Direct problem in differential Galois theory

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Joint work with M. Barkatou, T. Cluzeau and J.-A. Weil

Table of contents

Differential Galois theory

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i.e., a differential field extension $(K, \partial)/(k, \partial)$ *s.t.* $K^{\partial} = \mathbb{C}$ *and* $\exists U \in Gl_n(K)$, *with* $\partial(U) = AU$ *and* $K = k(U)$.

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Important properties :

- $G := Gal(A)$ *is an algebraic group* $\Rightarrow \mathfrak{g} := Lie(G)$
- $\mathfrak{g} \subset \mathrm{End}(\mathbb{C}$-*vector space of solutions*$)$

Direct problem

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$\rightsquigarrow$ algorithm to calculate the Lie algebra of the differential Galois group of an absolutely irreducible differential system

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Reduced forms

$\partial Y = AY$, $A = (a_{i,j}) \in \mathbb{C}(x)^{n^2}$, group G, $\mathfrak{g} = Lie(G)$.

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Wei-Normann decomposition of $A = \sum_{h=1}^{r} \alpha_h M_h$

- $\alpha_1, \ldots, \alpha_r$ *is a* $\mathbb{C}$*-basis of* $\sum_{i,j=1,\ldots,n} \mathbb{C} a_{i,j}$
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Proposition (Kolchin-Kovacic)

$\mathfrak{g} \subset Lie(A)$ *and*
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$\partial Z = BZ$ is a reduced form of $\partial Y = AY$

The point of view of differential modules

$(\mathcal{M}, \nabla)$ differential k-module of dimension n
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Hypotheses

1. $(\mathcal{M}, \nabla)$ absolutely irreducible$\Rightarrow$ $\mathcal{M} \otimes_k \mathcal{M}^*$ is a direct sum of irreducibles
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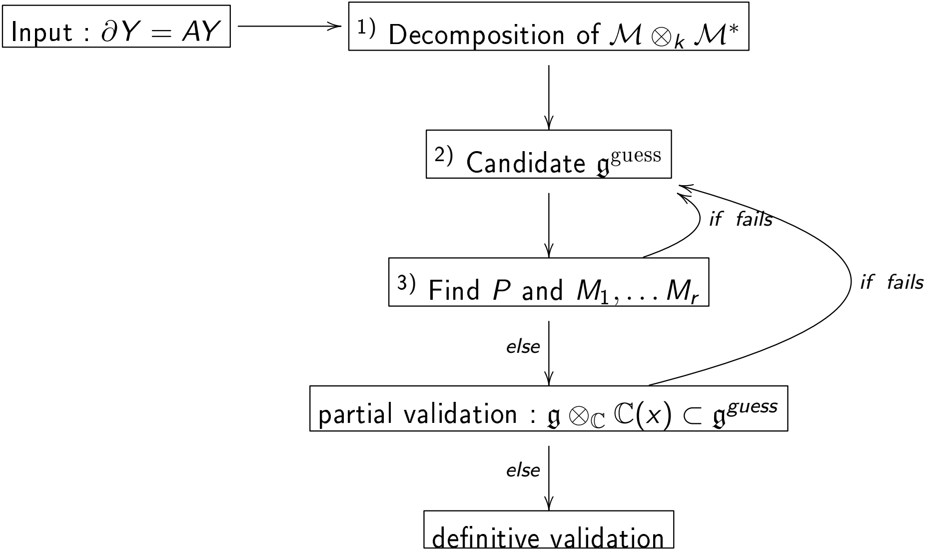
The algorithm

1. Decomposing $\mathcal{M} \otimes_k \mathcal{M}^*$
2. Find a candidate $\mathfrak{g}^{guess} \subset \mathcal{M} \otimes_k \mathcal{M}^*$ for $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}(x)$

The algorithm



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- *$\partial Y = AY$ is a reduced form $\Leftrightarrow \forall$ $Constr(\mathcal{M})$ and $\forall \Phi$ rational solution of $\partial Y = Constr(A)Y$, Φ is a constant vector.*
- *$\exists P \in GL_n(\bar{k})$ s.t. $\partial Z = P[A]Z$ is a reduced form $\forall$ $Constr(\mathcal{M})$ and $\forall \Phi$ rational solution of $\partial Y = Constr(A)Y$, P sends Φ over $\Phi(x_0)$.*