/b; p)^2}{\theta(aq^{-1}, bq^2, a/b; p)^2} q^3.
\end{aligned}$$

Theorem (Schlosser-Y, '2017)

Let $B = B(b_1, \ldots, b_n)$ be a Ferrers board. Then we have

$$\prod_{i=1}^{n} [z + b_i - i + 1]_{aq^{2(i-b_i-1)}, bq^{i-b_i-1}; q, p}$$
$$= \sum_{k=0}^{n} r_{n-k}(a, b; q, p; B) \prod_{j=1}^{k} [z - j + 1]_{aq^{2(j-1)}, bq^{j-1}; q, p}$$

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Corollary

For a Ferrers board $B = B(b_1, \ldots, b_n)$,

$$r_n(a, b; q, p; B) = \prod_{i=1}^{n} [b_i - i + 1]_{aq^{2(i-b_i-1)}, bq^{i-b_i-1}; q,p}.$$

In particular, for $B = B(n, n, \ldots, n) = [n] \times [n]$,

$$r_n(a, b; q, p; B) = [n]_{aq^{-2n}, bq^{-n}; q,p} [n-1]_{aq^{2-2n}, bq^{1-n}; q,p} \ldots [1]_{aq^{-2}, bq^{-1}; q,p}.$$

Proposition

Let B be a Ferrers board with l columns of height at most m, and $B \cup m$ denote the board obtained by adding the $(l+1)$-th column of length m to B. Then for any $k \leq l$, we have

$$\begin{aligned} r_k(a, b; q, p; B \cup m) = {} & W_{aq^{2(l-m)}, bq^{l-m}; q,p}(m-k)\, r_k(a, b; q, p; B) \\ & + [m-k+1]_{aq^{2(l-m)}, bq^{l-m}; q,p}\, r_{k-1}(a, b; q, p; B). \end{aligned}$$

Elliptic Stirling numbers of the second kind

Consider the staircase board $St_n = B(0, 1, 2, \ldots, n-1)$.
The product formula for $St_n \Rightarrow$

$$([z]_{a,b;q,p})^n = \sum_{k=0}^{n} r_{n-k}(a, b; q, p; St_n) \prod_{j=1}^{k} [z - j + 1]_{aq^{2(j-1)}, bq^{j-1}; q,p}.$$

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$r_{n-k}(a, b; q, p, St_n)$ is the *elliptic Stirling numbers of the second kind* $\mathcal{S}_{a,b;q,p}(n, k)$ defined by Zsófia Kereskényiné Balogh and Michael Schlosser.

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Considering whether there is a rook in the last column or not, we obtain the following recursion

$$\mathcal{S}_{a,b;q,p}(n+1, k) = W_{a,b;q,p}(k-1)\mathcal{S}_{a,b;q,p}(n, k-1) + [k]_{a,b;q,p}\mathcal{S}_{a,b;q,p}(n, k).$$

Elliptic file numbers

Definition

Given a skyline board $B = B(c_1, \dots, c_n)$, we define an elliptic analogue of the k-th file number of B by

$$f_k(a, b; q, p; B) = \sum_{Q \in \mathcal{C}_k(B)} wt_f(Q),$$

where

$$wt_f(Q) = \prod_{(i,j) \in U_B(Q)} w_{a,b;q,p}(1 - j).$$

Proposition

For any skyline board B, let $B \cup m$ denote a board obtained by attaching a column of height m to B. Then

$$
\begin{aligned}
& f_k(a, b; q, p; B \cup m) \\
= & W_{aq^{-2m}, bq^{-m}; q, p}(m) f_k(a, b; q, p; B) + [m]_{aq^{-2m}, bq^{-m}; q, p} f_{k-1}(a, b; q, p; B).
\end{aligned}
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Theorem (Schlosser-Y, '2017)

For any skyline board $B = B(c_1, \ldots, c_n)$, we have

$$\prod_{i=1}^{n} [z + c_i]_{aq^{-2c_i}, bq^{-c_i}; q,p} = \sum_{k=0}^{n} f_{n-k}(a, b; q, p; B) ([z]_{a,b;q,p})^k.$$

Elliptic r-Stirling numbers of the first kind

The *r-Stirling numbers of the first kind*, denoted by $\mathfrak{c}^{(r)}(n,k)$, are defined, for all positive r, by the number of permutations of the set $\{1,\dots,n\}$ having k cycles, such that the numbers $1,2,\dots,r$ are in distinct cycles. (Broder, '84)

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The r-Stirling numbers of the first kind have the following generating function

$$\sum_{k=0}^{n} \mathfrak{c}^{(r)}(n,k) z^k = \begin{cases} z^r(z+r)(z+r+1)\cdots(z+n-1), & n \geq r \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Elliptic r-Stirling numbers of the first kind

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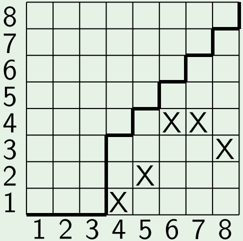
Consider the board $R_n = B(c_1,\ldots,c_n)$ such that $c_1 = \cdots = c_r = 0$, $c_i = i-1$, for $r+1 \leq i \leq n$. The product formula gives

$$z^r(z+r)(z+r+1)\cdots(z+n-1) = \sum_{k=0}^{n} f_{n-k}(R_n) z^k$$

File placements correspond to permutations of the set $\{1, \ldots, n\}$ having k cycles, such that the numbers $1, 2, \ldots, r$ are in distinct cycles.

Example ($n = 8, r = 3, n - k = 5$)



$\longleftrightarrow$ (1 4 7 6)(2 5)(3 8)

Elliptic r-Stirling numbers of the first kind

For $R_n = B(\underbrace{0, \ldots, 0}_{r}, r, r+1, \ldots, n-1)$, the product formula gives

$$([z]_{a,b;q,p})^r \prod_{i=1}^{n-r} [z+r+i-1]_{aq^{2(1-r-i)}, bq^{1-r-i};q,p} = \sum_{k=0}^{n} f_{n-k}(a, b; q, p; R_n)([z]_{a,b;q,p})^k.$$

Elliptic r-Stirling numbers of the first kind

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Let $\mathfrak{c}^{(r)}_{a,b;q,p}(n,k)$ denote $f_{n-k}(a,b;q,p;R_n)$.

- $\mathfrak{c}^{(r)}_{a,b;q,p}(n,k)$ defines an elliptic analogue of r-Stirling numbers of the first kind.
- $\mathfrak{c}^{(r)}_{a,b;q,p}(n+1,k) =$ $[n]_{aq^{-2n}, bq^{-n}; q,p} \mathfrak{c}^{(r)}_{a,b;q,p}(n,k) + W_{aq^{-2n}, bq^{-n}; q,p}(n) \mathfrak{c}^{(r)}_{a,b;q,p}(n,k-1)$.

Thank you