Reconstructing plane quartics from their invariants

Jeroen Sijsling
Universität Ulm

joint work with
Reynald Lercier and Christophe Ritzenthaler
Université de Rennes 1

Banff International Research Station, June 1st 2017

Curves

K: an algebraically closed field.
X: a smooth, complete, geometrically irreducible curve over K, described by equations.

Fixing a genus g, we want to describe equations, invariants and reconstruction methods to get some grip on the moduli space of curves of genus g.

Curves

K: an algebraically closed field.
X: a smooth, complete, geometrically irreducible curve over K, described by equations.

Fixing a genus g, we want to describe equations, invariants and reconstruction methods to get some grip on the moduli space of curves of genus g.

$g = 0$: $X \cong \mathbb{P}^1$.
$g = 1$: X can be described by a Weierstrass equation, and given a j-invariant, we can find a Weierstrass equation giving rise to that invariant.

Hyperelliptic reconstruction

$g = 2$: All nice curves of this genus are hyperelliptic, and Igusa constructed invariants over $\mathbb{Z}$.

Methods of reconstruction were first developed by Clebsch (1872) and Mestre (1991) and subsequently highly refined by Lercier and Ritzenthaler (2011), who also apply these methods to hyperelliptic curves of genus 3. We treat it as a black box, but the main ingredients are:

Hyperelliptic reconstruction

$g = 2$: All nice curves of this genus are hyperelliptic, and Igusa constructed invariants over $\mathbb{Z}$.

Methods of reconstruction were first developed by Clebsch (1872) and Mestre (1991) and subsequently highly refined by Lercier and Ritzenthaler (2011), who also apply these methods to hyperelliptic curves of genus 3. We treat it as a black box, but the main ingredients are:

1. Construct a quadric Q and a curve H of degree $g + 1$ inside $\mathbb{P}^2$ whose coefficients are expressions in the invariants;
2. Take the curve X obtained by taking the degree 2 cover of Q that ramifies over the $2g + 2$ points in $Q \cap H$;
3. Show that the resulting curve has the requested invariants.

Plane quartics and the isomorphisms between them

In this talk, we consider the case of non-hyperelliptic curves of genus 3. The canonical embedding describes these curves as the zero locus of ternary quartic forms. Moreover:

Proposition

Let X_1 and X_2 be plane quartics, with corresponding ternary forms F_1 and F_2. Then X_1 and X_2 are isomorphic if and only if F_1 and F_2 can be transformed into each other by a linear substitution.

Some formalism and notation

V: a vector space over K of dimension 2.
$V^* = \mathrm{Hom}(V, K)$: the dual vector space of V.
The variables z_1, z_2 of binary forms on V live here. (After a choice of basis.)

W: a vector space over K of dimension 3.
$W^* = \mathrm{Hom}(W, K)$: the dual vector space of W.
The variables x_1, x_2, x_3 of ternary forms on W live here.

$\mathrm{Sym}^d(V^*)$: the d-th symmetric power of V^*. Binary forms live here.

$\mathrm{Sym}^4(W^*)$: the fourth symmetric power of W^*. Ternary quartic forms live here.

Ring of invariants

We consider the graded ring of invariants $K[\mathrm{Sym}^4(W^*)]$. It contains 7 algebraically independent elements

$$I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}$$

that were first constructed by Dixmier (1987). To obtain the full ring of invariants, we have to adjoin 6 more elements

$$J_9, J_{12}, J_{15}, J_{18}, I_{21}, J_{21}$$

constructed by Ohno (2005, unpublished work). The fundamental invariants

$$I_3, I_6, I_9, J_9, I_{12}, J_{12}, I_{15}, J_{15}, I_{18}, J_{18}, I_{21}, J_{21}, I_{27}$$

are known as the Dixmier–Ohno invariants.

Coordinates on the moduli space

The choice of these generators of the invariant ring gives the embedding in the composition

$$\pi_{\mathrm{aff}} : \operatorname{Spec} K[\mathrm{Sym}^4(W^*)] \twoheadrightarrow \operatorname{Spec} K[\mathrm{Sym}^4(W^*)]^{\mathrm{SL}(W)} \hookrightarrow \mathbb{A}^{13}_K .$$

We also get a rational map

$$\pi_{\mathrm{proj}} : \operatorname{Proj} K[\mathrm{Sym}^4(W^*)] \dashrightarrow \operatorname{Proj} K[\mathrm{Sym}^4(W^*)]^{\mathrm{SL}(W)} \hookrightarrow \mathbb{P}_K(3:6:6:9:9:12:12:15:15:18:18:21:21:27).$$

Separating orbits

Recall the classical result on separating orbits:

Theorem

Let $X \in \operatorname{Proj} K[\operatorname{Sym}^4(W^*)](K)$ be a smooth plane quartic curve, and let $F \in \operatorname{Spec} K[\operatorname{Sym}^4(W^*)](K)$ be a corresponding ternary quartic form. Then X (resp. F) is up to isomorphism determined by its fundamental invariants; in other words, its preimage under the map π_{proj} (resp. π_{aff}) is a single $\mathrm{SL}(W)$-orbit.

Separating orbits

Recall the classical result on separating orbits:

Theorem

Let $X \in \operatorname{Proj} K[\mathrm{Sym}^4(W^*)](K)$ be a smooth plane quartic curve, and let $F \in \operatorname{Spec} K[\mathrm{Sym}^4(W^*)](K)$ be a corresponding ternary quartic form. Then X (resp. F) is up to isomorphism determined by its fundamental invariants; in other words, its preimage under the map π_{proj} (resp. π_{aff}) is a single $\mathrm{SL}(W)$-orbit.

Most of our statements go through under the hypothesis $I_{12} \neq 0$.

The nicest vector space of dimension 3

That is $W = \mathrm{Sym}^2(V)$. The spaces W and W^* inherit bases from V and V^*, namely

$$\begin{array}{llll} w_1 = v_1^2, & w_2 = 2v_1v_2, & w_3 = v_2^2 & \text{and} \\ x_1 = z_1^2, & x_2 = (1/2)z_1z_2, & x_3 = z_2^2. & \end{array}$$

Invariance of the discriminant translates into the following:

Proposition

There is a degree 2 surjection

$$h : \mathrm{SL}(V) \to \mathrm{SO}(w_2^2 - w_1w_3) \subset \mathrm{SL}(W)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

The nicest vector space of dimension 3

That is $W = \mathrm{Sym}^2(V)$. The spaces W and W^* inherit bases from V and V^*, namely

$$\begin{aligned} w_1 &= v_1^2, & w_2 &= 2v_1v_2, & w_3 &= v_2^2 \quad \text{and} \\ x_1 &= z_1^2, & x_2 &= (1/2)z_1z_2, & x_3 &= z_2^2. \end{aligned}$$

Invariance of the discriminant translates into the following:

Proposition

There is a degree 2 surjection

$$h : \mathrm{SL}(V) \to \mathrm{SO}(w_2^2 - w_1w_3) \subset \mathrm{SL}(W)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

We would therefore be reduced to the invariant theory of $\mathrm{SL}(V)$ if we could pass from $\mathrm{SL}(W)$ to its subgroup $\mathrm{SO}(w_2^2 - w_1w_3)$...

Sections

But in fact we can (almost) pare down our group to $SO(w_2^2 - w_1 w_3)$! This uses ideas by Katsylo (1996) and Van Rijnswou (2001).

The main tool is the following.

Definition

A quadric contravariant of ternary quartics is an $SL(W)$-equivariant homogeneous polynomial map

$$\gamma : \mathrm{Sym}^4(W^*) \to \mathrm{Sym}^2(W).$$

Sections

But in fact we can (almost) pare down our group to $\mathrm{SO}(w_2^2 - w_1 w_3)$! This uses ideas by Katsylo (1996) and Van Rijnswou (2001).
The main tool is the following.

Definition

A quadric contravariant of ternary quartics is an $\mathrm{SL}(W)$-equivariant homogeneous polynomial map

$$\gamma : \mathrm{Sym}^4(W^*) \to \mathrm{Sym}^2(W).$$

Up to scalars, there exists a unique quadric contravariant ρ of degree 4, with discriminant I_{12}. If we restrict ourselves to generic ternary quartic forms F, then we may suppose that

$$\rho(F) = u(w_2^2 - w_1 w_3),$$

and any two such curves are in the same $\mathrm{SL}(W)$-orbit if and only if they are in the same orbit of $\langle \zeta_3 \rangle\ \mathrm{SO}(w_2^2 - w_1 w_3)$.

Sections

Motivated by this, we let Z be the subvariety of $\operatorname{Spec} K[\mathrm{Sym}^4(W^*)]$ whose set of K-points equals

$$Z(K) = \left\{ F \in \mathrm{Sym}^4(W^*) : \rho(F) = u(w_2^2 - w_1 w_3) \right\}.$$

Then Z is a section of $X = \operatorname{Spec} K[\mathrm{Sym}^4(W^*)]_{I_{12}}$ for the inclusion $H = \langle \zeta_3 \rangle \, \mathrm{SO}(w_2^2 - w_1 w_3) \subset \mathrm{SL}(W) = G$. This means that

- $\mathrm{Stab}(Z) = H$;
- G-equivalence reduces to H-equivalence on an open $Z_1 \subset Z$;
- $Y = \overline{G \cdot Z}$.

Sections

Techniques developed by Gatti–Viniberghi (1978) can be applied to rigorously relate various invariants:

Theorem

The restriction arrow

$$K[\mathrm{Sym}^4(W^*)]_{I_{12}}^{\mathrm{SL}(W)} \to K[Z]^{\langle \zeta_3 \rangle\, \mathrm{SO}(w_2^2 - w_1 w_3)}$$

is an isomorphism. Moreover, we have

$$K[Z]^{\mathrm{SO}(w_2^2 - w_1 w_3)} = K[Z]^{\langle \zeta_3 \rangle\, \mathrm{SO}(w_2^2 - w_1 w_3)}[u]$$

where u is the function that sends an element $F \in Z(K)$ to the scalar u in $\rho(F) = u(w_2^2 - w_1 w_3)$.

Magic (also known as Lie theory)

We can understand the term $K[Z]^{\langle \zeta_3 \rangle \, \mathrm{SO}(w_2^2 - w_1 w_3)}$ by first understanding $K[Z]^{\mathrm{SO}(w_2^2 - w_1 w_3)} = K[Z]^{\mathrm{SL}(V)}$. Since

$$Z \subset \operatorname{Spec} K[\mathrm{Sym}^4(W^*)] = \operatorname{Spec} K[\mathrm{Sym}^4(\mathrm{Sym}^2(V^*))],$$

we can apply the following result.

Theorem (Van Rijnswou (2001))

There exists an explicit $\mathrm{SL}(V)$*-equivariant linear map*

$$\ell : \mathrm{Sym}^4(\mathrm{Sym}^2(V^*)) \to \mathrm{Sym}^8(V^*) \oplus \mathrm{Sym}^4(V^*) \oplus \mathrm{Sym}^0(V^*).$$

Magic (also known as Lie theory)

We can understand the term $K[Z]^{\langle \zeta_3 \rangle \, \mathrm{SO}(w_2^2 - w_1 w_3)}$ by first understanding $K[Z]^{\mathrm{SO}(w_2^2 - w_1 w_3)} = K[Z]^{\mathrm{SL}(V)}$. Since

$$Z \subset \operatorname{Spec} K[\mathrm{Sym}^4(W^*)] = \operatorname{Spec} K[\mathrm{Sym}^4(\mathrm{Sym}^2(V^*))],$$

we can apply the following result.

Theorem (Van Rijnswou (2001))

There exists an explicit $\mathrm{SL}(V)$*-equivariant linear map*

$$\ell : \mathrm{Sym}^4(\mathrm{Sym}^2(V^*)) \to \mathrm{Sym}^8(V^*) \oplus \mathrm{Sym}^4(V^*) \oplus \mathrm{Sym}^0(V^*).$$

In other words, we can understand $K[Z]^{\mathrm{SL}(V)}$ by studying certain joint invariants. The corresponding fundamental invariants were determined by Marc Olive (2014).

Shifting the problem

In good mathematical tradition, this allows us to solve our problem of reconstructing curves by shifting it! We restrict to quartic forms in Z and use the diagram

$$\begin{tikzcd} \operatorname{Spec} K[\operatorname{Sym}^4(W^*)] \supset Z \arrow[r, "\ell"] \arrow[d, "\pi_{\mathrm{aff}}"] & Z' \subset \operatorname{Spec} K[\operatorname{Sym}^8(V^*) \oplus \operatorname{Sym}^4(V^*) \oplus \operatorname{Sym}^0(V^*)] \arrow[d, "\pi_{\mathrm{aff}}"] \\ \mathbb{A}^M \arrow[r, dashed] & \mathbb{A}^N \end{tikzcd}$$

Here the dashed arrow is a rational map that expresses joint invariants in terms of Dixmier–Ohno invariants (and the function u such that $\rho(F) = u(F)(v_2^2 - v_1 v_3)$).

Shifting the problem

In good mathematical tradition, this allows us to solve our problem of reconstructing curves by shifting it! We restrict to quartic forms in Z and use the diagram

$$\begin{tikzcd}
\operatorname{Spec} K[\operatorname{Sym}^4(W^*)] \supset Z \arrow[r, "\ell"] \arrow[d, "\pi_{\mathrm{aff}}"] & Z' \subset \operatorname{Spec} K[\operatorname{Sym}^8(V^*) \oplus \operatorname{Sym}^4(V^*) \oplus \operatorname{Sym}^0(V^*)] \arrow[d, "\pi_{\mathrm{aff}}"] \\
\mathbb{A}^M \arrow[r, dashed] & \mathbb{A}^N
\end{tikzcd}$$

Here the dashed arrow is a rational map that expresses joint invariants in terms of Dixmier–Ohno invariants (and the function u such that $\rho(F) = u(F)(v_2^2 - v_1 v_3)$).

Theorem (Lercier–Ritzenthaler–Sijsling, 2016)

Via the isomorphism ℓ, a joint invariant j of degree d on Z' allows an expression of the form P_j/u^{2d}, where P_j is a polynomial in the Dixmier–Ohno invariants that is homogeneous of degree $9d$.

Proof of the main theorem

Step 1: Given the generic quadric

$$Q = ax_2^2 + bx_2x_1 + cx_2x_3 + dx_1^2 + ex_1x_3 + fx_3^2$$

determine an integral matrix T in the coefficients of Q with the property that T transforms Q to a multiple of $x_2^2 - x_1x_3$.

Proof of the main theorem

Step 1: Given the generic quadric

$$Q = ax_2^2 + bx_2x_1 + cx_2x_3 + dx_1^2 + ex_1x_3 + fx_3^2$$

determine an integral matrix T in the coefficients of Q with the property that T transforms Q to a multiple of $x_2^2 - x_1x_3$.

Step 2: Given a joint invariant j and the function b_0 with values in $\mathrm{Sym}^0(W^*) = K$, show that jT and b_0T are in $K[\mathrm{Sym}^4(W^*)]$. (Use $\det(T^\sigma) = -\det(T)$.)

Proof of the main theorem

Step 1: Given the generic quadric

$$Q = ax_2^2 + bx_2x_1 + cx_2x_3 + dx_1^2 + ex_1x_3 + fx_3^2$$

determine an integral matrix T in the coefficients of Q with the property that T transforms Q to a multiple of $x_2^2 - x_1x_3$.

Step 2: Given a joint invariant j and the function b_0 with values in $\mathrm{Sym}^0(W^*) = K$, show that jT and $b_0 T$ are in $K[\mathrm{Sym}^4(W^*)]$. (Use $\det(T^\sigma) = -\det(T)$.)

Step 3: Show that $K[\mathrm{Sym}^4(W^*)]^{\mathrm{SL}(W)}$ is a UFD. (This is true because irreducible factors of an invariant function are themselves invariant, which in turn follows from the fact that $\mathrm{SL}(W)$ is irreducible and only admits the trivial character.)

Proof of the main theorem

Step 4: Show that up to scalar $I_9 = u^2 b_0$ on Z and $(I_9/b_0 T)^3 = (I_{12}/\det(T)^2)^2$ on X.

Proof of the main theorem

Step 4: Show that up to scalar $I_9 = u^2 b_0$ on Z and $(I_9/b_0 T)^3 = (I_{12}/\det(T)^2)^2$ on X.

Step 5: Show that I_{12} is irreducible (via interpolation).

Proof of the main theorem

Step 4: Show that up to scalar $I_9 = u^2 b_0$ on Z and $(I_9/b_0 T)^3 = (I_{12}/\det(T)^2)^2$ on X.

Step 5: Show that I_{12} is irreducible (via interpolation).

Step 6: Write $j = P/u^n$ so that $j^3 = P^3/I_{12}^n$. Then we have

$$(jT)^3 = \det(T)^{4d} j^3 = \det(T)^{4d} \frac{P^3}{I_{12}^n} = \frac{P^3}{(I_9/b_0 T)^{3d} I_{12}^{n-2d}}.$$

Using the UFD property, we get

$$I_9^{3d} (jT)^3 I_{12}^{n-2d} = P^3 (b_0 T)^{3d}.$$

Substituting a generic quartic shows that I_{12} does not divide $b_0 T$, so $n \leq 2d$. We are done!

Calculating the expressions

The polynomials P_j in the expression P_j/u^{2d} can be determined via evaluation-interpolation:

- Generate a large family of random quartics F;
- Given a quartic F in the family, normalize the covariant ρ and transform F along to get into Z;
- Evaluate the function m/u^{2d} in the monomials m of degree $9d$ in the Dixmier–Ohno invariants;
- Determine P_j by solving the resulting linear equation.

This can be done over many finite fields, after which a result over $\mathbb{Q}$ can be interpolated and checked (by using the Hilbert series of the ring of Dixmier–Ohno invariants).

An algorithm

We can now reconstruct a ternary quartic form $F \in Z(K)$ from a tuple of Dixmier–Ohno invariants I over K by the following algorithm.

1. Normalize $u = 1$;
2. Calculate the fundamental joint invariants of the corresponding element $(b_8, b_4, b_0) = \ell(F)$ of $Z' \subset \operatorname{Spec} K[\operatorname{Sym}^8(V^*) \oplus \operatorname{Sym}^4(V^*) \oplus \operatorname{Sym}^0(V^*)]$ via the polynomials P_j;
3. Reconstruct the octic part b_8 by the hyperelliptic machinery;
4. Use the joint invariants that are linear in the coefficients of b_4 to determine that form (generically);
5. Calculate b_0 as $I_9 / u^{2d} = I_9$;
6. Travel back to find $F = \ell^{-1}((b_8, b_4, b_0))$.

An example

With more care in the projective case, we can in fact reconstruct over an at most quadratic extension of the field of moduli. Combining our algorithms with a reduction method by Elsenhans and Stoll, we could calculate the following.

Example

Let

$$I = \left(0,\ 0,\ 0,\ 0,\ \frac{-7\cdot 19}{2^2\cdot 3^2\cdot 5^2},\ 0,\ \frac{-2\cdot 11\cdot 19}{3\cdot 5^2},\ 0,\ \frac{7\cdot 19^2}{3\cdot 5^3},\ \frac{2^6\cdot 3^3\cdot 19^2}{5^3},\right.$$
$$\left.\frac{-2^9\cdot 3^5\cdot 19^2\cdot 31}{5^5},\ \frac{-2^{11}\cdot 3^5\cdot 17\cdot 19^2}{5^5},\ \frac{-19^2\cdot 6553}{2^{39}\cdot 3^6\cdot 5^5\cdot 11}\right).$$

A corresponding quartic curve is given by

$$X: -4x_1^4 + 12x_1^3x_2 + 62x_1^3x_3 + 108x_1^2x_2^2 - 144x_1^2x_2x_3 - 12x_1^2x_3^2 - 20x_1x_2^3 +$$
$$90x_1x_2^2x_3 + 210x_1x_2x_3^2 - 125x_1x_3^3 + 30x_2^4 + 160x_2^3x_3 - 135x_2x_3^3 - 180x_3^4 = 0\,.$$