Mathieu Moonshine and the generic space of states of K3 theories

Automorphic Forms, Mock Modular Forms and String Theory

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Plan:

1. K3 theories and their generic space of states
2. A mysterious piece of evidence: Mathieu Moonshine
3. A proof: Refining the elliptic genus

[W17] *Hodge-elliptic genera and how they govern K3 theories*; arXiv:1705.09904 [hep-th]

[Taormina/W15] *A twist in the M_{24} moonshine story*, Confluentes Mathematici 7, 1 (2015), 83-113; arXiv:1303.3221 [hep-th]

[Taormina/W13] *Symmetry-surfing the moduli space of Kummer K3s*, Proceedings of Symposia in Pure Mathematics 90 (2015), 129-153; arXiv:1303.2931 [hep-th]

[Taormina/W11] *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, JHEP **1308**:152 (2013); arXiv:1107.3834 [hep-th]

1. Assumptions: Superconformal field theories

(2-dim. Euclidean) unitary SCFT (in fact, spacetime SUSY plus $N = (2,2)$ worldsheet SUSY)

Data:
- SPACE OF STATES: a unitary $\mathbb{C}$-vector space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$
  (the Neveu-Schwarz sector)
- some OBSERVABLES: commuting linear operators H, J_0; $\widetilde{H}$, $\widetilde{J}_0$ on $\mathbb{H}$
  $(2H = L_0 - 2J_0$ etc.)
- ...

such that:
- H, J_0, $\widetilde{H}$, $\widetilde{J}_0$ are self-adjoint, diagonalizable, $H \geq 0$, $\widetilde{H} \geq 0$,
  $\operatorname{spec}(H - \widetilde{H}) \cup \operatorname{spec}(J_0 - \widetilde{J}_0) \subset \mathbb{Z}$, $\operatorname{spec}(J_0) \cup \operatorname{spec}(\widetilde{J}_0) \subset \frac{1}{2}\mathbb{Z}$
- there is a well-defined $\widetilde{\text{R}}$-partition function

$$Z(\tau, z) = \operatorname{tr}_{\mathbb{H}}\left((-1)^{J_0 - \widetilde{J}_0} y^{J_0 - c/6} \overline{y}^{\widetilde{J}_0 - c/6} q^H \overline{q}^{\widetilde{H}} \right),$$

where $\tau, z \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, $q := \exp(2\pi i \tau)$, $y := \exp(2\pi i z)$,

$$Z(\tau, z) = Z(\tau + 1, z) = \left(\exp\left(\frac{\pi i z^2}{\tau} - \frac{\pi i \overline{z}^2}{\overline{\tau}} \right) \right)^{c/3} Z\left(-\tfrac{1}{\tau}, \tfrac{z}{\tau}\right).$$

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$c \in \mathbb{R}$: CENTRAL CHARGE; $\chi(\mathbb{H}) := Z(\tau, 0)$ WITTEN INDEX.

1. K3 theories

Definition

A K3 THEORY is a superconformal field theory as above at $c = 6$ with Witten index $\chi(\mathbb{H}) = 24$.

Result:

[Seiberg88,Cecotti90,Aspinwall/Morrison94,Nahm/W01]

There is an 80-dimensional moduli space $\mathcal{M}_{\text{K3}}$ of K3 theories,

$$\mathcal{M}_{\text{K3}} = {}_{\mathrm{O}^+(4,20;\mathbb{Z})}\backslash^{\mathrm{O}(4,20;\mathbb{R})}\!\big/_{\mathrm{O}(4)\times\mathrm{O}(20)}.$$

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Let $\mathbb{H}_0$: maximal $\mathbb{C}$-vector space such that, as a representation of $\left\{H,\, J_0,\, \widetilde{J}_0\right\}$, for every K3 theory, $\mathbb{H}_0 \hookrightarrow \ker(\widetilde{H})$

– the GENERIC SPACE OF STATES of K3 theories.

2. Towards Mathieu Moonshine: The elliptic genus

CFT ELLIPTIC GENUS of an SCFT at central charge c as above:

$$\mathcal{E}_{\mathrm{CFT}}(\mathbb{H};\tau,z) := \mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-c/6} q^{H} \overline{q}^{\widetilde{H}}\right).$$

Properties:

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For K3 theories,

$$\mathcal{E}_{\mathrm{CFT}}(\mathbb{H};\tau,z) = 8\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2.$$

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Mathieu Moonshine **[Eguchi/Ooguri/Tachikawa10, Gannon12]**
The elliptic genus of K3 theories agrees
with the character of a particular $N=4$ supermodule of M_{24}.

2. Towards Mathieu Moonshine: Symmetry surfing

Proposals for K3 theories:

- if the generic chiral algebra of theories in $\mathcal{M}_{K3}$ is the $N=4$ superconformal algebra at $c=6$, then $\mathbb{H}_0$ is known **[Ooguri89,W00]**
- symmetry surfing **[Taormina/W11,13,15]**:
  $\mathbb{H}_0$ carries an M_{24} action which combines the actions of the finite symplectic symmetry groups of all K3 surfaces;
  symmetry surfing is confirmed for $\mathbb{Z}_2$ orbifold conformal field theories, where the combined action of all symmetry groups yields an action of a maximal subgroup of M_{24} **[Taormina/W15, Gaberdiel/Keller/Paul16]**
- $\mathbb{H}_0$ seems to carry deeper structure which all K3 theories share – Mathieu Moonshine serves as mysterious evidence in favour of this idea

2. The complex elliptic genus of Calabi-Yau D-folds

Let M denote a compact Calabi-Yau D-fold, $\mathcal{T} := T^{1,0}M$.

Expectation: (true for K3 theories)

For non-linear sigma models on M,

$$\mathcal{E}_{\mathrm{CFT}}(\mathbb{H};\tau,z) = \mathcal{E}(M;\tau,z),$$ the complex elliptic genus of M.

Definition

- For holomorphic vector bundles $T_{\ell,m} \to M$, $\ell, m \in \mathbb{Z}$, $\mu \in \mathbb{Q}$:

  HOLOM. EULER CHAR. of $E_{q,-y} := y^\mu \bigoplus_{\ell,m} q^\ell (-y)^m T_{\ell,m}$:

  $$\chi(E_{q,-y}) := y^\mu \sum_{\ell,m} q^\ell (-y)^m \chi(T_{\ell,m}).$$

- $$\mathbb{E}_{q,-y} := y^{-\frac{D}{2}} \Lambda_{-y}\mathcal{T}^* \otimes \bigotimes_{n=1}^{\infty} \left[\Lambda_{-yq^n}\mathcal{T}^* \otimes \Lambda_{-y^{-1}q^n}\mathcal{T} \otimes S_{q^n}\mathcal{T}^* \otimes S_{q^n}\mathcal{T} \right],$$

  where for any bundle $E \to M$, $\Lambda_x E := \bigoplus_{m=0}^{\infty} x^m \Lambda^m E$, $S_x E := \bigoplus_{m=0}^{\infty} x^m S^m E$

  $$= y^{-\frac{D}{2}} \bigoplus_{\ell,m} q^\ell (-y)^m \mathcal{T}_{\ell,m},$$

  COMPLEX ELLIPTIC GENUS of M: $\mathcal{E}(M;\tau,z) := \chi(\mathbb{E}_{q,-y})$.

3. Refinements of the elliptic genus

M, $\mathbb{H}$, $\mathbb{E}_{q,-y} = y^{-\frac{D}{2}} \bigoplus_{\ell,m} q^{\ell}(-y)^m \mathcal{T}_{\ell,m}$ as before, $\nu \in \mathbb{C}$, $u := \exp(2\pi i \nu)$.

Elliptic genus

$$\mathcal{E}_{\mathrm{CFT}}(\mathbb{H};\tau,z) = \mathrm{tr}_{\ker(\widetilde{H})}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-c/6} q^{H}\right).$$

$$\mathcal{E}(M;\tau,z) = y^{-\frac{D}{2}} \sum_j (-1)^j \sum_{\ell,m} q^{\ell}(-y)^m \dim H^j(M,\mathcal{T}_{\ell,m}).$$

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HODGE ELLIPTIC GENERA

- **[Kachru/Tripathy16]**

$$\mathcal{E}^{\mathrm{HEG}}_{\mathrm{CFT}}(\mathbb{H}; \tau, z, \nu) := \mathrm{tr}_{\ker(\widetilde{H})}\left((-1)^{J_0 - \widetilde{J}_0} y^{J_0 - c/6} u^{\widetilde{J}_0 - c/6} q^H \right).$$

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3. The refined elliptic genera for K3 surfaces M

Results – if M is a K3 surface and $\mathbb{H}$ belongs to a K3 theory:

- **[Kachru/Tripathy16]** (using the Bochner principle):
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  formula for $\mathcal{E}^{\mathrm{HEG,ch}}(M;\tau,z,\nu)$ – it is elliptic, but not modular,
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Concluding remark

The space $\mathbb{H}_0$ of generic states of K3 theories is modelled by the cohomology of the chiral de Rham complex. As a representation of the $N=4$ superconformal algebra, it agrees with the Mathieu Moonshine Module, supporting the idea of symmetry surfing.

Thank you
for your attention!