Simple Walks in the Three Quarter Plane

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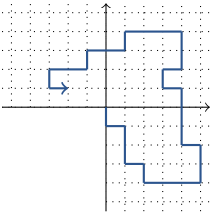

Introduction

Simple Walks

We consider the **simple walks** (*i.e.* walks with a set of steps $\mathcal{S} = \{\mathrm{W}, \mathrm{N}, \mathrm{E}, \mathrm{S}\}$) in the lattice plane. We constrain the walks to **avoid the negative quadrant**.

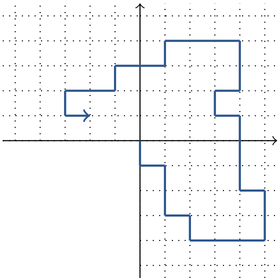

Figure: Simple walk in the three quarter plane.

Introduction

Objective

The goal is to compute the **number of paths** $c(i, j; n)$ of length n, starting at $(0, 0)$ and ending at (i, j), with $(i \geq 0$ or $j \geq 0)$ and $n \geq 0$.

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Example

For example, $c(0, 0; 0) = 1$ (the empty walk);
$c(0, 0; 2) = 4$ $(\rightarrow\leftarrow, \leftarrow\rightarrow, \downarrow\uparrow, \uparrow\downarrow)$;
$c(0, 0; n) = 0$ for an odd n.

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Mireille Bousquet-Mélou (Square lattice walks avoiding a quadrant, [1]) has already studied this problem.

The objective here is to:

- Develop analytic approach in the three quarter plane;
- Generalize to sets of steps which have infinite group.

Method

Usual way to compute $c(i, j; n)$

A usual way to compute $c(i, j; n)$ is the following:

1. Consider the **generating function** of $c(i, j; n)$:

$$C(x, y) = \sum_{\substack{i \geq 0 \text{ or } j \geq 0 \\ n \geq 0}} c(i, j; n) x^i y^j t^n;$$

2. Find a **functional equation** that $C(x, y)$ satisfies.
3. Solve the **functional equation**. Here, we use an analytic approach by transforming the functional equation into a **boundary value problem**.

Functional Equation

Cut the domain into three parts

We decompose the domain of possible ends of the walks into three parts:

$$C(x,y) = L(x,y) + D(x,y) + S(x,y).$$

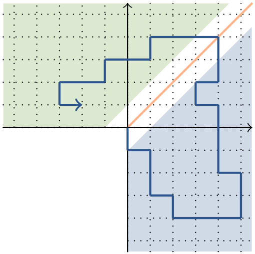

Figure: Three possible endpoints of the walks.

$$\begin{cases} L(x,y) = \sum_{\substack{i\geq 0 \\ j\leq i-1 \\ n\geq 0}} c(i,j;n)x^i y^j t^n, \\ D(x,y) = \sum_{\substack{i\geq 0 \\ n\geq 0}} c(i,i;n)x^i y^i t^n, \\ S(x,y) = \sum_{\substack{i\leq 0 \\ j\geq i+1 \\ n\geq 0}} c(i,j;n)x^i y^j t^n. \end{cases}$$

Starting on the diagonal (i_0, i_0), $i_0 \geq 0$. (1)

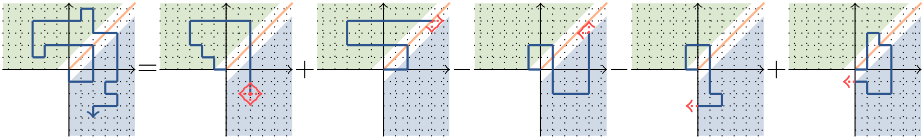

Figure: Different ways to end in the lower part starting on the diagonal.

$$L(x,y) = t(x + x^{-1} + y + y^{-1})L(x,y) + t(x + y^{-1})D(x,y) \\ - t(x^{-1} + y)LD(x,y) - tx^{-1}L(0,y) + tx^{-1}\sum_{n\geq 0} c(0,-1;n)y^{-1}t^n.$$

Starting on the diagonal (i_0, i_0), $i_0 \geq 0$. (2)

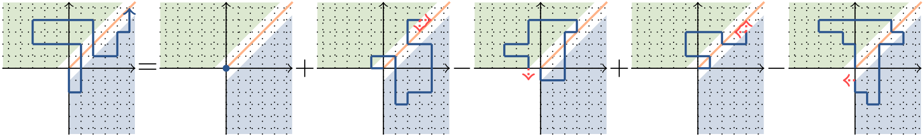

Figure: Different ways to end on the diagonal starting on the diagonal.

$$D(x,y) = x^{i_0} y^{i_0} + 2t(x^{-1} + y)LD(x,y) - 2tx^{-1} \sum_{n \geq 0} c(0,-1;n) y^{-1} t^n.$$

Starting on the diagonal (i_0, i_0), $i_0 \geq 0$. (3)

Functional Equation - Starting on the diagonal

$$L(x,y)K(x,y)xy = \frac{1}{2}x^{i_0+1}y^{i_0+1} - tyL(0,y) + (t(x^2y+x) - \frac{1}{2}xy)D(x,y)$$

with

$$K(x,y) = 1 - t(x + x^{-1} + y + y^{-1}).$$

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Functional equation - Simple walks in the quarter plane

$$Q(x,y)K(x,y)xy = x^{i_0+1}y^{j_0+1} - txQ(x,0) - tyQ(0,y),$$

with

$$Q(x,y) = \sum_{i,j,n \geq 0} q(i,j;n)x^i y^j t^n.$$

Starting off of the diagonal (i_0, j_0), $i_0 \geq 0$ and $j_0 \leq i_0 - 1$. (1)

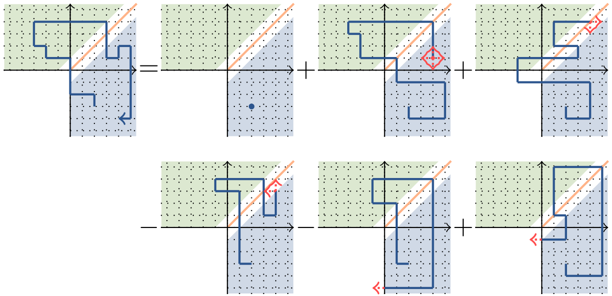

Figure: Different ways to ends in the lower part starting in the lower part.

$$\begin{aligned} L(x,y) =& x^{i_0} y^{j_0} + t(x + x^{-1} + y + y^{-1}) L(x,y) + t(x + y^{-1}) D(x,y) \\ & - t(x^{-1} + y) LD(x,y) - t x^{-1} L(0,y) + t x^{-1} \sum_{n \geq 0} c(0,-1;n) y^{-1} t^n. \end{aligned}$$

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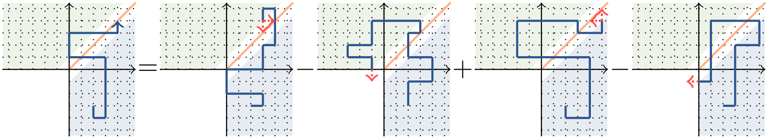

Figure: Different ways to end on the diagonal starting in the lower part.

$$D(x,y) = t(x+y^{-1})UD(x,y) - ty^{-1}\sum_{n\geq 0} c(-1,0;n)x^{-1}t^n$$
$$+\, t(x^{-1}+y)LD(x,y) - tx^{-1}\sum_{n\geq 0} c(0,-1;n)y^{-1}t^n.$$

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With $K(x,y) = 1 - t(x + x^{-1} + y + y^{-1})$;

Functional Equation - Starting in the lower part

$$L(x,y)K(x,y)xy = x^{i_0+1}y^{j_0+1} - tyL(0,y) + (t(x^2y + x) - xy)D(x,y)$$
$$+ t(x^2y + x)\sum_{\substack{i\geq 0\\ n\geq 0}} c(i-1, i; n)x^{i-1}y^i t^n - t\sum_{n\geq 0} c(-1, 0; n)t^n$$

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Functional Equation - Starting in the lower part

$$L(x,y)K(x,y)xy = x^{i_0+1}y^{j_0+1} - tyL(0,y) + (t(x^2y+x) - xy)D(x,y) + t(x^2y+x)\sum_{\substack{i\geq 0\\ n\geq 0}} c(i-1,i;n)x^{i-1}y^i t^n - t\sum_{n\geq 0} c(-1,0;n)t^n$$

Functional Equation - Starting in the upper part

$$L(x,y)K(x,y)xy = -tyL(0,y) + (t(x^2y+x) - xy)D(x,y) + t(x^2y+x)\sum_{\substack{i\geq 0\\ n\geq 0}} c(i-1,i;n)x^{i-1}y^i t^n - t\sum_{n\geq 0} c(-1,0;n)t^n$$

Resolution when we start on the diagonal

Functional Equation - Starting on the diagonal

$$L(x,y)K(x,y)xy = \frac{1}{2}x^{i_0+1}y^{i_0+1} - tyL(0,y) + (t(x^2y+x) - \frac{1}{2}xy)D(x,y).$$

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Change of variable

$$\varphi : \left\{ \begin{array}{ccc} x & \to & xy, \\ y & \to & x^{-1}. \end{array} \right.$$

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Figure: Simple walk and Gessel's walk.

Resolution when we start on the diagonal

New Functional Equation

$$\widetilde{L}(x,y)\widetilde{K}(x,y)xy = \frac{1}{2}xy - t\widetilde{L}(x,0) + x\left(ty\left(xy+x\right) - \frac{1}{2}y\right)\widetilde{D}(y),$$

with

$$\begin{cases} \widetilde{L}(x,y) & = & \sum\limits_{\substack{i\geq 1\\ j\leq 0\\ n\geq 0}} c(j,j-i;n)x^i y^j t^n, \\ \widetilde{D}(y) & = & \sum\limits_{\substack{i\geq 0\\ n\geq 0}} c(i,i;n)y^i t^n, \\ \widetilde{K}(x,y) & = & 1 - t\left(x^{-1} + xy + x + x^{-1}y^{-1}\right). \end{cases}$$

Roots and Branches of the Kernel

Cancel the Kernel

$$-xy\widetilde{K}(x,y) = \widehat{a}(y)x^2 + \widehat{b}(y)x + \widehat{c}(y) = a(x)y^2 + b(x)y + c(x).$$

Discriminant: $\widehat{d}(y) = \widehat{b}(y)^2 - 4\widehat{a}(y)\widehat{c}(y)$ and $d(x) = b(x)^2 - 4a(x)c(x)$.

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Branches of the Kernel $i = 0, 1$

$$\widetilde{X}_i(y) = \frac{-\widehat{b}(y) \pm \sqrt{\widehat{d}(y)}}{2\widehat{a}(y)};$$

$$\widetilde{Y}_i(x) = \frac{-b(x) \pm \sqrt{d(x)}}{2a(x)}.$$

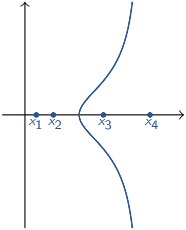

Figure: $\widetilde{X}([y_1, y_2])$.

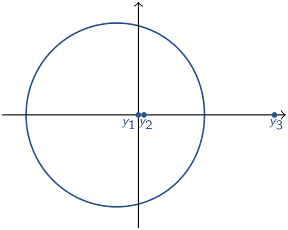

Figure: $\widetilde{Y}([x_1, x_2])$.

Boundary Value Problem

History

- These problem appeared and were studied in the XVIII$^{\text{th}}$ century and the XIX$^{\text{th}}$ century;
- Riemann first mentioned the problem;
- Hilbert then H. Poincaré studied the problem;
- The Sokhotski-Plemelj formulae are elementary tools to solve the problem.
- Reference authors on BVP : Muskhelischvili, Gakhov and Litvintchuk.

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Link with the walks in the plane

In the 70's Malyshev in Russia then Fayolle and Iasnogorodski in France first used an analytic method via BVP to solve a functional equation satisfies by generating functions of walks.

Boundary Value Problem

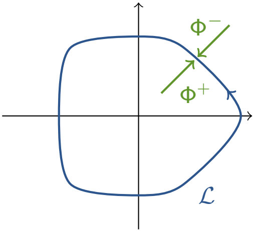
BVP - Definition

A function Φ satisfies a BVP on a simple smooth oriented contour $\mathcal{L}$ if:

- Φ is **sectionally holomorphic**: holomorphic in $\mathbb{C} \setminus \mathcal{L}$ where it has left limit Φ^+ and right limit Φ^-. Furthermore, Φ is of finite degree at infinity.
- Φ satisfies the following **boundary condition on** $\mathcal{L}$:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \mathcal{L},$$

with G and g are Hölder functions on $\mathcal{L}$, and G does not vanish on $\mathcal{L}$.



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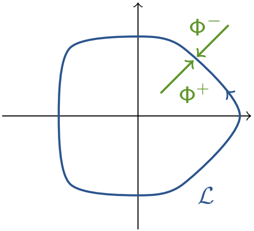

We know some techniques and methods to find a function Φ which satisfies a BVP.

Generating function $\widetilde{D}(y)$ stated as a BVP

Functional Equation - Starting on the diagonal

$$\widetilde{L}(x,y)\widetilde{K}(x,y)xy = \frac{1}{2}xy - t\widetilde{L}(x,0) + x\left(ty\left(xy+x\right) - \frac{1}{2}y\right)\widetilde{D}(y),$$

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Riemann-Carleman with shift BVP

By evaluating the functional equation in $\widetilde{Y_0}$ and $\widetilde{Y_1}$, we have the following **boundary value problem**: For $y \in \widetilde{Y}([x_1, x_2])$,

$$R(y)\widetilde{D}(y) - R(\overline{y})\widetilde{D}(\overline{y}) = y - \overline{y},$$

with

$$R(y) = y - 2t\widetilde{X_0}(y)y(y+1).$$

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It does not look like the BVP we have introduced !

Boundary Value Problem - Riemann-Hilbert on a segment

Riemann-Hilbert BVP

$$\widetilde{D}\left(v^+(u)\right) = \frac{R\left(v^-(u)\right)}{R\left(v^+(u)\right)}\widetilde{D}\left(v^-(u)\right) + \frac{v^+(u) - v^-(u)}{R\left(v^+(u)\right)}.$$

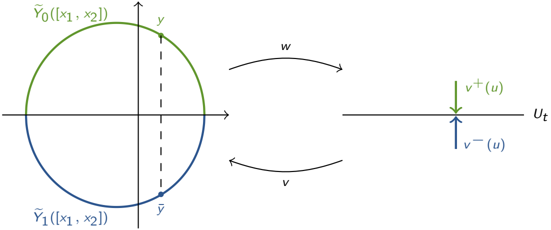

Figure: Conformal gluing function.

Result - Contour integral expression of $\widetilde{D}(y)$

Theorem [Raschel, T., 2017]

For y inside the curve $\widetilde{Y}([x_1, x_2])$,

$$\widetilde{D}(y) = \frac{\Psi(w(y))}{2i\pi} \times \int_{\widetilde{Y}([x_1,x_2])} \frac{tw'(t)dt}{R(t)\Psi^+(w(t))(w(t)-w(y))},$$

with: for z inside $\widetilde{Y}([x_1, x_2])$ and $s \in \widetilde{Y}([x_1, x_2])$,

$$\begin{cases} \Psi(z) &= e^{\Gamma(z)}, \\ \Psi^+(s) &= e^{\Gamma^+(s)}, \\ \Gamma(z) &= \frac{1}{2i\pi} \int_{Y([x_1,x_2])} \frac{\log(tR(\bar{t})/R(t))dt}{t-z}. \end{cases}$$

Γ^+ can be computed with the Sokhotski-Plemelj formulae.

Set-up

Remember - Functional Equation

$$L(x,y)K(x,y)xy = \frac{1}{2}x^{i_0+1}y^{i_0+1} - tyL(0,y) + (t(x^2y+x) - \frac{1}{2}xy)D(x,y).$$

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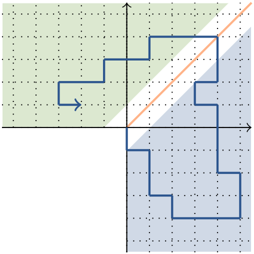

Symmetry of the cut and the walk

$$\Rightarrow S(x,y) = L(y,x).$$

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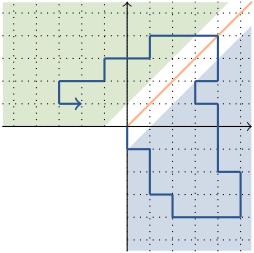

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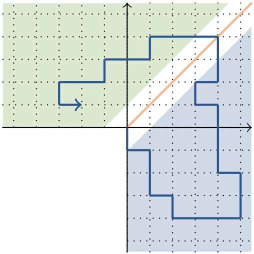

- We have an expression of $\widetilde{D}(y)$;
- With a change of variable we get an expression of $D(x,y)$;

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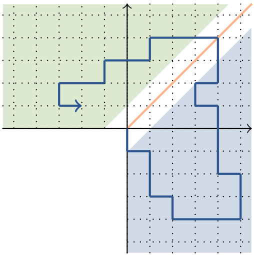

- We have an expression of $\widetilde{D}(y)$;
- With a change of variable we get an expression of $D(x,y)$;
- With the functional equation we get an expression of $L(x,y)$;

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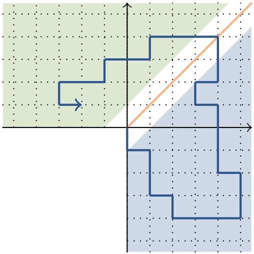

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3. Study the asymptotic of $c(i,j,n)$;
4. Study the class (algebraic, D-finite) of the generating functions $C(x,y)$, $L(x,y)$, $D(x,y)$;

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3. Study the asymptotic of $c(i,j,n)$;
4. Study the class (algebraic, D-finite) of the generating functions $C(x,y)$, $L(x,y)$, $D(x,y)$;
5. Solve the starting off of the diagonal functional equation;

Future works and possible applications

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7. Solve problems in other cones.

Reference

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The Sokhotski-Plemelj Formulae.

Theorem

Let $\mathcal{L}$ be a simple smooth line or curve in the complex plane, and φ be a Hölder function on $\mathcal{L}$. The function

$$\Phi(z) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\varphi(t)dt}{t-z}, \; z \notin \mathcal{L},$$

is continuous on $\mathcal{L}$ from the left and from the right, with the exception of the ends. Moreover the corresponding limiting values, denoted respectively by ϕ^+ and ϕ^-, are Hölder functions on $\mathcal{L}$, and they satisfy the so-called Sokhotski-Plemelj formulae, for $t \in \mathcal{L}$,

$$\begin{cases} \phi^+(t) & = & \frac{1}{2}\varphi(t) + \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\varphi(s)ds}{s-t}, \\ \phi^-(t) & = & -\frac{1}{2}\varphi(t) + \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\varphi(s)ds}{s-t}, \end{cases}$$

where the integrals are understood in the sense of Cauchy-principal value.

Cauchy's formulae

Theorem

Let $C(x, y)$ be holomorphic in $\mathcal{D}(0, 1)$.
Then for any $i_0 \geq 1$ or $j_0 \geq 1$:

$$c(i_0, j_0) = \frac{1}{(2i\pi)^2} \int \int \frac{C(x, y)}{x^{i_0} y^{j_0}} dxdy,$$

where the domain of integration is $\{x \in \mathbb{C} : |x| = \varepsilon\} \times \{y \in \mathbb{C} : |y| = \varepsilon\}$, for any $\varepsilon \in [0, 1)$.