The Structure of Mapping Objects in the Category of Orbifolds

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Geometric Structures on Lie Groupoids
BIRS

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References

Main references for this talk:

- W. Chen, On a notion of maps between orbifolds, I. Function spaces, *Communications in Contemporary Mathematics* **8** (2006), pp. 569–620.
- Vesta Coufal, Dorette Pronk, Carmen Rovi, Laura Scull, Courtney Thatcher, Orbispaces and their mapping spaces via groupoids: a categorical approach, *Contemporary Mathematics* **641** (2015), pp. 135–166.
- Dorette Pronk, Laura Scull, A Bicategory of Orbigroupoids with Small Hom-Groupoids, in progress.
- Dorette Pronk, Laura Scull, Exponential Objects for Orbigroupoids, in progress.

Today's Questions

- Given two orbifolds $\mathcal{G}$ and $\mathcal{H}$, can we put a topology on the hom-groupoid $\mathbf{OMap}(\mathcal{G}, \mathcal{H})$ of good maps and 2-cells?
- When does this hom-groupoid represent a (possibly infinite dimensional) orbifold?
- When does it have the universal property to be a categorical exponential?

Outline

1. Orbigroupoids
2. Maps Between Orbigroupoids
3. Orbispaces
4. Small mapping groupoids
5. The Topology on Mapping Groupoids
6. Mapping Objects with Compact Domain

Orbigroupoids

- **The smooth case**
  A smooth **orbigroupoid** is a Lie groupoid with
  - structure maps that are local diffeomorphisms;
  - a proper diagonal $\mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$.
- **The topological case**
  An **orbigroupoid** is a topological groupoid with
  - structure maps that are étale;
  - a proper diagonal $\mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$.

Orbigroupoids

Remarks

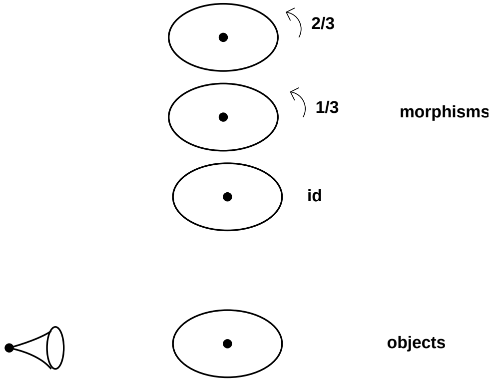
1. The isotropy groups of an orbigroupoid are finite.
2. The quotient space,

$$\mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0 \twoheadrightarrow \mathcal{G}_0/\mathcal{G}_1$$

   is also called the **underlying space** of the orbigroupoid.
3. Properness of the groupoid implies that this quotient space is Hausdorff.
4. For this talk I work with the topological case.
5. And I take **Top** to be a Cartesian closed category of topological spaces.

Examples: a G-point $*_G$

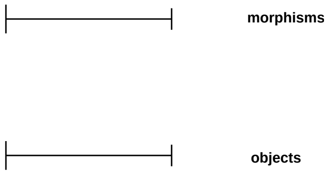
Examples
A Cone of Order 3



This is a translation groupoid, $\mathbb{Z}/3 \ltimes D$.

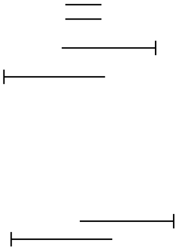
Examples

The Unit Interval



Examples

A Split Unit Interval

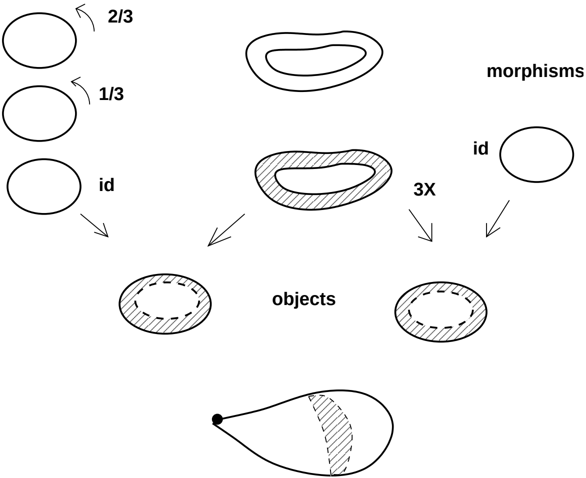


morphisms

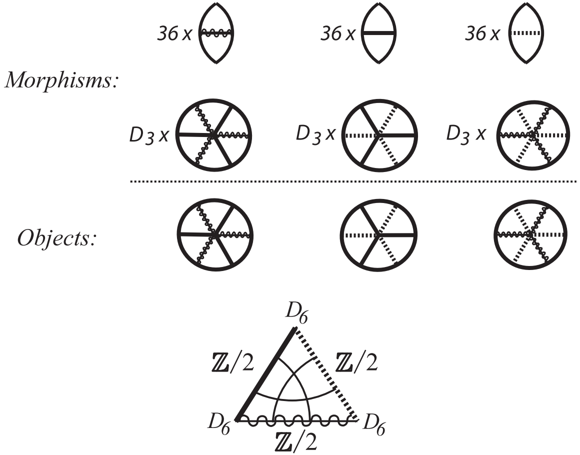
objects

Examples

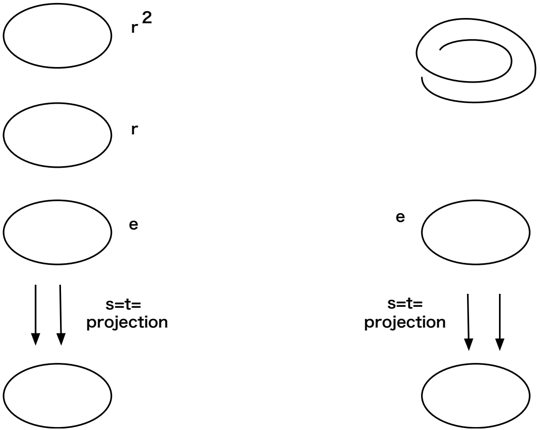
The Teardrop Groupoid



Examples: The Triangular Billiard Groupoid $\mathbb{T}$



Examples: The $\mathbb{Z}/3$ Circles, $S^1_{\mathbb{Z}/3}$ and $\tilde{S}^1_{\mathbb{Z}/3}$



Good Maps Between Orbigroupoids

There are two approaches to obtain a bicategory of orbigroupoids that is appropriate for homotopy theory.

- Hilsum-Skandalis bibundles with bundle isomorphisms
- The bicategory of fractions of continuous groupoid homomorphisms with respect to essential equivalences
- The two approaches give biequivalent bicategories of orbigroupoids.

Maps Between Orbigroupoids

Continuous Groupoid Homomorphisms

$$
\begin{array}{ccccc}
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 & \xrightarrow{\;m\;} & \mathcal{G}_1 & \overset{s,\,t}{\underset{u}{\rightrightarrows\!\!\!\!\leftarrow}} & \mathcal{G}_0 \\
\Big\downarrow{\scriptstyle f_1 \times f_1} & & \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_0} \\
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 & \xrightarrow{\;m\;} & \mathcal{H}_1 & \overset{s,\,t}{\underset{u}{\rightrightarrows\!\!\!\!\leftarrow}} & \mathcal{H}_0
\end{array}
$$

2-Cells

A **2-cell**

$$\alpha\colon f \Rightarrow f' : \mathcal{G} \rightrightarrows \mathcal{H}$$

is given by, a continuous function

$$\alpha\colon \mathcal{G}_0 \to \mathcal{H}_1$$

such that

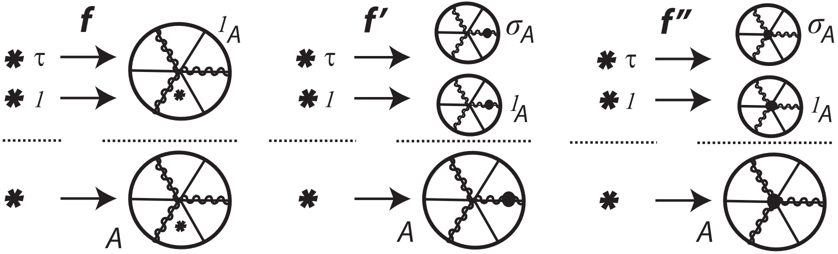
- $s \circ \alpha = f_0$ and $t \circ \alpha = f_0'$;
- (**naturality**) the following square commutes in $\mathcal{H}$ for each $g \in \mathcal{G}_1$,

$$\begin{array}{ccc}
f_0(sg) & \xrightarrow{f_1(g)} & f_0(tg) \\
{\scriptstyle \alpha(sg)}\big\downarrow & & \big\downarrow{\scriptstyle \alpha(tg)} \\
f_0'(sg) & \xrightarrow[f_1'(tg)]{} & f_0'(tg)
\end{array}$$

The Groupoid **GMap**$(\mathcal{G}, \mathcal{H})$

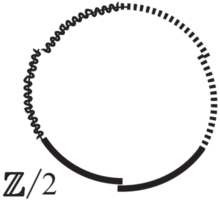
- Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids.
- The continuous groupoid homomorphisms from $\mathcal{G}$ to $\mathcal{H}$ and continuous natural transformations between them form a topological groupoid **GMap**$(\mathcal{G}, \mathcal{H})$.

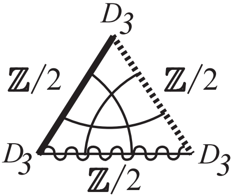
Example: **GMap**$(*_{\mathbb{Z}/2}, \mathbb{T})$



Example: **GMap**$(*_{\mathbb{Z}/2}, \mathbb{T})$

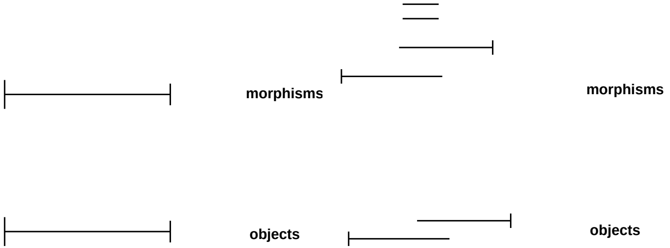
- We obtain a copy of the original orbigroupoid $\mathbb{T}$ together with a copy of the (trivial) $\mathbb{Z}/2$-circle, $S^1_{\mathbb{Z}/2}$,





From Orbigroupoids to Orbispaces

- The following two groupoids both represent the unit interval as orbispace



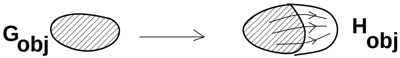
- They are not isomorphic in the category of orbigroupoids and groupoid homomorphisms.
- However, the groupoid homomorphism from the second to the first is an essential equivalence.

Essential Equivalences

- A morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ is an **essential equivalence** when it is essentially surjective and fully faithful.
- It is **essentially surjective** when $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \longrightarrow \mathcal{H}_0$ in

$$\begin{array}{ccccc} \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 & \longrightarrow & \mathcal{H}_1 & \xrightarrow{t} & \mathcal{H}_0 \\ \downarrow & & \downarrow s & & \\ \mathcal{G}_0 & \xrightarrow[\varphi_0]{} & \mathcal{H}_0 & & \end{array}$$

is an **open surjection**.

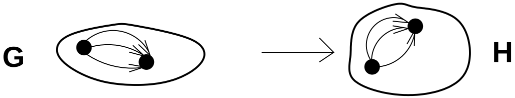


Essential Equivalences

The morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ is **fully faithful** when

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & H_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{\varphi_0 \times \varphi_0} & H_0 \times H_0 \end{array}$$

is a **pullback**,



Morita Equivalence

- The essential equivalence maps between topological groupoids generate the **Morita equivalence relation** in the sense that $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if and only if they are connected by a span of essential equivalence maps,

$$\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}$$

- To define a bicategory of orbispaces, we use a **bicategory of fractions** to invert the essential equivalences.

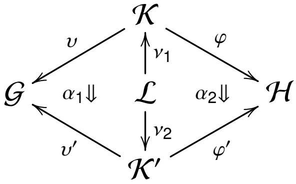
Generalized Maps

- Maps are **generalized maps** defined by spans

$$\mathcal{G} \xleftarrow{\upsilon} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$$

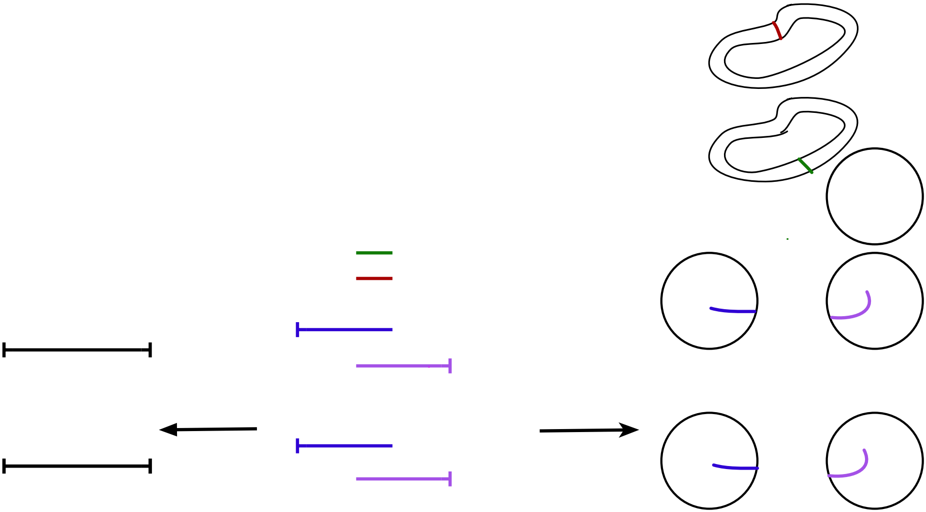
where υ is an essential equivalence

- A **2-cell** between two generalized maps is an (equivalence class of) diagrams



where $\upsilon\nu_1$ is an essential equivalence.

Example



Challenges

- Recall: we want to define a **topological groupoid** **OMap**$(\mathcal{G}, \mathcal{H})$ with
  - objects given by generalized maps $\mathcal{G} \xleftarrow{v} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$
  - arrows given by 2-cells; i.e., equivalence classes of diagrams
- The collection of generalized maps $\mathcal{G} \to \mathcal{H}$ as described is a **proper class.**
- How do we get a good description of the space of arrows, **OMap**$(\mathcal{G}, \mathcal{H})_1$? This is a **quotient of the space of 2-cell diagrams**!

Challenges

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Solution to the size issue

- We introduce a smaller class of arrows to be inverted: essential covering maps.
- This class does not satisfy the right bicalculus of fractions conditions (they are not closed under composition).
- There is an adjusted version of these conditions and an adjusted bicategory of fractions construction to fix this.
- However, you don't need the full structure of the new bicategory of fractions to define the mapping groupoids in terms of essential covering maps and verify that they are equivalent, as categories, to the ones defined using essential equivalences.

The Space of Arrows Issue

- We originally planned to obtain the topology on **OMap**$(\mathcal{G}, \mathcal{H})$ as a pseudo-colimit of the groupoids **GMap**$(\mathcal{G}', \mathcal{H})$, indexed over the diagram of essential equivalences over $\mathcal{G}$, $\mathcal{G}' \to \mathcal{G}$.
- This requires some work in the general case, and has been shown to work in some special cases (see Angel and Colman, Free and based path groupoids, arXiv)
- While working on the pseudo-colimit, we looked closely at the bicategory of fractions.
- Then we realized that the equivalence relation was not so unwieldy after all.
- So we switched to using its properties for a more direct approach.

Essential Coverings

- A collection $\mathcal{U}$ of open subsets of $\mathcal{G}_0$ is an **essential covering** of $\mathcal{G}_0$ if the map $(j_{\mathcal{U}})_0 \colon \coprod_{U \in \mathcal{U}} U \to \mathcal{G}_0$ is essentially surjective.
- Note that an essential covering does not necessarily cover all of $\mathcal{G}_0$, but it meets every orbit.

Essential Covering Maps

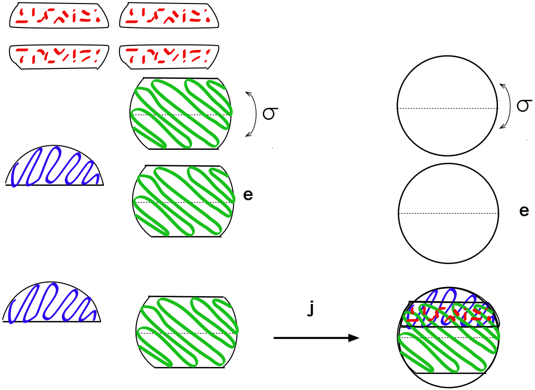
Any essential covering $\mathcal{U}$ gives rise to a groupoid $\mathcal{G}^*(\mathcal{U})$ with a groupoid homomorphism $j_{\mathcal{U}}\colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$:

- $\mathcal{G}^*(\mathcal{U})_0 = \coprod_{U\in\mathcal{U}} U$;
- $(j_{\mathcal{U}})_0\colon \mathcal{G}^*(\mathcal{U})_0 \to \mathcal{G}_0$ is defined by inclusions on the connected components;
- $\mathcal{G}^*(\mathcal{U})_1$ is defined as the pullback,

$$\begin{array}{ccc} \mathcal{G}(\mathcal{U})_1 & \xrightarrow{(j_{\mathcal{U}})_1} & \mathcal{G}_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \coprod_{U\in\mathcal{U}} U \times \coprod_{U\in\mathcal{U}} U & \xrightarrow[(j_{\mathcal{U}})_0\times(j_{\mathcal{U}})_0]{} & \mathcal{G}_0 \times \mathcal{G}_0. \end{array}$$

- This makes the map $j_{\mathcal{U}}\colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ an essential equivalence.

Example



Essential Covering Maps

Definition

For an orbigroupoid $\mathcal{G}$, the collection of **essential covering maps** is obtained as follows:

- Take all essential coverings of $\mathcal{G}_0$ which form a subset of the powerset of $\mathcal{G}_0$; i.e., all open subsets in the cover are distinct. (The cover is non-repeating.)
- Take all maps $w \colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ for which there is a natural isomorphism $\alpha_w \colon w \Rightarrow j_{\mathcal{U}}$.

Properties of Essential Covering Maps

For any orbigroupoid $\mathcal{G}$, there is a **set** of essential covering maps with codomain $\mathcal{G}$.

Properties of Essential Covering Maps

- For each essential equivalence $\mathcal{K} \xrightarrow{\nu} \mathcal{G}$ of orbigroupoids there is an essential covering $\mathcal{U}$ of $\mathcal{G}$ such that $j_{\mathcal{U}}$ factors through ν,

$$\begin{array}{ccc} \mathcal{G}^*(\mathcal{U}) & & \\ {\scriptstyle \varepsilon}\downarrow & \overset{\alpha}{} \searrow^{j_{\mathcal{U}}} & \\ \mathcal{K} & \xrightarrow[\nu]{} & \mathcal{G} \end{array}$$

- Given two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$, each generalized map

$$\mathcal{G} \xleftarrow[\nu]{} \mathcal{K} \xrightarrow[\varphi]{} \mathcal{H}$$

is isomorphic to one of the form,

$$\mathcal{G} \xleftarrow[j_{\mathcal{U}}=\nu\varepsilon]{} \mathcal{G}^*(\mathcal{U}) \xrightarrow[\varphi'=\varphi\varepsilon]{} \mathcal{H}$$

Properties of Essential Covering Maps

Any 2-cell from

$$\mathcal{G} \xleftarrow[w]{} \mathcal{G}^*(\mathcal{U}) \xrightarrow[\varphi]{} \mathcal{H}$$

to

$$\mathcal{G} \xleftarrow[w']{} \mathcal{G}^*(\mathcal{V}) \xrightarrow[\psi]{} \mathcal{H}$$

can be represented by a diagram of the form

$$\begin{array}{ccccc} & \overset{w}{\swarrow} & \mathcal{G}^*(\mathcal{U}) & \overset{\varphi}{\searrow} & \\ & & \uparrow j_{\mathcal{U}}^{\mathcal{W}} & & \\ \mathcal{G} & \alpha & \mathcal{G}^*(\mathcal{W}) & \beta & \mathcal{H} \\ & & \downarrow j_{\mathcal{V}}^{\mathcal{W}} & & \\ & \underset{w'}{\nwarrow} & \mathcal{G}^*(\mathcal{V}) & \underset{\psi}{\nearrow} & \end{array}$$

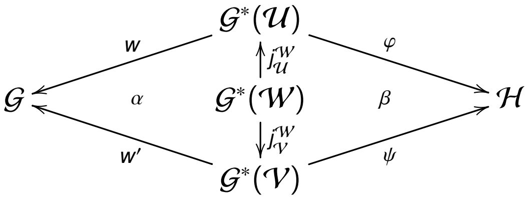
The essential covering $\mathcal{W}$ can be viewed as an essential refinement of $\mathcal{U}$ and $\mathcal{V}$.

The Mapping Groupoid, **OMap**$(\mathcal{G}, \mathcal{H})$

Let **OMap**$(\mathcal{G}, \mathcal{H})$ be the groupoid such that each object corresponds to a span,

$$\mathcal{G} \xleftarrow{w} \mathcal{G}^*(\mathcal{U}) \xrightarrow{f} \mathcal{H}$$

and each arrow corresponds to an equivalence class of diagrams,



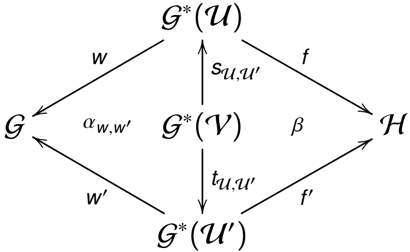
Goal: Give a relatively simple description of the topology on this groupoid.

The Space of Objects

- Write
$$\mathbf{CMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{G}) \subseteq \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{G})$$
for the full subgroupoid of $\mathbf{GMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{G})$ on the essential covering maps, with the subspace topology.
- Then we obtain
$$\mathbf{OMap}(\mathcal{G},\mathcal{H})_0 = \coprod_{\mathcal{U}} \mathbf{CMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{G})_0 \times \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{H})_0,$$
where the coproduct is taken over all non-repeating essential covers of $\mathcal{G}_0$.

The Equivalence Relation

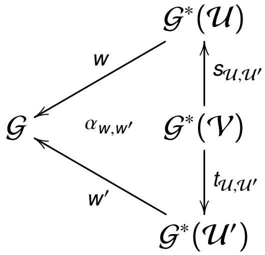
Given any two generalized maps $(w, f), (w', f') \colon \mathcal{G} \to \mathcal{H}$ and **ANY** common essential refinement every 2-cell $(w, f) \Rightarrow (w', f')$ can be represented **uniquely** by a diagram of the form



for **this particular chosen** common refinement.

Main Result

- It is possible to choose the refinements



  in such a way that the map from the space of all diagrams to the subspace of representatives with these refinements is continuous.
- Hence, we have a retract, and the quotient topology is the subspace topology.

Choice of Refinements

- Choose an essential common refinement

$$\begin{array}{ccc} \mathcal{G}^*(\mathcal{W}_{\mathcal{U},\mathcal{U}'}) & \xrightarrow{s_{\mathcal{U},\mathcal{U}'}} & \mathcal{G}^*(\mathcal{U}) \\ {\scriptstyle t_{\mathcal{U},\mathcal{U}'}}\downarrow & \alpha_{\mathcal{U},\mathcal{U}'} & \downarrow{\scriptstyle j_{\mathcal{U}}} \\ \mathcal{G}^*(\mathcal{U}') & \xrightarrow[j_{\mathcal{U}'}]{} & \mathcal{G} \end{array}$$

  for each pair $\mathcal{U}, \mathcal{U}'$ of essential coverings of $\mathcal{G}_0$;
- For each essential covering map $w\colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$, choose a 2-cell $\beta_w\colon w \Rightarrow j_{\mathcal{U}}$.
- Choose the composites of the 2-cells $\alpha_{\mathcal{U},\mathcal{U}'}$ with the β_w's to define the $\alpha_{w,w'}\colon ws_{\mathcal{U},\mathcal{U}'} \Rightarrow w' t_{\mathcal{U},\mathcal{U}'}$.

Composition with an Essential Equivalence

Proposition

If $\varphi\colon \mathcal{G}' \to \mathcal{G}$ is an essential equivalence, then the induced map $\varphi^*\colon \mathbf{GMap}(\mathcal{G},\mathcal{H}) \to \mathbf{GMap}(\mathcal{G}',\mathcal{H})$ is fully faithful in the sense that

$$\begin{array}{ccc} \mathbf{GMap}(\mathcal{G},\mathcal{H})_1 & \longrightarrow & \mathbf{GMap}(\mathcal{G}',\mathcal{H})_1 \\ \downarrow & & \downarrow \\ \mathbf{GMap}(\mathcal{G},\mathcal{H})_0 \times \mathbf{GMap}(\mathcal{G},\mathcal{H})_0 & \longrightarrow & \mathbf{GMap}(\mathcal{G}',\mathcal{H})_0 \times \mathbf{GMap}(\mathcal{G}',\mathcal{H})_0 \end{array}$$

is a pullback of spaces.

The Space of Arrows

Write $P_{\mathcal{U},\mathcal{U}'}$ for the pseudo pullback of groupoids,

$$\begin{array}{ccc} P_{\mathcal{U},\mathcal{U}'} & \longrightarrow & \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{H}) \\ \downarrow & \cong & \downarrow s^*_{\mathcal{U},\mathcal{V}} \\ \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}'),\mathcal{H}) & \xrightarrow[t^*_{\mathcal{U},\mathcal{U}'}]{} & \mathbf{GMap}(\mathcal{G}^*(\mathcal{W}_{\mathcal{U},\mathcal{U}'}),\mathcal{H}). \end{array}$$

Then,

$$\mathbf{OMap}(\mathcal{G},\mathcal{H})_1 \cong \coprod_{\mathcal{U},\mathcal{U}'} \mathbf{CMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{G})_0 \times \mathbf{CMap}(\mathcal{G}^*(\mathcal{U}'),\mathcal{G})_0 \times (P_{\mathcal{U},\mathcal{U}'})_0.$$

In particular, this space is Hausdorff.

Composition

Proposition

Composition by a generalized map $(w, f) = \mathcal{G} \xleftarrow{w} \mathcal{G}^*(\mathcal{U}) \xrightarrow{f} \mathcal{H}$ *induces continuous groupoid maps between mapping groupoids,*

$$(w, f)_* \colon \mathbf{OMap}(\mathcal{K}, \mathcal{G}) \to \mathbf{OMap}(\mathcal{K}, \mathcal{H})$$

and

$$(w, f)^* \colon \mathbf{OMap}(\mathcal{H}, \mathcal{L}) \to \mathbf{OMap}(\mathcal{G}, \mathcal{L}).$$

Morita Invariance

Theorem

Whenever $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent and $\mathcal{H}$ and $\mathcal{H}'$ are Morita equivalent, the corresponding mapping groupoids

$$\mathbf{OMap}(\mathcal{G},\mathcal{H}) \text{ and } \mathbf{OMap}(\mathcal{G}',\mathcal{H}')$$

are (Morita) equivalent.

Orbit-compact Domains

When $\mathcal{G}_0/\mathcal{G}_1$ is compact,

- We only need to consider finite essential coverings with compact closures.
- We obtain a groupoid $\textbf{OMap}_c(\mathcal{G}, \mathcal{H}) \hookrightarrow \textbf{OMap}(\mathcal{G}, \mathcal{H})$.

The Space of Objects

- The space of essentially compact covering maps $\mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ is discrete.
- Hence the space of objects has the form,

$$\mathbf{OMap}_c(\mathcal{G},\mathcal{H})_0 \cong \coprod_{\mathcal{U},w} \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{G})_0.$$

- Similarly,

$$\mathbf{OMap}(\mathcal{G},\mathcal{H})_1 \cong \coprod_{\mathcal{U},\mathcal{U}',w,w'} (P_{\mathcal{U},\mathcal{U}'})_0.$$

Main Theorem

Theorem

When $\mathcal{G}_0/\mathcal{G}_1$ is compact,

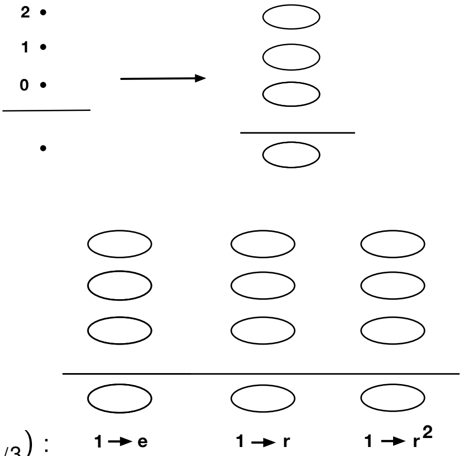
- *The inclusion* $\mathbf{OMap}_c(\mathcal{G},\mathcal{H}) \hookrightarrow \mathbf{OMap}(\mathcal{G},\mathcal{H})$ *is an essential equivalence.*
- *The groupoid* $\mathbf{OMap}_c(\mathcal{G},\mathcal{H})$ *is étale and proper.*
- $\mathbf{OMap}_c(\mathcal{G},\mathcal{H})$ *is an exponential object:*

$$\mathbf{OrbiGrpds}(\mathfrak{E}^{-1})(\mathcal{G},\mathbf{OMap}_c(\mathcal{K},\mathcal{H})) \simeq \mathbf{OrbiGrpds}(\mathfrak{E}^{-1})(\mathcal{G}\times\mathcal{K},\mathcal{H}).$$

Example: **OMap**$(*_{\mathbb{Z}/3}, S^1_{\mathbb{Z}/3})$

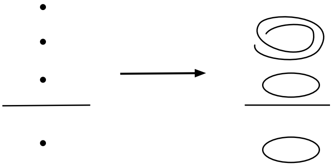
$S^1_{\mathbb{Z}/3}$:

OMap$(*_{\mathbb{Z}/3}, S^1_{\mathbb{Z}/3})$:



Example: **OMap**$(*_{\mathbb{Z}/3}, \tilde{S}^1_{\mathbb{Z}/3})$

$\tilde{S}^1_{\mathbb{Z}/3}$:



OMap$(*_{\mathbb{Z}/3}, \tilde{S}^1_{\mathbb{Z}/3})$: