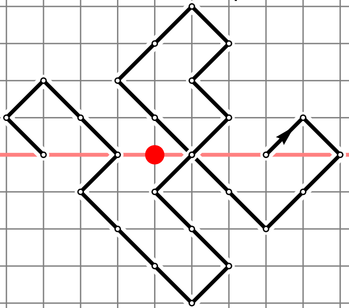
Lattice walks at the Interface of Algebra, Analysis and Combinatorics
BIRS, Banff, Canada - September 20th, 2017

Winding angles of simple walks on $\mathbb{Z}^2$

Timothy Budd

Based on arXiv:1709.04042 and w.i.p.



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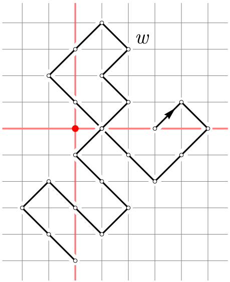
Combinatorial problem involving winding angles

- Let w be a simple diagonal walk on $\mathbb{Z}^2 \setminus \{\text{origin}\}$ of length $|w| \geq 0$.



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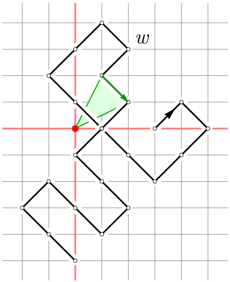

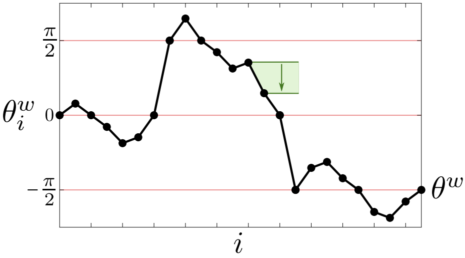

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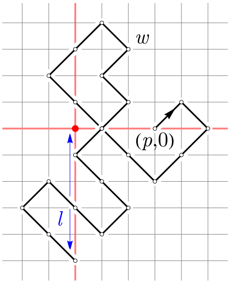
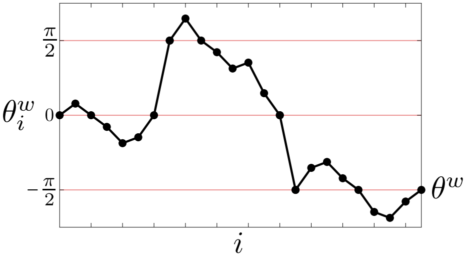
- Winding angle sequence $(\theta_0^w, \theta_1^w, \dots, \theta_{|w|}^w)$, $\theta_0^w = 0$, $\theta^w := \theta_{|w|}^w$.
- Can we compute the following generating function?

$$W_{\ell,p}^{(\alpha)}(t) := \sum_w t^{|w|} \mathbf{1}_{\{w_0=(p,0),\ |w_{|w|}|=\ell,\ \theta^w=\alpha\}}. \qquad (p,\ell \geq 1, \alpha \in \tfrac{\pi}{2}\mathbb{Z})$$

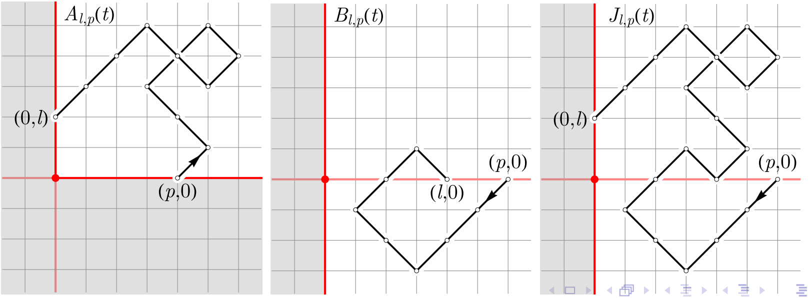




Building blocks

- Three types of building blocks: type A, B, J.

$$\sum_{m=1}^{\infty} A_{l,m}(t) B_{m,p}(t) = J_{l,p}(t).$$



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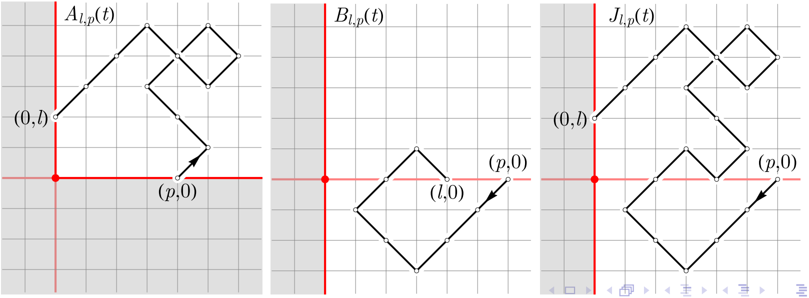

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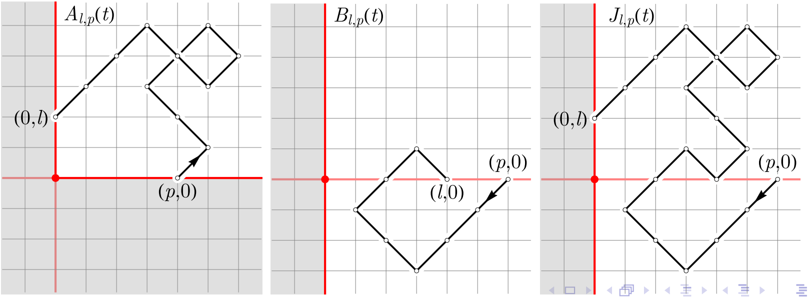
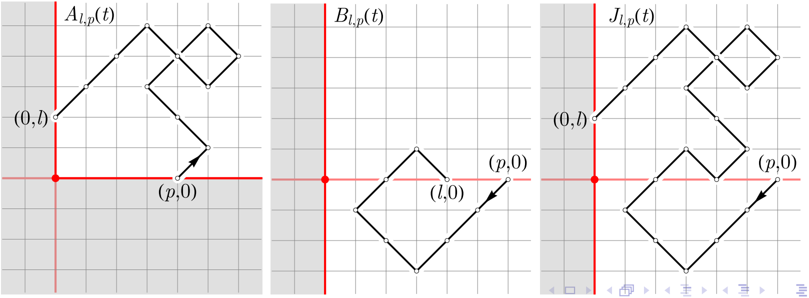
- To formalize this: fix $k = 4t \in (0,1)$ and choose convenient Hilbert space + basis.



Building blocks (operators)

- Let basis $(e_p)_{p=1}^{\infty}$ of $\ell^2(\mathbb{C})$ be such that $\langle e_l, e_p \rangle = p\,\mathbf{1}_{\{l=p\}}$ and let

$$\langle e_l, \mathbf{A}_k e_p \rangle = lp\, A_{l,p}(t), \quad \langle e_l, \mathbf{B}_k e_p \rangle = B_{l,p}(t), \quad \langle e_l, \mathbf{J}_k e_p \rangle = l\, J_{l,p}(t).$$



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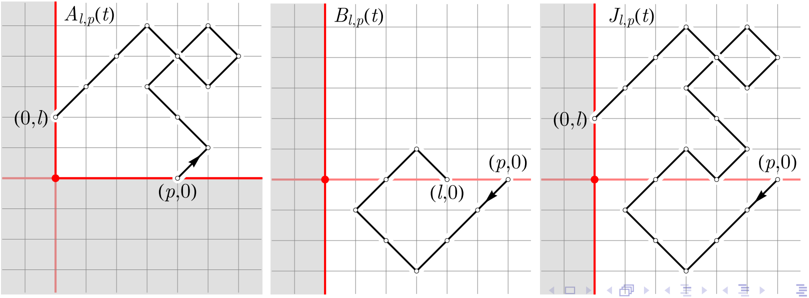

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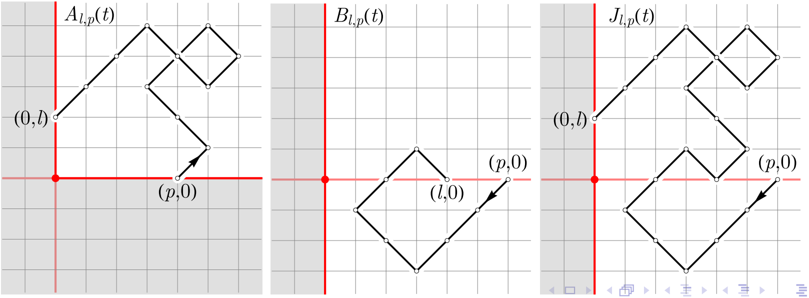
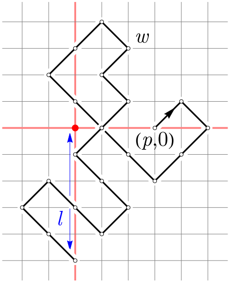
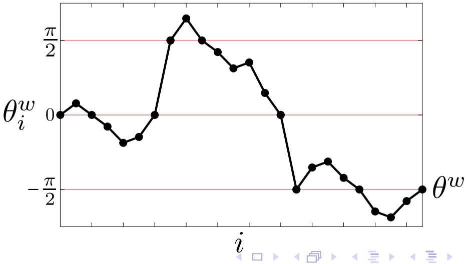
- $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{J}_k$ are bounded, self-adjoint and commuting! Simultaneous eigenvalue decomposition?



Putting the building blocks together

$$W^{(\alpha)}_{\ell,p}(t) := \sum_{w} t^{|w|}\mathbf{1}_{\{w_0=(p,0),\ |w_{|w|}|=\ell,\ \theta^w=\alpha\}}. \qquad (p,\ell \geq 1, \alpha \in \tfrac{\pi}{2}\mathbb{Z})$$





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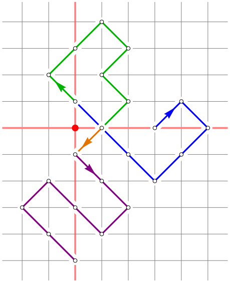
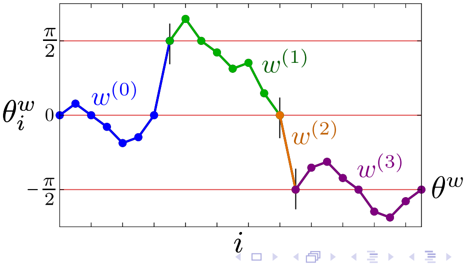
- w is encoded by a simple walk $(\alpha_j)_{j=0}^N$ on $\frac{\pi}{2}\mathbb{Z}$ from 0 to α together with a sequence $(w^{(0)}, \ldots, w^{(N)})$ of "matching" walks with $w^{(0)}, \ldots, w^{(N-1)}$ of type J and $w^{(N)}$ of type B.





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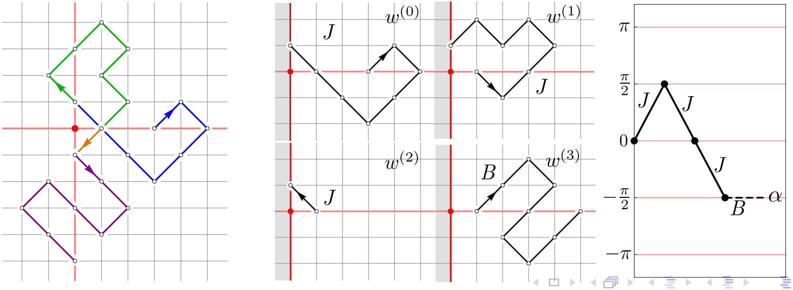

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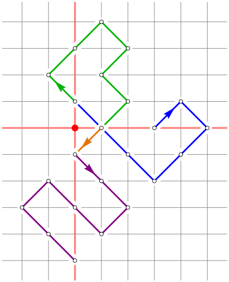
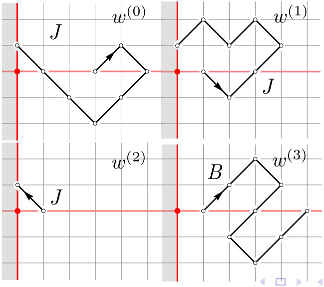
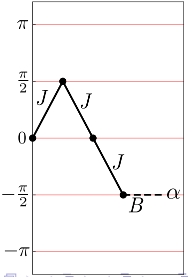
- Hence $W^{(\alpha)}_{\ell,p}(t) = \langle e_l, \mathbf{Y}^{(\alpha)}_k e_p\rangle$ where $\mathbf{Y}^{(\alpha)}_k$ is formally given by

$$\mathbf{Y}^{(\alpha)}_k = \sum_{N=0}^{\infty} \#\{\text{simple walks from 0 to } \alpha \text{ of length } N\}\cdot \mathbf{J}_k^N\,\mathbf{B}_k$$

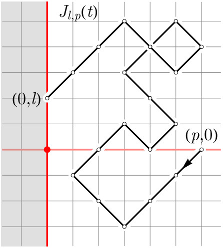


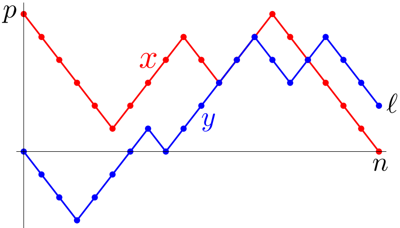




The operator $\mathbf{J}_k$

$$J_{\ell,p}(t) = \sum_{n=1}^{\infty} t^n \frac{p}{n} \binom{n}{\frac{n-p}{2}} \binom{n}{\frac{n-\ell}{2}} \mathbf{1}_{\{n-p \text{ and } n-\ell \text{ nonnegative and even}\}}$$





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- Not only is $\mathbf{J}_k$ self-adjoint, $\langle e_\ell, \mathbf{J}_k e_p \rangle = \ell\, J_{\ell,p}(t)$, but also $\mathbf{J}_k = \mathbf{R}_k^\dagger \mathbf{R}_k$ with (recall $k = 4t$)

$$\mathbf{R}_k e_p := \sum_{n=1}^{\infty} e_n \left(\frac{k}{4}\right)^{n/2} \frac{p}{n} \binom{n}{\frac{n-p}{2}} \mathbf{1}_{\{n-p \geq 0 \text{ and even}\}}$$

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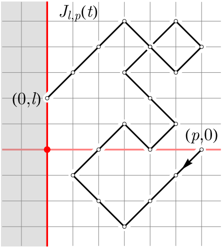

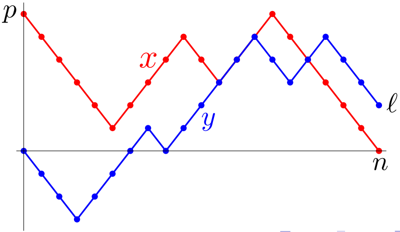

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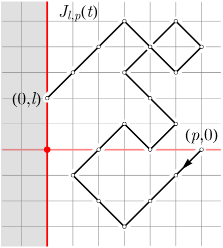
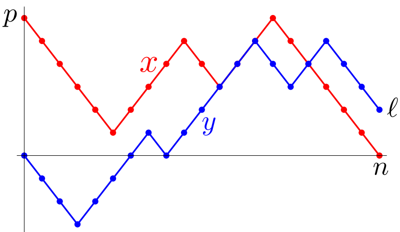
$$= \sum_{n=1}^{\infty} e_n \; [z^n]\psi_k(z)^p, \qquad \psi_k(z) := \frac{1-\sqrt{1-k\,z^2}}{\sqrt{k}\,z}.$$





Dirichlet space $\mathcal{D}$

- $\mathcal{D} = \mathcal{D}(\mathbb{D})$ is Hilbert space of analytic functions f on the unit disk $\mathbb{D} \subset \mathbb{C}$ with $f(0) = 0$ and finite norm w.r.t. $(\mathrm{d}A(x+iy) := \frac{1}{\pi}\mathrm{d}x\mathrm{d}y)$

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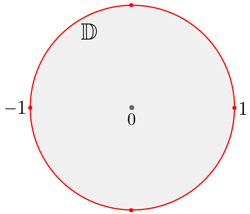

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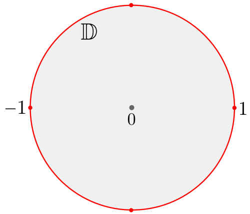

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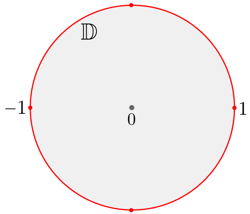

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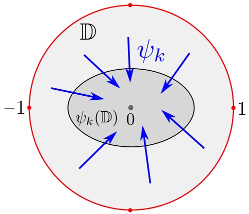

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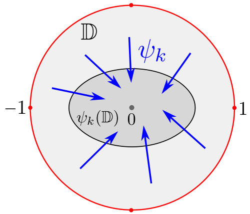
- By conformal invariance of the Dirichlet inner product,

$$\langle f, \mathbf{J}_k g \rangle_{\mathcal{D}} = \left\langle f, \mathbf{R}_k^{\dagger} \mathbf{R}_k g \right\rangle_{\mathcal{D}} = \langle f \circ \psi_k, g \circ \psi_k \rangle_{\mathcal{D}} = \langle f, g \rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}.$$



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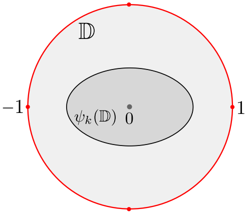

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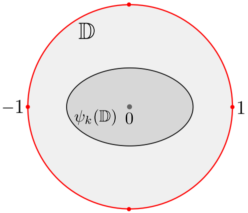

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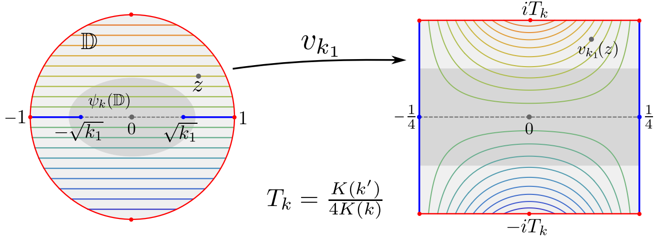
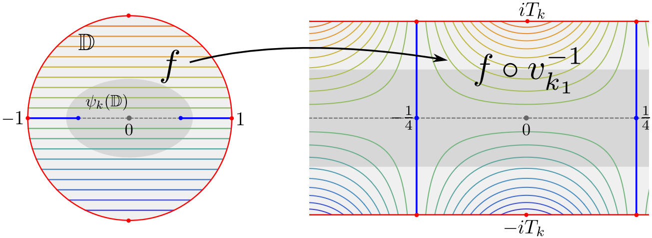
- Look for a nice conformal mapping.
- An elliptic integral does the job ($k' = \sqrt{1-k^2}$, $k_1 = \frac{1-k'}{1+k'}$)

$$v_{k_1}(z) = \frac{1}{4K(k_1)} \int_0^z \frac{\mathrm{d}x}{\sqrt{(k_1 - x^2)(1 - k_1 x^2)}} = \frac{\operatorname{arcsn}\left(\frac{z}{\sqrt{k_1}}, k_1\right)}{4K(k_1)}$$



- The push-forward of $f \in \mathcal{D}$ extends to an analytic function on the strip $\mathbb{R} + i(-T_k, T_k)$ that is even around $\pm 1/4$, hence 1-periodic.



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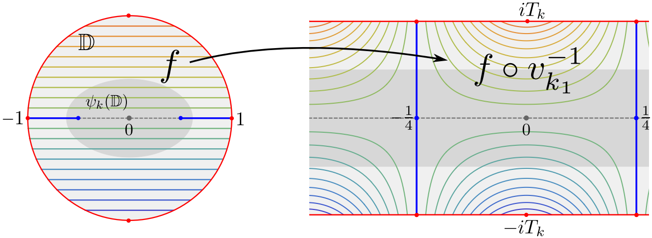

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$$f_m(z) = \cos(2\pi m(v_{k_1}(z) + 1/4)) - \cos(\pi m/2), \quad m \geq 1$$

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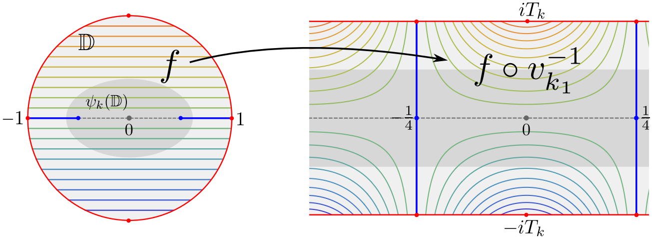

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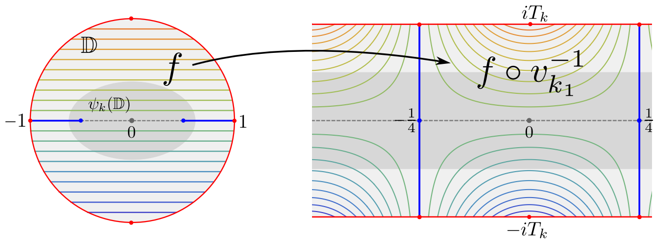
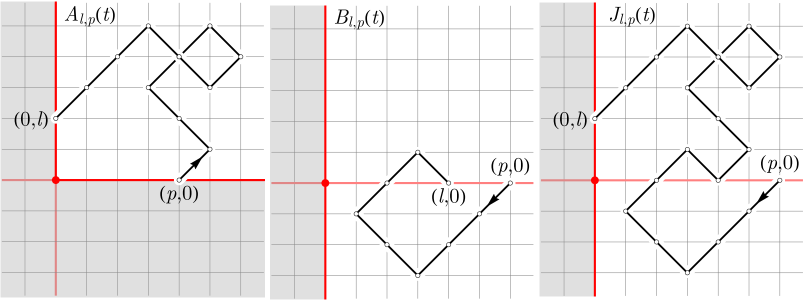
- Conclusion: $\mathbf{J}_k$ has eigenvectors $(f_m)_{m \geq 1}$ and eigenvalues

$$\frac{\langle f_m, f_m \rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}}{\langle f_m, f_m \rangle_{\mathcal{D}(\mathbb{D})}} = \frac{\sinh(2m\pi T_k)}{\sinh(4m\pi T_k)} = \frac{1}{q_k^{m/2} + q_k^{-m/2}}, \qquad \underset{\text{“nome”}}{q_k = e^{-\pi \frac{K(k')}{K(k)}}} \; .$$



- May work out eigenvalues of $\mathbf{A}_k$ and $\mathbf{B}_k$ too (eigenvectors $(f_m)_{m\geq 1}$):



$\mathbf{A}_k \; : \; \frac{\pi}{2K(k)} \frac{m}{q_k^{-m/2} - q_k^{m/2}}$ $\qquad$ $\mathbf{B}_k \; : \; \frac{2K(k)}{\pi} \frac{1}{m} \frac{1-q_k^m}{1+q_k^m}$ $\qquad$ $\mathbf{J}_k \; : \; \frac{1}{q_k^{m/2}+q_k^{-m/2}}$

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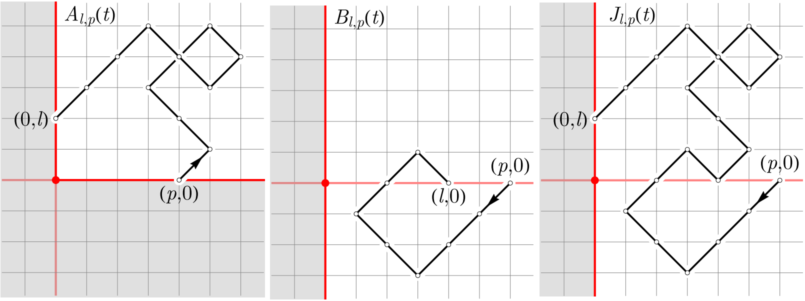


$\mathbf{A}_k : \ \dfrac{\pi}{2K(k)} \dfrac{m}{q_k^{-m/2} - q_k^{m/2}}$ $\quad$ $\mathbf{B}_k : \ \dfrac{2K(k)}{\pi} \dfrac{1}{m} \dfrac{1-q_k^m}{1+q_k^m}$ $\quad$ $\mathbf{J}_k : \ \dfrac{1}{q_k^{m/2}+q_k^{-m/2}}$

- Recall $W^{(\alpha)}_{\ell,p}(t) = \langle e_l, \mathbf{Y}^{(\alpha)}_k e_p \rangle$, $\alpha \in \frac{\pi}{2}\mathbb{Z}$, where

$$\mathbf{Y}^{(\alpha)}_k = \sum_{N=0}^{\infty} \#\{\text{simple walks from } 0 \text{ to } \alpha \text{ of length } N\} \cdot \mathbf{J}_k^N \, \mathbf{B}_k .$$

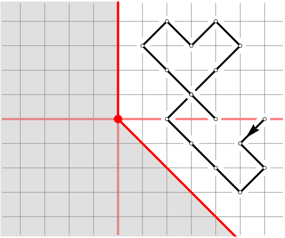
It has eigenvalues

$$\mathbf{Y}^{(\alpha)}_k f_m = \frac{2K(k)}{\pi} \frac{1}{m} q_k^{m|\alpha|/\pi} \, f_m .$$

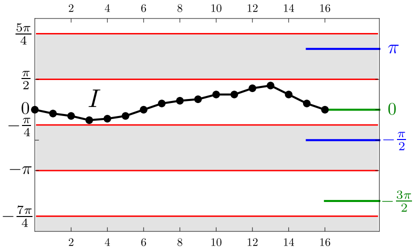
Reflection principle

- For $I = (\beta_-, \beta_+)$, $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$, $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$ and p, ℓ even, let

$$W_{\ell,p}^{(\alpha,I)}(t) = \sum_w t^{|w|} \mathbf{1}_{\{w_0=(p,0),\, |w_{|w|}|=\ell,\, \theta^w=\alpha,\, \theta^w_i \in I \text{ for } 1\le i<|w|\}}.$$



$\alpha = 0, I = (-\frac{\pi}{4}, \frac{\pi}{2})$



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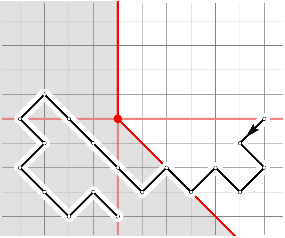

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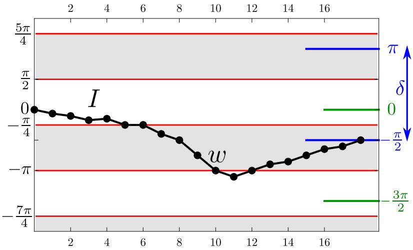

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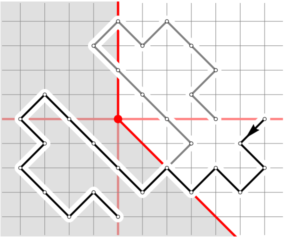

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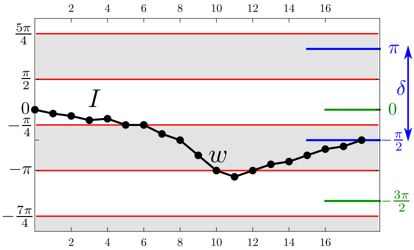

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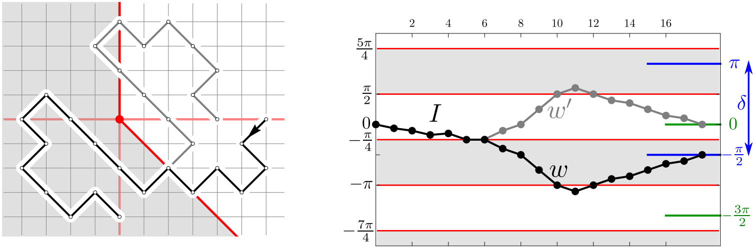

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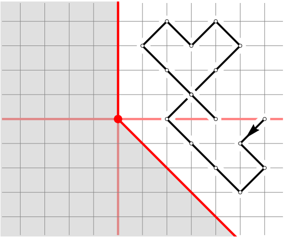
- $W^{(0,(-\pi/4,\pi/2))}_{\ell,p}(t) = \langle e_\ell, \mathbf{X} e_p \rangle_{\mathcal{D}}$ and $\mathbf{X}$ has e.v. $\frac{2K(k)}{\pi m} \frac{1-q_k^m}{1+q_k^{m/2}+q_k^m}$.



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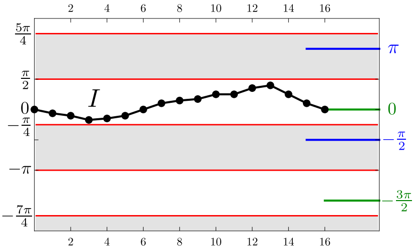
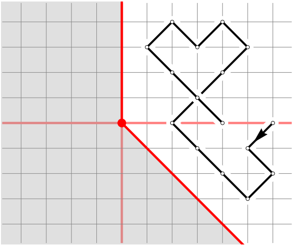
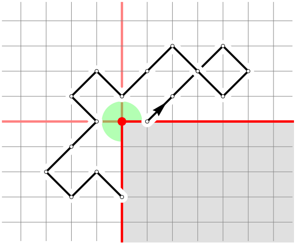


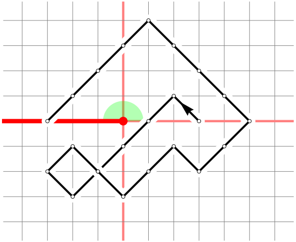
More examples See [TB,'17, Theorem 1] for the general case.



$$\langle e_\ell, \bullet\, e_p\rangle_{\mathcal{D}}, \quad \frac{2K(k)}{\pi m}\,\frac{1-q_k^m}{1+q_k^{m/2}+q_k^m}$$



$$\frac{1}{\ell p}\,\langle e_\ell, \bullet\, e_p\rangle_{\mathcal{D}}, \quad \frac{\pi m}{2K(k)}\,\frac{1}{q_k^{-m\alpha/\pi}-q_k^{m\alpha/\pi}}$$



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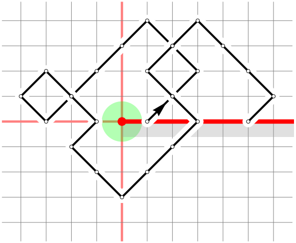

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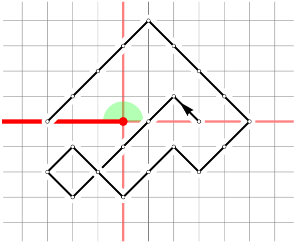

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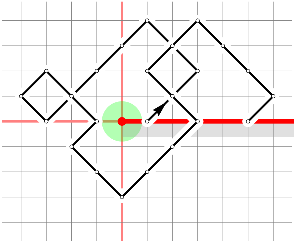

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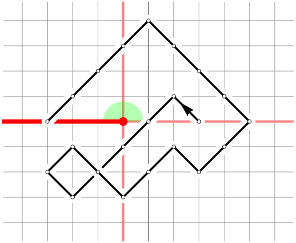
$$k \to k_1 = \frac{1-k'}{1+k'}, \quad k' = \sqrt{1-k^2}, \quad \text{since} \quad q_{k_1} = q_k^2.$$



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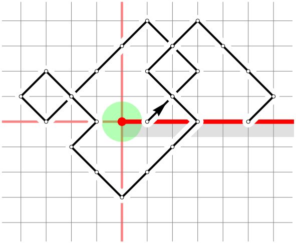

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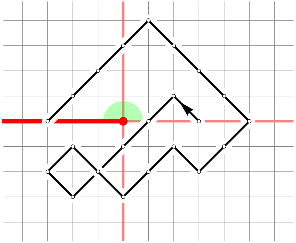
- Deserves a combinatorial explanation!



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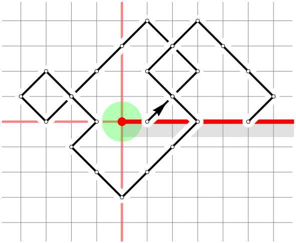

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A partial explanation

- Consider loops w, i.e. $w_0 = w_{|w|} \in \{(1,0),(2,0),\dots\}$, with winding angle $\theta^w = \alpha \in 2\pi\mathbb{Z}$ and $\theta^w_i < \alpha$ for $i < |w|$.
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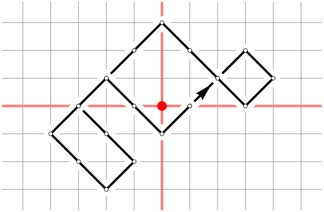

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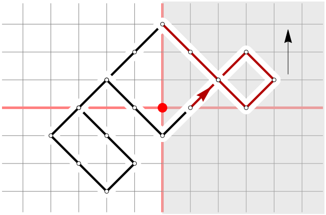

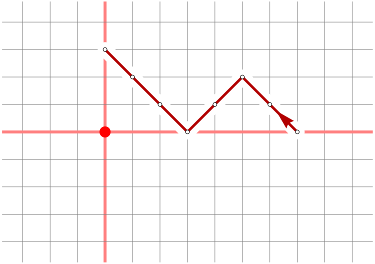

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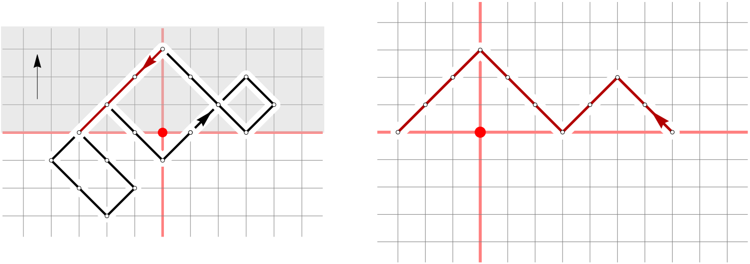

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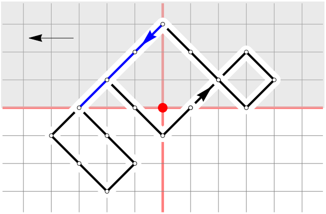

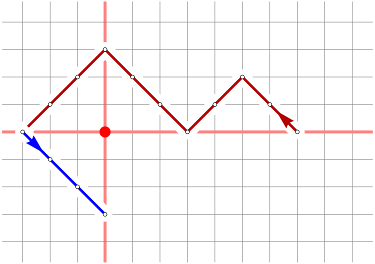

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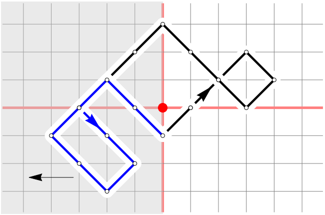

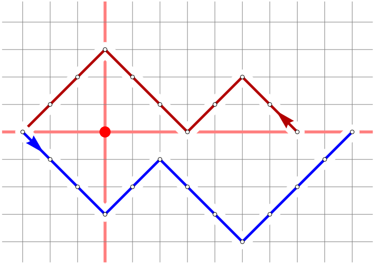

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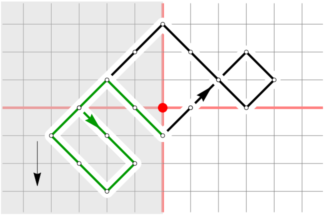

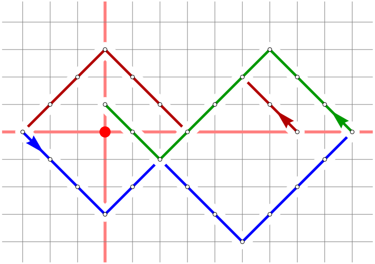

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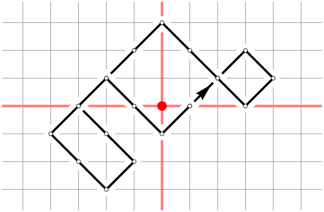
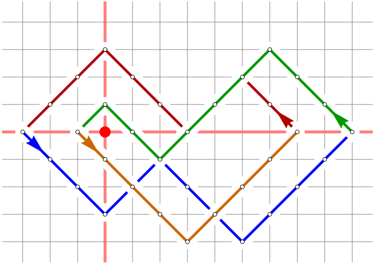
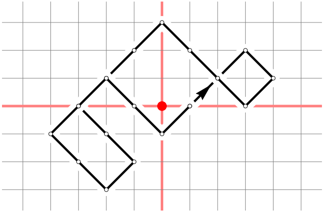
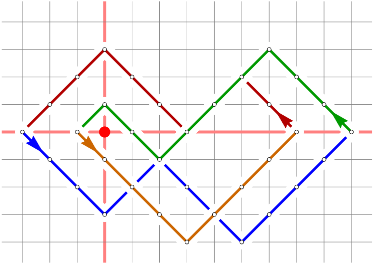




A partial explanation

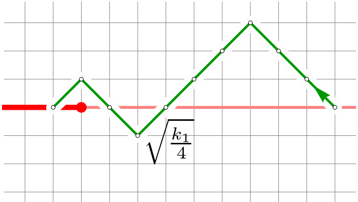


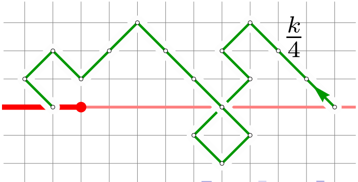


- Substituting $x \to \sqrt{k_1(t)/4}$ in g.f. of Dyck paths on the slit plane with fixed endpoints yields the corresponding g.f. for diagonal walks.

$$k_1(t) = \frac{1-k'}{1+k'}, \quad k' = \sqrt{1-k^2}, \quad k = 4t$$

- Open problem: give a bijective explanation of this fact!





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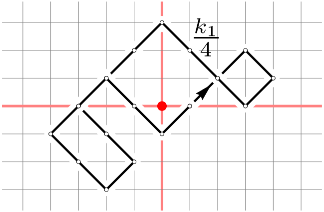

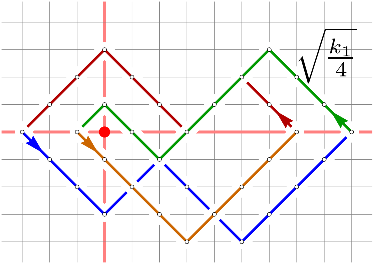

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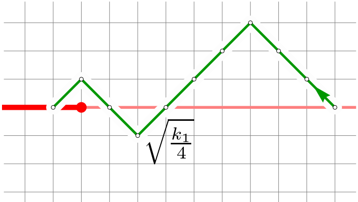
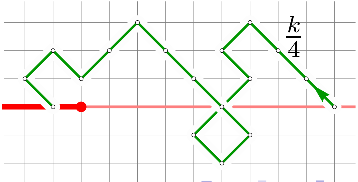
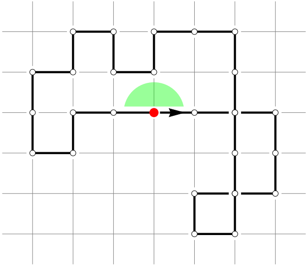


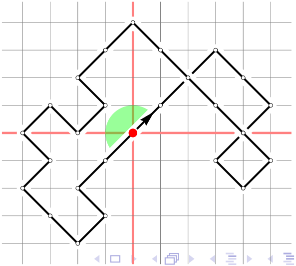


Application: Excursions

- Consider set $\mathcal{E}$ of excursions from the origin (rectilinear or diagonal).

$$F^{(\alpha)}(t) := \sum_{w \in \mathcal{E}} t^{|w|} \mathbf{1}_{\{\theta^w = \alpha\}}, \qquad \alpha \in \frac{\pi}{2}\mathbb{Z}.$$





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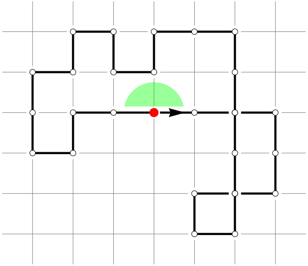
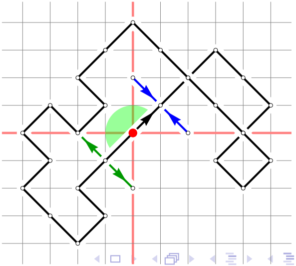
- Not quite covered by main result since walks do not avoid $(0,0)$. However, a combinatorial trick (exercise!) shows

$$F^{(\alpha)}(t) = 4\sum_{m,l,p=1}^{\infty} (-1)^{l+p+m+1} m\, W_{2l,2p}^{(|\alpha|+m\pi/2)}(t)$$





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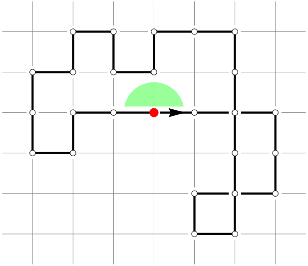
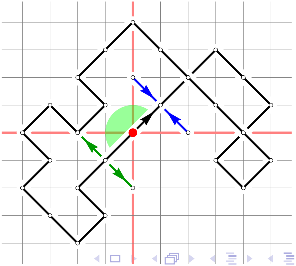
$$= \frac{2\pi}{K(k)}\sum_{n=1}^{\infty}\frac{q_k^n(1-q_k^n)^2}{1-q_k^{4n}} q_k^{2n|\alpha|/\pi}$$



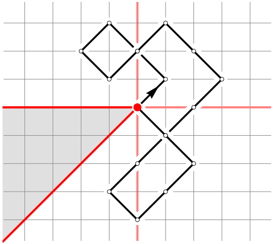


Excursions in cones

- For $I = (\beta_-, \beta_+)$, $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$, $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$, a reflection principle shows

$$F^{(\alpha,I)}(t) := \sum_{w\in\mathcal{E}} t^{|w|}\mathbf{1}_{\{w_1=(1,1),\, \theta^w=\alpha,\, \theta_i^w\in I \text{ for all } i\}}$$

$$= \frac{1}{4}\sum_{n\in\mathbb{Z}}\left(F^{(\alpha+n\delta)}(t) - F^{(2\beta_+-\alpha+n\delta)}(t)\right), \quad \delta := 2(\beta_+ - \beta_-)$$



$$\alpha = -\pi/2$$
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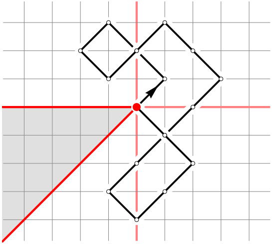
$$= \frac{\pi}{8\delta}\sum_{\sigma\in(0,\delta)\cap\frac{\pi}{2}\mathbb{Z}}\left(\cos\left(\frac{4\sigma\alpha}{\delta}\right) - \cos\left(\frac{4\sigma(2\beta_+-\alpha)}{\delta}\right)\right)F\left(t, \frac{4\sigma}{\delta}\right),$$

where

$$F(t,b) := \sum_{\alpha\in\frac{\pi}{2}\mathbb{Z}} F^{(\alpha)}(t)e^{ib\alpha} = \frac{1}{\cos\left(\frac{\pi b}{2}\right)}\left[1 - \frac{\pi\tan\left(\frac{\pi b}{4}\right)}{2K(k)}\frac{\theta_1'\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}\right]$$



$$\alpha = -\pi/2$$
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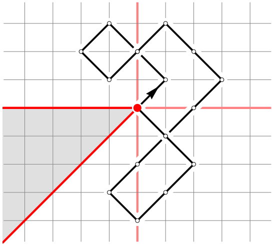
$$F^{(\alpha,I)}(t) = \frac{\pi}{8\delta} \sum_{\sigma\in(0,\delta)\cap\frac{\pi}{2}\mathbb{Z}} \left(\cos\left(\frac{4\sigma\alpha}{\delta}\right) - \cos\left(\frac{4\sigma(2\beta_+ - \alpha)}{\delta}\right)\right) F\left(t, \frac{4\sigma}{\delta}\right),$$

$$F(t,b) = \frac{1}{\cos\frac{\pi b}{2}}\left[1 - \frac{\pi\tan\frac{\pi b}{4}}{2K(k)} \frac{\theta_1'\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}\right] \qquad (b \in \mathbb{R}\setminus\mathbb{Z})$$

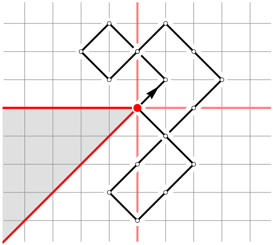


$$\begin{aligned} \alpha &= -\pi/2 \\ \beta_- &= -\pi \\ \beta_+ &= 3\pi/4 \end{aligned}$$

$$F^{(\alpha,l)}(t) = \frac{\pi}{8\delta} \sum_{\sigma \in (0,\delta)\cap \frac{\pi}{2}\mathbb{Z}} \left(\cos\left(\frac{4\sigma\alpha}{\delta}\right) - \cos\left(\frac{4\sigma(2\beta_+ - \alpha)}{\delta}\right)\right) F\left(t, \frac{4\sigma}{\delta}\right),$$

$$F(t,b) = \frac{1}{\cos\frac{\pi b}{2}} \left[1 - \frac{\pi \tan\frac{\pi b}{4}}{2K(k)} \frac{\theta_1'\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4}, \sqrt{q_k}\right)} \right] \qquad (b \in \mathbb{R} \setminus \mathbb{Z})$$

$$= \frac{1}{\cos\frac{\pi b}{2}} \left[1 - \tan\frac{\pi b}{4} \left(2Z(u,k) + \frac{\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)}{\mathrm{sn}(u,k)} \right) \right], \qquad u = \frac{K(k)b}{2}.$$



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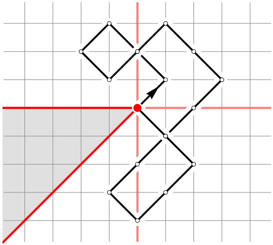
- $t\mapsto F(t,b)$ is algebraic for $b\in\mathbb{Q}\setminus\mathbb{Z}$ and transcendental for $b\in\mathbb{Z}$!



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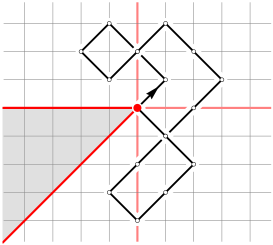
- $t \mapsto F^{(\alpha,I)}(t)$ is algebraic if $\beta_+ - \beta_- \in \frac{\pi}{2}\mathbb{Z} + \frac{\pi}{4}$

(or if $\beta_\pm \in \frac{\pi}{2}\mathbb{Z}$ and either $\beta_+ - \beta_- \in \pi\mathbb{Z} + \frac{\pi}{2}$ or $\alpha \in \pi\mathbb{Z} + \frac{\pi}{2}$ or $\beta_+ - \alpha \in \pi\mathbb{Z}$).



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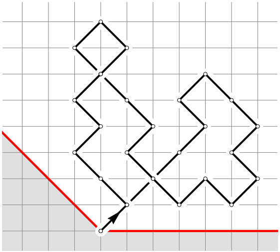
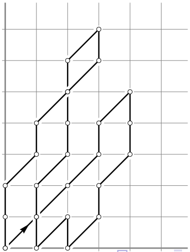
Gessel's sequence

- Special case $\alpha = 0$, $I = (-\pi/4, \pi/2)$:
$$F^{(0,I)}(t) = \frac{1}{4} F\left(t, \frac{4}{3}\right) = \frac{1}{2}\left[\frac{\sqrt{3}\pi}{2K(4t)} \frac{\theta_1'\left(\frac{\pi}{3}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi}{3}, \sqrt{q_k}\right)} - 1\right]$$





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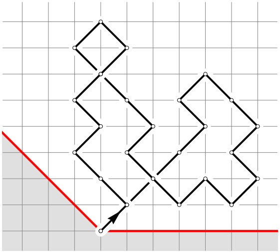
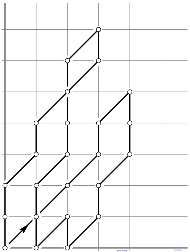
- Gessel's conjecture, proved in [Kauers, Koutschan, Zeilberger, '09], [Bostan, Kurkova, Raschel, '13], [Bousquet-Mélou, '16], [Bernardi, Bousquet-Mélou, Raschel, '17]:

$$F^{(0,I)}(t) = \sum_{n=0}^{\infty} t^{2n+2}\, 16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n} = \frac{1}{2}\left[{}_2F_1\left(-\frac{1}{2}, -\frac{1}{6}; \frac{2}{3}; (4t)^2\right) - 1\right]$$





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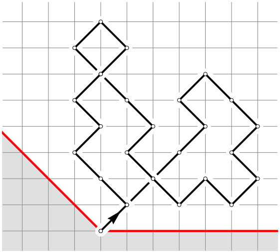
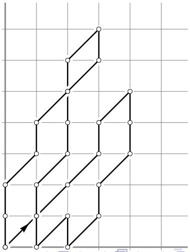
- Another proof: check that both satisfy same algebraic equation.





Background: planar map combinatorics

- Walks with small steps: $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$
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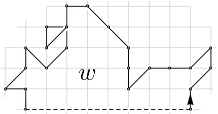

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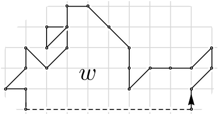

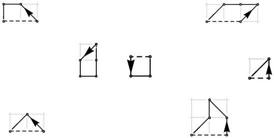

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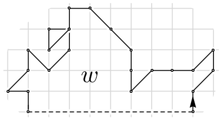

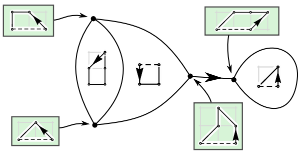

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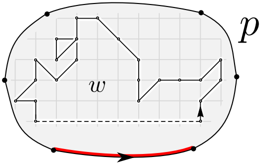

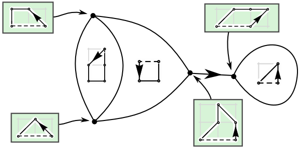

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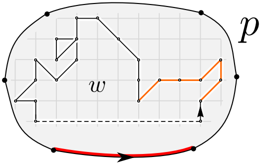

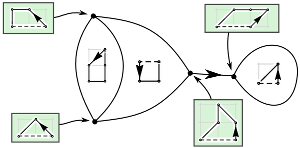

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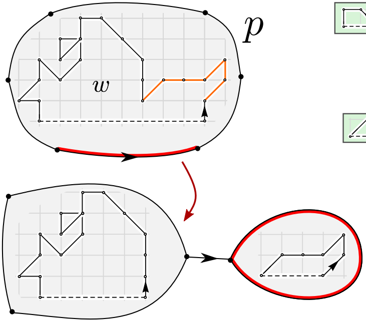

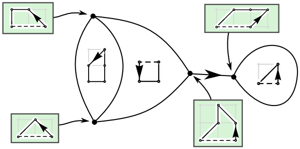

Background: planar map combinatorics

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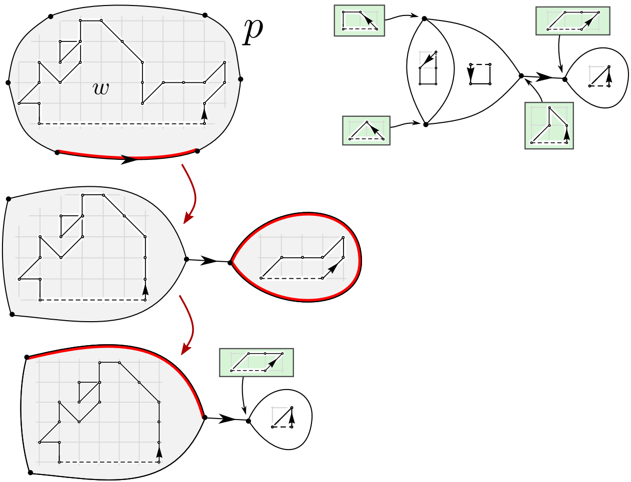

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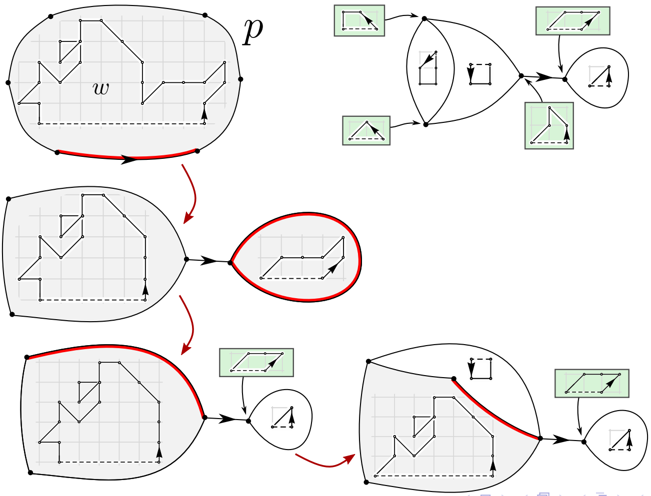

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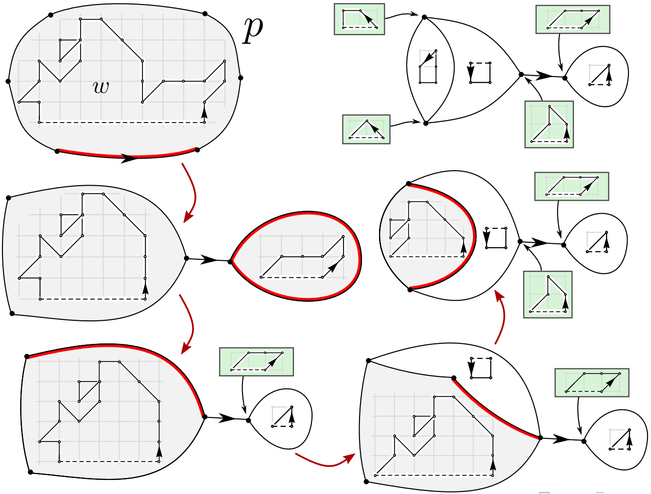

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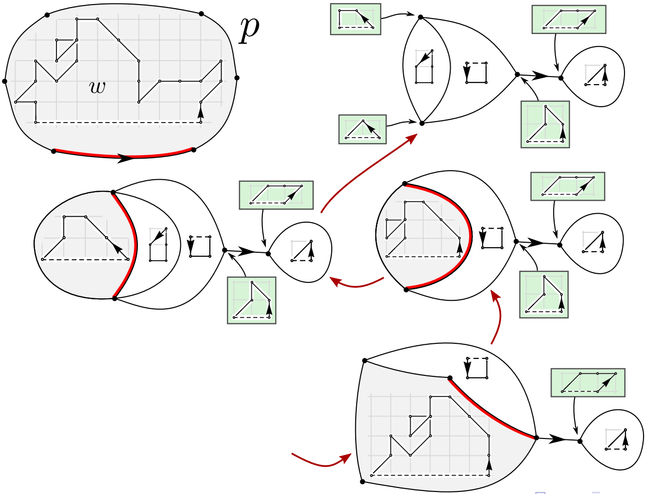

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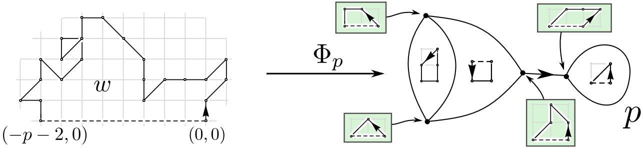

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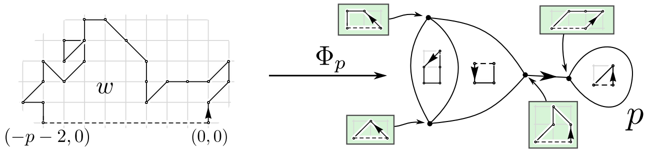

- Φ_p is a bijection with rooted planar maps of perimeter p with
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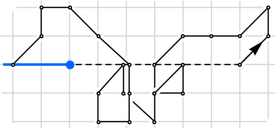

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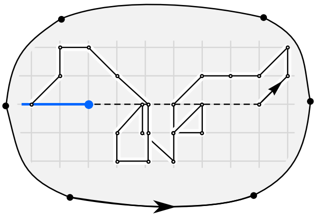

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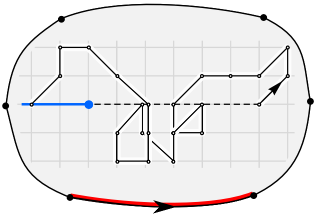

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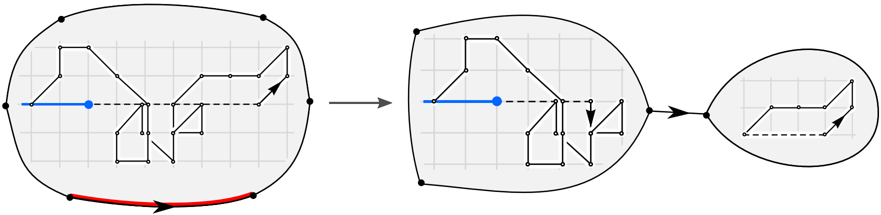

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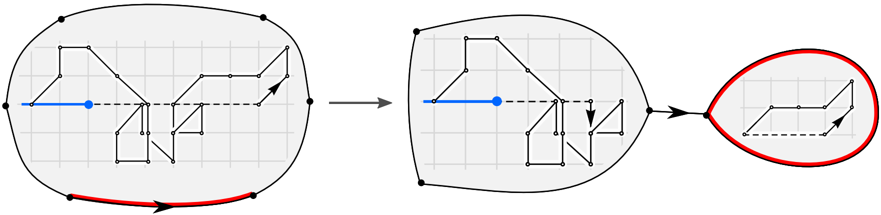

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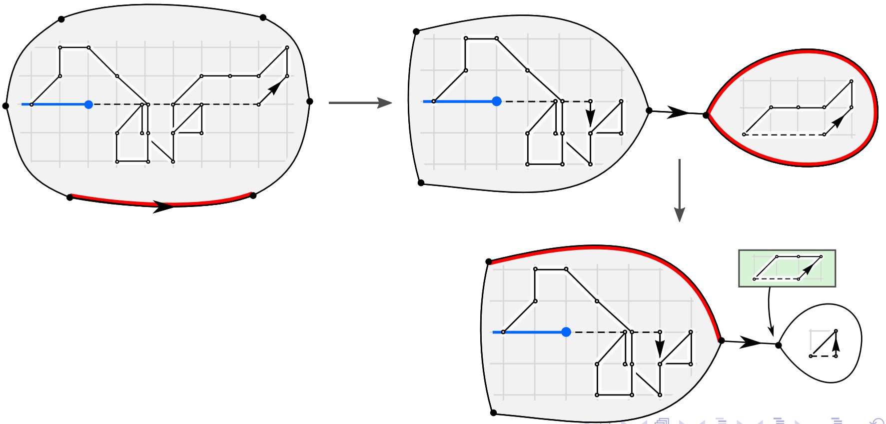

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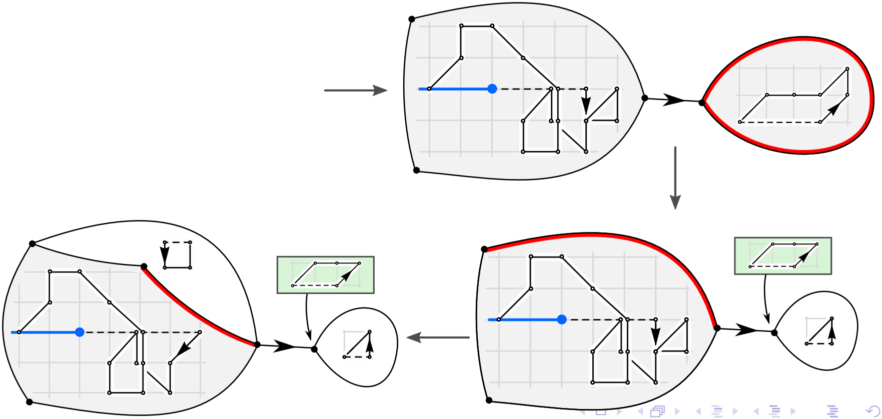

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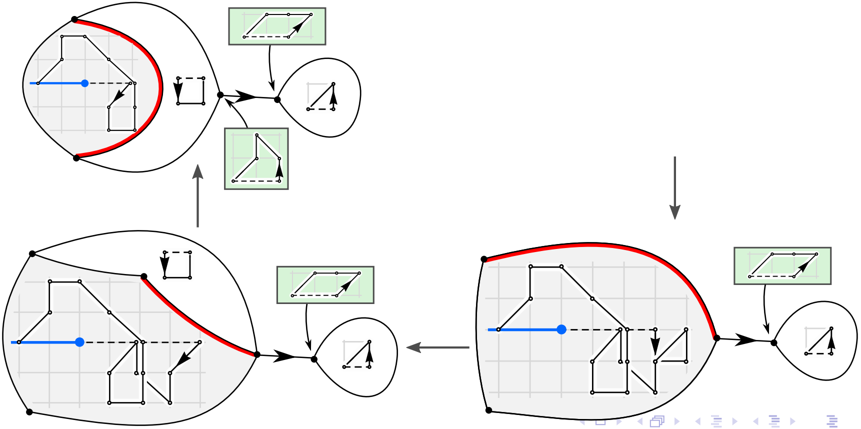

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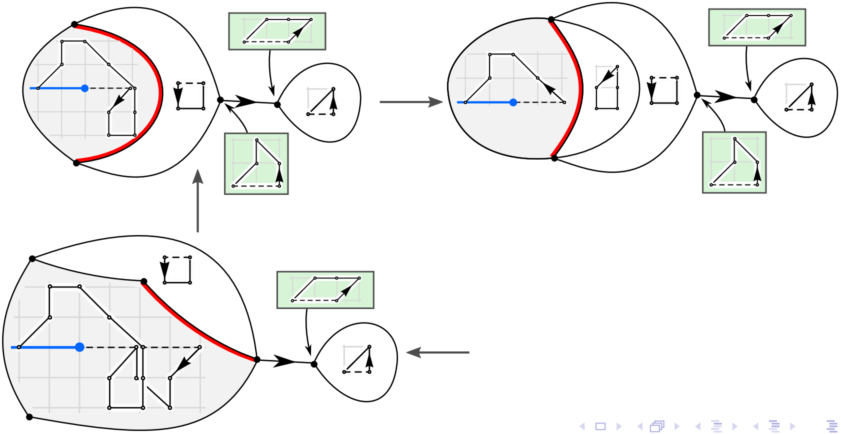

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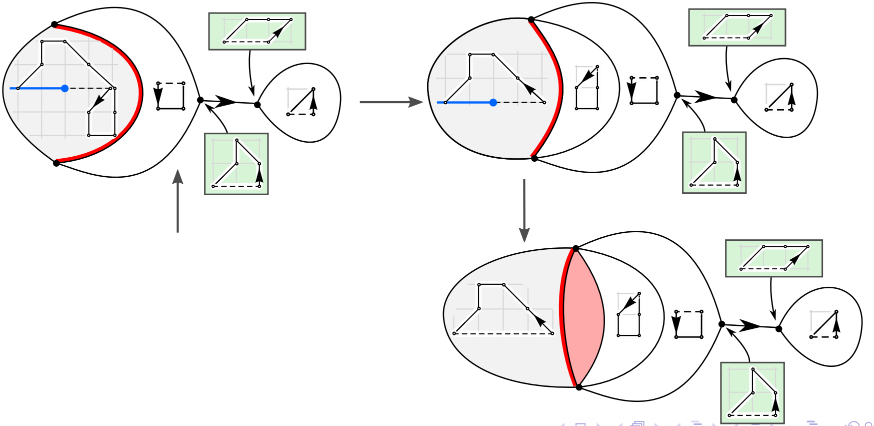

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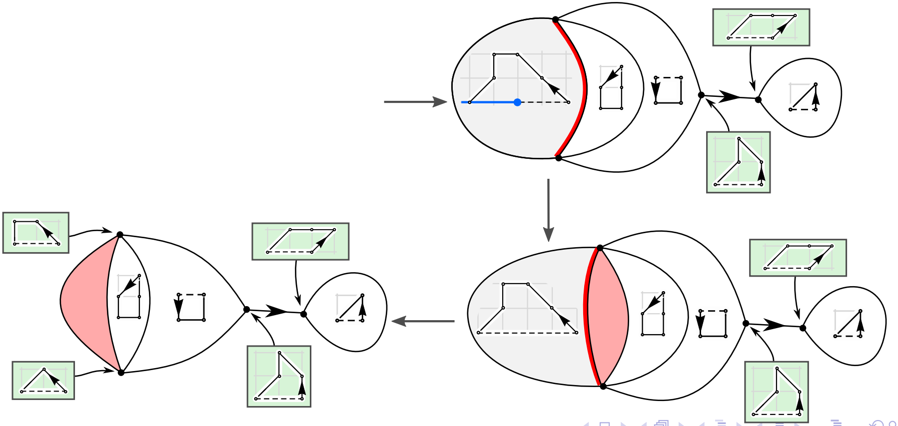

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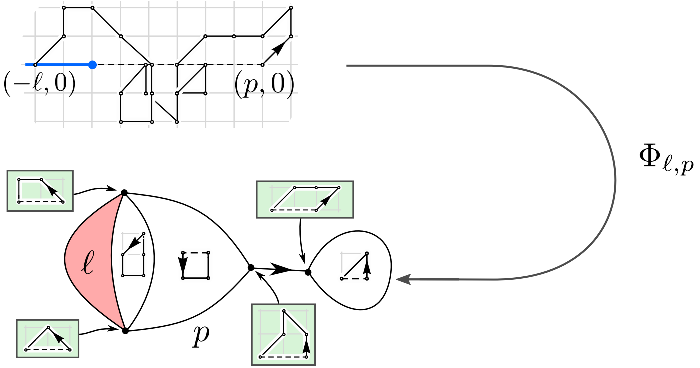
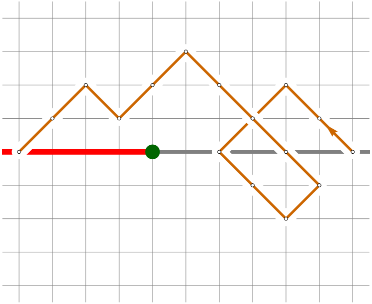
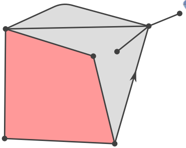


From walks to loop-decorated maps





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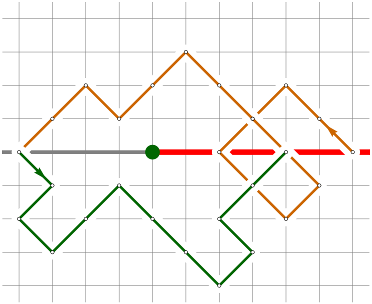

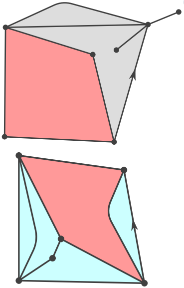

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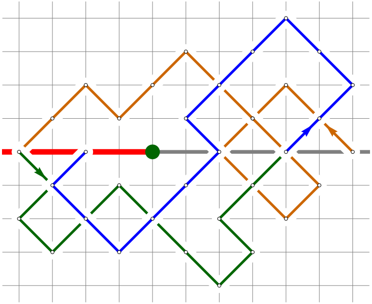

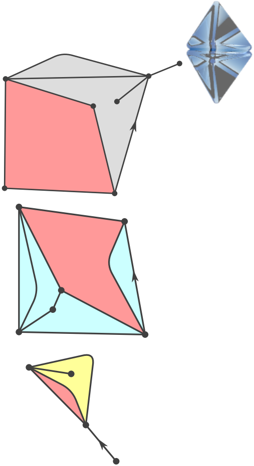

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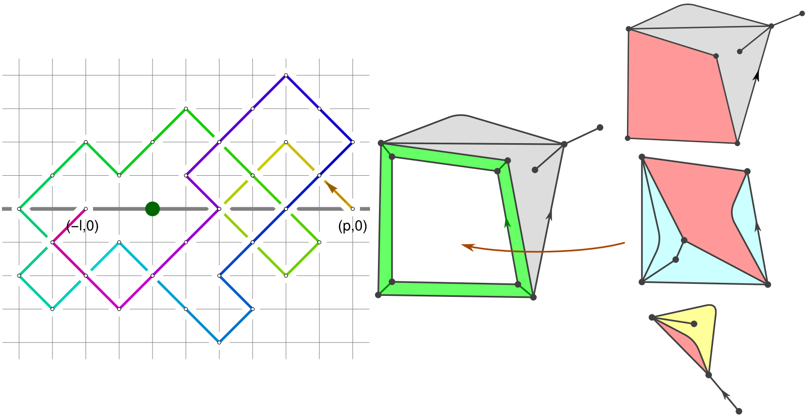

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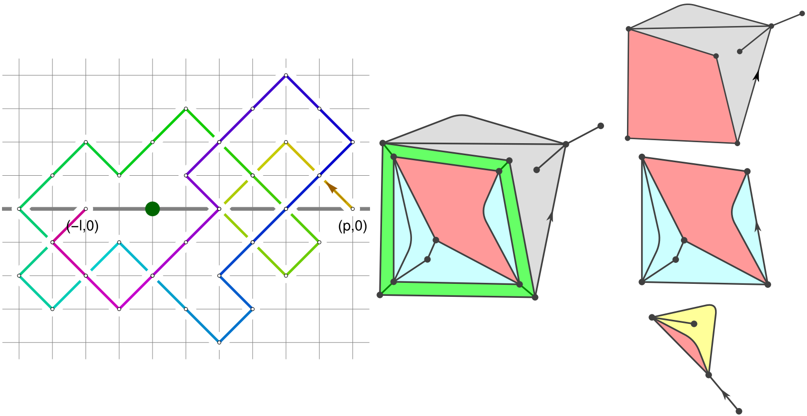

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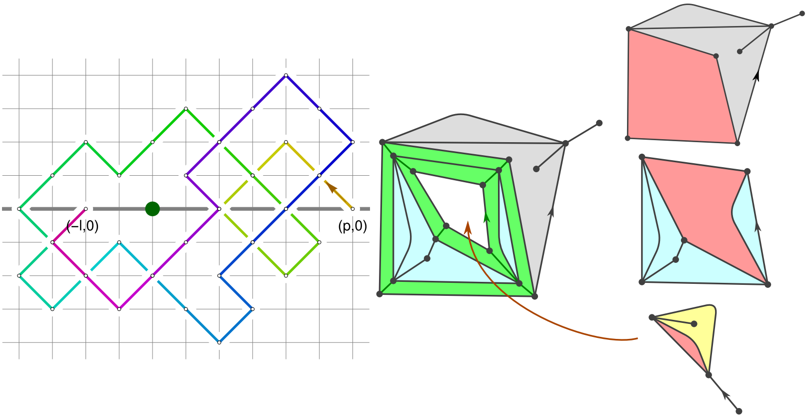

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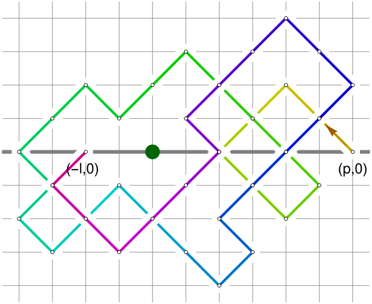

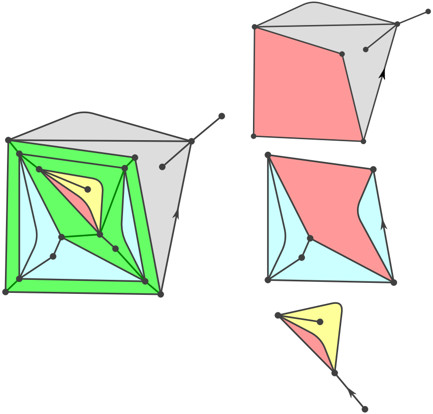

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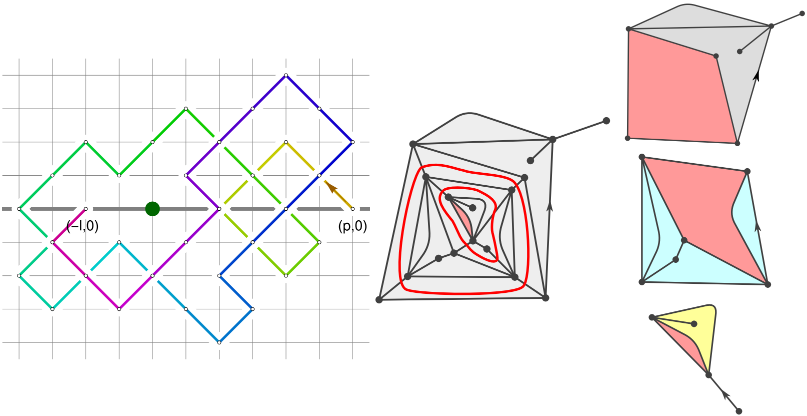

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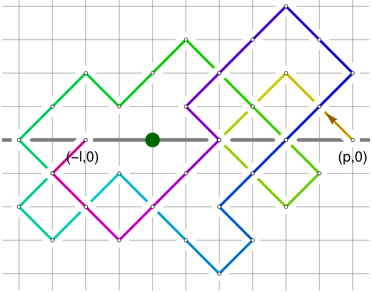

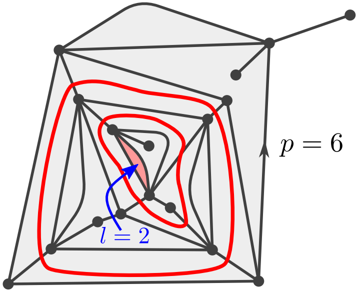

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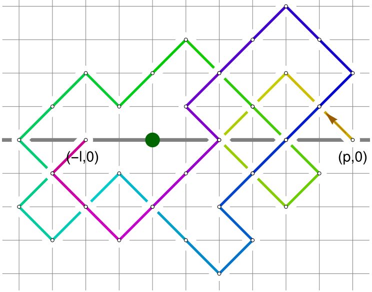

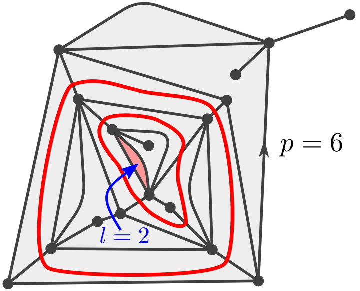

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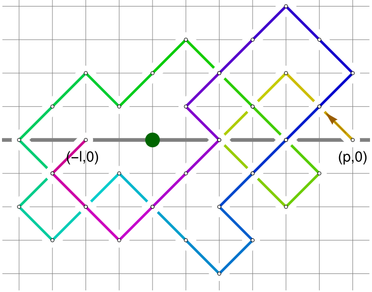

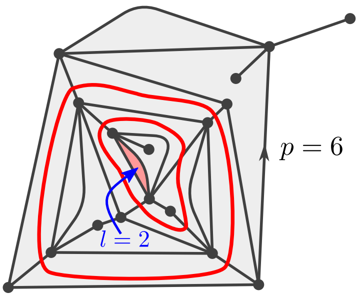

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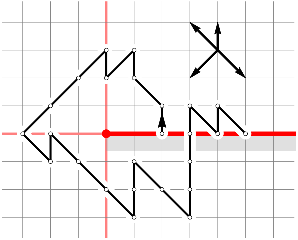
- A very similar enumeration problem ($O(n)$ loop model on random planar maps) has been solved in the mathematical physics literature.

[Borot, Bouttier, Guitter, '11] [Borot, Bouttier, Duplantier, '16]

Other walks with small steps?

- Generalization to walks with step set $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$.

$$K(x,y) = xyt\Big(\sum_{(i,j)\in\mathcal{S}} x^i y^j - \frac{1}{t}\Big)$$



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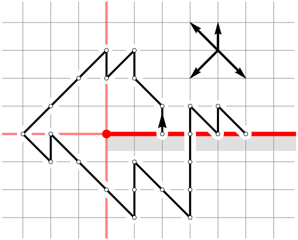
- If $\mathcal{S}$ is non-singular then $K(x,y)=0$ defines an elliptic curve, which determines a point in moduli space with corresponding nome q.

[Fayolle, Iasnogordski, Malyshev]



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[Fayolle, Iasnogordski, Malyshev]

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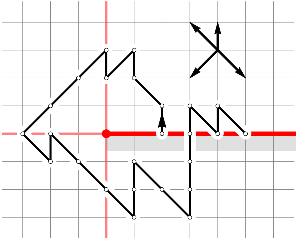

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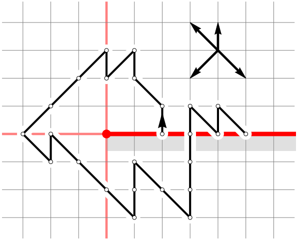
- The operator $\mathbf{X}$ on $\mathcal{D}$ defined by

$$\frac{1}{p}\langle e_l, \mathbf{X} e_p\rangle_{\mathcal{D}} = \sum_{\mathcal{S}\text{-walk } w} t^{|w|}\mathbf{1}_{\{w_0=(p,0),\,|w_{|w|}|=l,\,\theta^w=\alpha,\,\theta_i^w>0\}}$$



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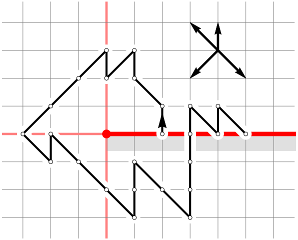
[Fayolle, Iasnogordski, Malyshev]

$$q = e^{-i\pi\frac{\omega_2}{\omega_1}}, \quad \omega_1 = i\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-d(x)}}, \quad \omega_2 = \int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{d(x)}} \quad (d(x_i) = 0)$$

- The operator $\mathbf{X}$ on $\mathcal{D}$ defined by

$$\frac{1}{p}\langle e_l, \mathbf{X} e_p\rangle_{\mathcal{D}} = \sum_{\mathcal{S}\text{-walk } w} t^{|w|}\mathbf{1}_{\{w_0=(p,0),\, |w_{|w|}|=l,\, \theta^w=\alpha,\, \theta^w_i>0\}}$$

has eigenvalues $q^{m|\alpha|/\pi}$, $m \geq 1$.



Other walks with small steps?

- Generalization to walks with step set $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$.

$$K(x,y) = xyt\Big(\sum_{(i,j)\in\mathcal{S}} x^i y^j - \frac{1}{t}\Big) = ay^2 + by + c, \quad d(x) = b^2 - 4ac$$

- If $\mathcal{S}$ is non-singular then $K(x,y)=0$ defines an elliptic curve, which determines a point in moduli space with corresponding nome q. [Fayolle, Iasnogordski, Malyshev]

$$q = e^{-i\pi\frac{\omega_2}{\omega_1}}, \quad \omega_1 = i\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-d(x)}}, \quad \omega_2 = \int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{d(x)}} \quad (d(x_i)=0)$$

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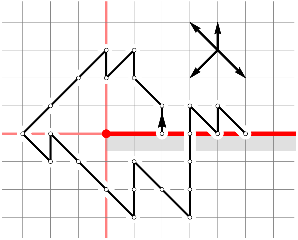

- Depending on symmetries of $\mathcal{S}$:

$$\alpha \in 2\pi\mathbb{Z}, \quad \alpha \in \pi\mathbb{Z}, \quad \text{or} \quad \alpha \in \tfrac{\pi}{2}\mathbb{Z}$$

Thanks for you attention!
Comments?