Rings of Arithmetic Differential Operators on Tubes

Richard Crew

The University of Florida

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References:

[DModsI] P. Berthelot, *$\mathcal{D}$-modules arithmetique I*, Ann. Sc. Ec. Norm. Sup. 4^e ser. **29** (1996) 185–272.

[DModsII] P. Berthelot, *$\mathcal{D}$-modules arithmetique II. Descente par Frobenius*, Mém. SMF **81**, 2002.

[UnitDisk] R. Crew, *Arithmetic $\mathcal{D}$-modules on the unit disk*, Comp. Math. **48** no. 1 (2012) 227–268.

[DModsAdic] R. Crew, *Arithmetic $\mathcal{D}$-modules on adic formal schemes* `arXiv:1701.01324` .

Motivation

In [DModsI] and [DModsII] Berthelot constructed a ring of arithmetic differential operators $\mathcal{D}^{\dagger}_{\mathcal{X}/\mathcal{S}\mathbb{Q}}$ relative to a smooth morphism $f : \mathcal{X} \to \mathcal{S}$ of p-adic formal schemes. A considerable amount of work by others has shown that a suitable category of left $\mathcal{D}^{\dagger}_{\mathcal{X}/\mathcal{S}\mathbb{Q}}$-modules has most of the desired properties of a p-adic coefficient system, at least when $\mathcal{S} = \mathrm{Spf}(\mathcal{V})$ where $\mathcal{V}$ is a complete mixed discrete valuation ring of mixed characteristic.

For various reasons one would want to extend this theory to the case where $\mathcal{X} \to \mathcal{S}$ is formally smooth but not necessarily of finite type, or even adic. For example the case $\mathcal{S} = \mathrm{Spf}(\mathcal{V})$ and $\mathcal{X} = \mathrm{Spf}(\mathcal{V}[[t]])$, for which the topology of $\mathcal{V}[[t]]$ is the (p, t)-adic topology was studied in [UnitDisk]. One would like to consider cases like $\mathcal{X} = \mathrm{Spf}(\mathcal{V}[[t_1, \ldots, t_n]])$, or more generally the case where $\mathcal{Y} \to \mathcal{S}$ is of finite type and $\mathcal{X}$ is the completion of $\mathcal{Y}$ along a closed subscheme.

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In [DModsAdic] we constructed such a theory for a fairly general class of morphisms, including the cases just mentioned. I will start by describing this construction, and then sketch how it may be used to construct a generalization of the category of convergent isocrystals. Similar ideas may be used for the category of overconvergent isocrystals.

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A morphism $f : \mathcal{X} \to \mathcal{S}$ of locally noetherian schemes is *universally noetherian* if for any morphism $\mathcal{Y} \to \mathcal{S}$ with $\mathcal{Y}$ noetherian, $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is noetherian.

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The class of universally noetherian morphisms is closed under composition, base change by a locally noetherian formal scheme, and fiber products. Furthermore:

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If $\mathcal{X} \to \mathcal{S}$ is universally noetherian and $\mathcal{Y} \to \mathcal{S}$ is any morphism with $\mathcal{Y}$ locally noetherian, then any morphism $\mathcal{X} \to \mathcal{Y}$ is universally noetherian.

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We could say that a morphism $f : X \to S$ of locally noetherian schemes is universally noetherian if $X \times_S Y$ is noetherian for any morphism $Y \to S$ with Y noetherian. However any scheme is a formal scheme in the discrete topology, and the two definitions coincide in the case of schemes.

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In fact if $f : \mathcal{X} \to \mathcal{S}$ is a morphism of locally noetherian schemes and $f_0 : X \to S$ is the corresponding morphism of reduced closed subschemes, then f is universally noetherian if and only if f_0 is.

If A, B are adic rings and $A \to B$ is a continuous we say that B is a universally noetherian A-algebra if $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is universally noetherian.

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- If $Y \subset \mathcal{X}$ is a closed subscheme and $\hat{\mathcal{X}}_Y$ is the completion of $\mathcal{X}$ along Y, the canonical morphism $\hat{\mathcal{X}}_Y \to \mathcal{X}$ is universally noetherian.

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- More generally if $f : \mathcal{X} \to \mathcal{S}$ is *formally of finite type* (i.e. the morphism $f_0 : X \to S$ of reduced closed subschemes is of finite type) then f is universally noetherian. This includes all the examples mentioned at the beginning.

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- If (A, I) is an adic noetherian ring $S \subseteq A$ is a multiplicative subset, then the completion of $S^{-1}A$ with respect to S^{-1} is a universally noetherian A-algebra.

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Suppose $f : \mathcal{X} \to \mathcal{S}$ is universally noetherian, $x \in \mathcal{X}$ and $s = f(x)$. The field extension $\kappa(x)/\kappa(s)$ is finitely generated.

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In fact if $\mathcal{X} \to \mathcal{S}$ is universally noetherian, so is the base change $\mathcal{X} \times_{\mathcal{S}} \kappa(s) \to \kappa(s)$ and the closed immersion $\kappa(x) \to \mathcal{X} \times_{\mathcal{S}} \kappa(s)$.

Differential Invariants and Smoothness

Suppose R is an adic noetherian ring and $R \to A$ is a universally noetherian R-algebra. Then $A \hat{\otimes}_R A$ is noetherian and the kernel $\hat{I}$ of $A \hat{\otimes}_R A \to A$ is finitely generated,

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$$\hat{\Omega}^1_{A/R} = \hat{I}/\hat{I}^2.$$

In fact one can show in this situation that $\hat{\Omega}^1_{A/R}$ is the completion of the usual module of 1-forms $\Omega^1_{A/R}$ for the topology arising from its A-module structure.

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and if $A \to B$ is surjective with kernel K there is a canonical short exact sequence

$$K/K^2 \to B \underset{A}{\hat{\otimes}} \hat{\Omega}^1_{A/R} \to \hat{\Omega}^1_{B/R} \to 0.$$

One can show that $\hat{\Omega}^1_{A/R}$ is generated by finitely elements of the form $\mathrm{d}x = 1 \mathbin{\hat{\otimes}} x + x \mathbin{\hat{\otimes}} 1 + \hat{I}^2$.

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For example if $\mathfrak{j} \subset R$ and $A = R\{T_1, \dots, T_d\}$ is the $\mathfrak{j}$-adic completion of the polynomial ring then $\hat{\Omega}^1_{A/R}$ is free on the $\mathrm{d}T_1, \dots \mathrm{d}T_d$.

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By the same token the rings of k-fold principal parts of order r are defined by

$$\begin{aligned} A(k) &= A \mathbin{\hat{\otimes}_R} A \mathbin{\hat{\otimes}_R} \cdots \mathbin{\hat{\otimes}_R} A \quad (k+1 \text{ times}) \\ I(k) &= \mathrm{Ker}(A(k) \to A) \\ \hat{P}^r_{A/R}(k) &= A(k)/I(k)^{r+1}. \end{aligned}$$

These definitions globalize without any problem to any universally noetherian *separated* morphism $f : \mathcal{X} \to \mathcal{S}$. We obtain a coherent sheaf $\Omega^1_{\mathcal{X}/\mathcal{S}}$ of $\mathcal{O}_{\mathcal{X}}$-modules (we adopt the convention that global constructs do not take a "hat" since there is no meaning to the corresponding uncompleted construction).

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First-order deformation theory works in the expected way. So does the construction of the usual (i.e. Grothendieck) ring of differential operators as the direct limit of the $Hom_{\mathcal{O}_{\mathcal{X}}}(\mathcal{P}^r_{\mathcal{X}/\mathcal{S}}(1), \mathcal{O}_{\mathcal{X}})$ with an appropriate composition law.

We say that a morphism $f : \mathcal{X} \to \mathcal{S}$ of locally noetherian adic formal schemes is *quasi-smooth* (resp. *quasi-étale*, *quasi-unramified*) if it is separated, locally noetherian and formally smooth (resp. formally étale, formally unramified).

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A quasi-smooth morphism is flat. This depends on a difficult theorem of Grothendieck on the flatness of a formally smooth local homomorphism of local rings. In particular a quasi-étale morphism is flat and quasi-unramified; I do not know if the converse is true. A quasi-smooth morphism is *not* necessarily open. For example the canonical morphism $\hat{\mathcal{X}}_Y \to \mathcal{X}$ is quasi-étale but not open.

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Theorem

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1. *$\Omega^1_{\mathcal{X}/\mathcal{S}}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of finite type, and*
2. *the natural morphism*

$$Sym^n_{\mathcal{O}_{\mathcal{X}}}(\Omega^1_{\mathcal{X}/\mathcal{S}}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$$

is an isomorphism for all $n \geq 0$.

If $\mathcal{X} \to \mathcal{S}$ is quasi-smooth then locally on $\mathcal{X}$ we may define the *formal dimension* of $\mathcal{X}/\mathcal{S}$ to be the rank of $\Omega^1_{\mathcal{X}/\mathcal{S}}$. If this is globally defined we denote this integer by $\mathrm{fdim}(\mathcal{X}/\mathcal{S})$.

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As a corollary of the last theorem we find that if $\mathcal{X} \to \mathcal{S}$ is quasi-smooth then the k-fold diagonal $\mathcal{X} \to \mathcal{X}_{\mathcal{S}}(k)$ is a regular immersion. We also get the following structure theorem for quasi-smooth morphisms:

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Suppose $f : \mathcal{X} \to \mathcal{S}$ is quasi-smooth and $d = \mathit{fdim}(\mathcal{X}/\mathcal{S})$. Locally on $\mathcal{X}$ there is a factorisation $f = p \circ g$ where $g : \mathcal{X} \to \mathbb{A}^d_{\mathcal{S}}$ is quasi-étale and $p : \mathbb{A}^d_{\mathcal{S}} \to \mathcal{S}$ is the canonical projection.

If $\mathcal{X} \to \mathcal{S}$ is quasi-smooth then locally on $\mathcal{X}$ we may define the *formal dimension* of $\mathcal{X}/\mathcal{S}$ to be the rank of $\Omega^1_{\mathcal{X}/\mathcal{S}}$. If this is globally defined we denote this integer by $\mathrm{fdim}(\mathcal{X}/\mathcal{S})$.

As a corollary of the last theorem we find that if $\mathcal{X} \to \mathcal{S}$ is quasi-smooth then the k-fold diagonal $\mathcal{X} \to \mathcal{X}_{\mathcal{S}}(k)$ is a regular immersion. We also get the following structure theorem for quasi-smooth morphisms:

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In fact this factorisation is defined wherever there are "local coordinates" relative to f, i.e. local sections $x_1, \ldots, x_d$ of $\mathcal{O}_{\mathcal{X}}$ such that $\mathrm{d}x_1, \ldots, \mathrm{d}x_d$ is a free basis of $\Omega^1_{\mathcal{X}/\mathcal{S}}$.

Since a finitely generated extension is formally étale if and only if it is finite and separable, we get:

Lemma

Suppose $f : \mathcal{X} \to \mathcal{S}$ is quasi-unramified, $x \in \mathcal{X}$ and $s = f(s)$. The field extension $\kappa(x)/\kappa(s)$ is finite and separable.

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This is the essential step in proving the following extension of a well-known criterion for a morphism to be unramified; it is a necessary ingredient in the proof of the fibration theorem to be proven later:

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- *For all $y \in \mathcal{S}$, the formal $\kappa(s)$-scheme $f^{-1}(s)$ is a disjoint union of a finite number of $\kappa(s)$-schemes, all of the form $\mathrm{Spf}(L)$ with $L/\kappa(s)$ finite and separable.*

Suppose now $f : \mathcal{X} \to \mathcal{S}$ is a morphism of locally noetherian adic formal schemes and $\mathcal{X}$, $\mathcal{S}$ have characterisitic $p > 0$. Let $q = p^f$ and denote by $F_{\mathcal{X}/\mathcal{S}} : \mathcal{X} \to \mathcal{X}^{(q)}$ the relative qth power Frobenius. If f is quasi-smooth, $F_{\mathcal{X}/\mathcal{S}}$ is flat.

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This can be proven by reduction to the smooth case, using the structure theorem for quasi-smooth morphisms. The problem in showing that $F_{\mathcal{X}/\mathcal{S}}$ is finite for any quasi-smooth morphism is that a quasi-étale morphism is not necessarily open.

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Here $\mathfrak{a}^{(p^m)} \subseteq \mathfrak{a}$ is the ideal generated by the p^mth powers of elements of $\mathfrak{a}$. We also speak of $(\mathfrak{a}, \mathfrak{b}, \gamma)$ as being an "m-PD-structure on $\mathfrak{a}$."

Given an m-PD-structure $(\mathfrak{a}, \mathfrak{b}, \gamma)$ the *partial divided powers* of an $x \in \mathfrak{a}$ are defined by

$$x^{\{k\}_{(m)}} = x^r \gamma_q(x^{p^m}) \qquad k = p^m q + r,\ 0 \leq r < p^m.$$

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Any ring R with m-PD-structure has a canonical *m-PD-adic filtration* $\mathfrak{a}^{\{k\}} \subset A$ analogous to the $\mathfrak{a}$-adic filtration, but having special compatibilities with the level m divided powers. In particular the m-PD-structure descends to $R/\mathfrak{a}^{\{k\}}$ and the canonical homomorphism $R \to R/\mathfrak{a}^{\{k\}}$ is compatible with the m-PD-structures.

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Suppose A is an R-algebra and $(\mathfrak{a}, \mathfrak{b}, \gamma)$ is an m-PD-structure on R. The central construction of the theory is the construction of the *m-PD-envelope* of an ideal $I \subset A$. This is an A-algebra $P_{(m),\alpha}(I)$ equipped with an m-PD-structure $(I^\bullet, I^\circ, [\ ])$ that is universal for A-algebras with an m-PD-structure compatible with $(\mathfrak{a}, \mathfrak{b}, \gamma)$.

We denote by $P^r_{(m),\alpha}(I)$ the quotient of $P_{(m),\alpha}(I)$ by that $(r+1)$-st step of the m-PD-adic filtration. Its structure is known if I is generated by a regular sequence $x_1, \ldots, x_d$, A/I is flat over R, and the quotient $A \to A/I$ has a section. Then $P^r_{(m),\alpha}(I)$ is a free A/I-module on the $x^{\{K\}_{(m)}}$ for $0 \leq |K| < r$.

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Suppose now A is a quasi-smooth R-algebra. The diagonal ideal $I \subset A \hat{\otimes}_R A$ is regular, and we may apply (at least locally) the preceding construction to the diagonal ideal $I \subset A \hat{\otimes}_R A$. The result is a collection of m-PD-rings which we will denote by $P^r_{A/R,(m)}$.

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If $x_1, \ldots, x_d$ are local coordinates we set $\xi_i = 1 \mathbin{\hat{\otimes}} x_i - x_i \mathbin{\hat{\otimes}} 1$ as before. The $\xi^{\{K\}_{(m)}}$ for $0 \leq |K| \leq r$ form a basis of $P^r_{A/R,(m)}$ for either of the A-module structures of $P^r_{A/R,(m)}$ coming from the corresponding ones of $A \mathbin{\hat{\otimes}}_R A$.

The A-module of *arithmetic differential operators of level m and order $\leq r$* is defined by analogy with the case of ordinary operators:

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If $x_1, \ldots, x_d$ are local coordinates we denote by $\{\partial^{\langle K\rangle_{(m)}}\}_{|K|\leq r}$ the basis of $\mathrm{Diff}^r_{A/R,(m)}$ dual to the basis $\{\xi^{\{K\}_{(m)}}\}_{|K|\leq r}$ of $P^r_{A/R,(m)}$. Then

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Since the formation of $P^r_{A/R,(m)}$ commutes with flat base change, this construction sheafifies easily. Thus to any quasi-smooth $\mathcal{X} \to \mathcal{S}$ and m-PD-structure on $\mathcal{S}$ we may associate a sheaf of rings $\mathcal{D}^{(m)}_{\mathcal{X}/\mathcal{S}}$; it is an inductive limit of coherent locally free $\mathcal{O}_{\mathcal{X}}$-modules.

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If $I \subset A$ is generated by a sequence of elements that is centralising in D then I is bilateralising and ID is centralising. Sums, products and powers of bilateralising ideals are bilateralising.

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In fact if $J = (p, f_1, \dots, f_r)$ is an ideal of definition then so is $J' = (p, f_1^{p^{m+1}}, \dots, f_r^{p^{m+1}})$. On the other hand Berthelot showed that any $f^{p^{m+1}}$ is central in $D^{(m)}_{A/R}/pD^{(m)}_{A/R}$.

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It follows that the m-bilateralising ideals of definition are cofinal in the set of all ideals of definition.

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Note that if $(p, f_1, \ldots, f_r) = J = (p, g_1, \ldots, g_s)$ then $(p, f_1^{p^{m+1}}, \ldots, f_r^{p^{m+1}}) = (p, g_1^{p^{m+1}}, \ldots, g_s^{p^{m+1}})$ as well, so this construction globalizes.

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If $J \subset A$ is open and bilateralising we define

$$D^{(m)}_{A/R,J} = D^{(m)}_{A/R}/JD^{(m)}_{A/R}$$

and give it the induced ring structure.

Suppose now $\mathcal{X} \to \mathcal{S}$ is quasi-smooth and $\mathcal{S}$ is given an m-PD-structure. The argument of the last lemma shows that $\mathcal{O}_{\mathcal{X}}$ has a fundamental system of ideals of definition whose sections on any open affine are m-bilateralising. For any such ideal $J \subset \mathcal{O}_{\mathcal{X}}$ the $D^{(m)}_{A/R,J}$ patch together to yield a sheaf of rings $\mathcal{D}^{(m)}_{\mathcal{X}/\mathcal{S},J}$.

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As a sheaf of rings it is left and right coherent. Furthermore if we denote by $X_J \subset \mathcal{X}$ the closed subscheme defined by J then $\mathcal{D}^{(m)}_{\mathcal{X}/\mathcal{S},J}$ is a quasi-coherent $\mathcal{O}_{X_J}$-algebra.

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We define

$$\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}} = \varprojlim_{J} \mathcal{D}^{(m)}_{\mathcal{X}/\mathcal{S},J}$$

where the inverse limit is over m-bilaterising ideals of definition of $\mathcal{O}_{\mathcal{X}}$.

With this definition it is not hard to show that $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$ is a coherent sheaf of rings, and establish versions of "theorem A" and "theorem B" for left or right coherent $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-modules.

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The description of coherent left $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-modules by means of m-PD-stratifications that is familiar in the smooth case extends to the present case as well. The same holds for the description of right $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-modules by means of costratifications.

Frobenius Descent

Let $q = p^s$. Since $p \in \mathfrak{a}$ the formal schemes $\mathcal{S}_0 = V(\mathfrak{a}\mathcal{O}_{\mathcal{S}})$ and $\mathcal{X}_0 = V(\mathfrak{a}\mathcal{O}_{\mathcal{X}})$ have characteristic p. Suppose $F : \mathcal{X} \to \mathcal{X}'$ is a morphism over $\mathcal{S}$ that lifts the qth power relative Frobenius of $\mathcal{X}_0/\mathcal{S}_0$. Since $\mathcal{X}_0 \to \mathcal{S}_0$ is quasi-smooth, the relative Frobenius is flat, and then so is F.

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The argument of the main part of [DModsII] shows that if M is a left $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}'/\mathcal{S}}$-module (not necessarily coherent) then F^*M has a natural structure of a left $\hat{\mathcal{D}}^{(m+s)}_{\mathcal{X}/\mathcal{S}}$-module. In fact for this argument to work one only needs that F is flat.

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If $\mathcal{X} \to \mathcal{S}$ is formally of finite type in addition to being quasi-smooth then Berthelot's descent theorem holds for the categories of left and right $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-modules: F^* (resp. $F^\flat$) induces an equivalence of categories of left (resp. right) $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}'/\mathcal{S}}$-modules with the category of left (resp. right) $\hat{\mathcal{D}}^{(m+s)}_{\mathcal{X}/\mathcal{S}}$. Here the finiteness of F is essential.

Quasi-nilpotence and m-HPD-stratifications

In the classical theory the quasi-nilpotence of a connection can be expressed by saying that the corresponding stratification extends to an HPD-stratification. Berthelot showed in [DModsI] how this extends to the case of m-PD-stratifications. The main point that the full m-PD-envelope of the diagonal ideal can be sheafified (i.e. is compatible with flat base change), and that it has a known structure.

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In the general case $\mathcal{X} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is still a formal scheme one must use J-adic completions as we did before, and as before this raises a number of technical problems. We first work in an affine setting: R is a ring with m-PD-structure, A is an R-algebra and $I \subseteq A$ is an ideal.

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Then IA_n is regular for all $n \geq 0$. If J is open, $p^{n+1} \in J$ for some n, and for any such n we set

$$P_{J,(m)}(I) = P_{(m)}(IA_n) \otimes_{A_n} A/J.$$

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It is easily checked that this definition is independent of n and if J' is any open ideal with $J' \subseteq J$, there is a canonical homorphism $P_{J',(m)}(I) \to P_{J,(m)}(I)$.

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It is easily checked that this definition is independent of n and if J' is any open ideal with $J' \subseteq J$, there is a canonical homorphism $P_{J',(m)}(I) \to P_{J,(m)}(I)$. Furthermore the canonical m-PD-structure on $P_{(m)}(IA_n)$ descends to $P_{J,(m)}(I)$.

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- Let $A' = A/I$. Then there is an ideal $J' \subset A'$ and an n such that $p^{n+1} \in J'$ and

$$JP_{(m)}(IA_n) = \sigma(J')P_{(m)}(IA_n).$$

With this hypothesis one can show that for $n \gg 0$ the m-PD-structure on $P_{J^n,(m)}(I)$ is compatible with the one on R. Passing to the limit we get an m-PD-structure $(\hat{I}^\bullet, \hat{I}^\circ, [\ ])$ on $\hat{P}_{(m)}(I)$.

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The m-PD-rings $P_{J,(m)}(I)$ globalize easily. Thus with our previous notation, if we are given ideals $\mathcal{I}$, $\mathcal{J} \subset \mathcal{O}_{\mathcal{X}}$ satisfying global version of the previous assumptions, there are quasi-coherent $\mathcal{O}_{X_{\mathcal{J}}}$-algebras $\mathcal{P}_{\mathcal{J},(m)}(\mathcal{I})$ endowed with m-PD-structures, and the m-PD-structure of $\mathcal{P}_{\mathcal{J}^n,(m)}(\mathcal{I})$ is compatible with that of $\mathcal{S}$ for $n \gg 0$. As before $\mathcal{P}_{(m)}(\mathcal{I})$ is defined as the inverse limit of the $\mathcal{P}_{\mathcal{J}^n,(m)}(\mathcal{I})$ over n.

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This procedure applies to the diagonal ideal of the $k+1$-fold fiber product $\mathcal{X}_{\mathcal{S}}(k)$, and the resulting sheaves of rings are denoted by $\mathcal{P}_{\mathcal{X}/\mathcal{S},(m)}(k)$. As usual we set $\mathcal{P}_{\mathcal{X}/\mathcal{S},(m)}(1) = \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)}$

A coherent left $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-module M is *topologically quasi-nilpotent* if for every m-bilateralising ideal of definition $J \subset \mathcal{O}_{\mathcal{X}}$ the operators $\partial^{\langle K \rangle_{(m)}}$ for $|K| > 0$ act nilpotently on the module M/JM; this condition is independent of the choice of local coordinates.

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In fact M is topologically quasi-nilpotent if and only if the m-PD-stratification on M extends to an m-HPD-stratification, i.e. an isomorphism

$$M \underset{\mathcal{O}_{\mathcal{X}}}{\hat{\otimes}} \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \xrightarrow{\sim} \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \underset{\mathcal{O}_{\mathcal{X}}}{\hat{\otimes}} M$$

satisfying the usual conditions: it reduces to the identity on the diagonal and its three pullbacks to $\mathcal{P}_{\mathcal{X}/\mathcal{S},(m)}(2)$ satisfy the cocycle condition.

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and the system of isomorphisms so obtained is transitive. If $\mathcal{Y} \to \mathcal{S}$ is quasi-smooth, $\tau_{f,g}$ is linear for the natural $\hat{\mathcal{D}}^{(m)}_{\mathcal{Y}/\mathcal{S}}$-module structures of $f^* M$ and $g^* M$.

Tubes

We now describe formal versions of the closed tubes that are used in rigid cohomology. In [DModsAdic] we gave a construction that is mainly useful when the base $\mathcal{S}$ is a p-adic formal scheme.

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A more general construction exploits the classical *Proj* construct in a formal setting. We fix an adic noetherian formal scheme $\mathcal{X}$, an ideal of definition $J \subset \mathcal{O}_{\mathcal{X}}$ and a $\mathbb{N}$-graded $\mathcal{O}_{\mathcal{X}}$-algebra $\mathcal{E} = \oplus_{\ell}\mathcal{E}_{\ell}$ with the following properties:

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- Each $\mathcal{E}_{\ell}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module;
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It follows that the rings of local sections of $\mathcal{E}_0$ and $\mathcal{E}$ are noetherian.

Then

$$Proj(\mathcal{E}) = \varinjlim_{n} Proj(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} (\mathcal{O}_{\mathcal{X}}/J^{n+1}))$$

is an adic noetherian formal scheme. When $\mathcal{X} = \mathrm{Spf}(A)$ is affine and $E = \Gamma(\mathcal{X}, \mathcal{E})$, the underlying point set of $\mathrm{Proj}(E)$ is the set of homogenous prime ideals not containing $E_+ = \oplus_{\ell > 0} E_\ell$ and open for the J-preadic topology.

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The usual properties of *Proj* relative base change and functoriality extend immediately to the formal case.

Suppose now $\mathcal{S}$ is an adic formal $\mathbb{Z}_p$-scheme with an m-PD-structure $(\mathfrak{a}, \mathfrak{b}, \gamma)$ and $\mathcal{X} \to \mathcal{S}$ is quasi-smooth. In addition to the previous assumptions on $\mathcal{E}$ we suppose that the following is given:

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If $\pi : Proj(\mathcal{E}) \to \mathcal{X}$ is the structure morphism we will show that $\mathcal{O}_{Proj(\mathcal{E})}$ has a natural left $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$ module structure compatible with its $\pi^{-1}(\mathcal{O}_\mathcal{X})$-algebra structure. In the affine case $\mathcal{X} = \mathrm{Spf}(A)$, $\mathcal{S} = \mathrm{Spf}(R)$ this amounts to giving each $E_{(f)}$ a left $\hat{D}^{(m)}_{A/R}$-structure compatible with its A-module structure.

For $\ell \geq 0$ let

$$\chi_n^\ell : P^n_{A/R,(m)} \otimes_A E_\ell \to E_\ell \otimes P^n_{A/R,(m)}$$

be the m-PD-stratification corresponding to the left $\hat{D}^{(m)}_{A/R}$-module structure and let

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Lemma

For any homogenous $e \in E_\ell$, $\theta_n^\ell(e)$ is invertible in $(E \otimes_A P^n_{A/R,(m)})_f$.

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In fact in $(E \otimes_A P^r_{A/R,(m)})_f$ we can write

$$\theta_n^\ell(e) = e \left(1 + \sum_{0<|K|<n} \frac{\partial^{\langle K \rangle_{(m)}}(e)}{e} \otimes_A \xi^{\{K\}_{(m)}} \right)$$

and it suffices to observe that $\xi^{\{K\}_{(m)}}$ for $|K| > 0$ is nilpotent in $P^n_{A/R,(m)}$.

Since $P^n_{A/R,(m)}$ is a finite A-module there are isomorphisms

$$(E \otimes_A P^n_{A/R,(m)})_{(f)} \simeq E_{(f)} \otimes_A P^n_{A/R,(m)}.$$

Then the lemma allows us to extend θ^ℓ_n to a ring homomorphism

$$\theta_n : E_{(f)} \to E_{(f)} \otimes_A P^n_{A/R,(m)}.$$

by setting

$$\theta_n(x/f^k) = \frac{\theta_n(x)}{\theta_n(f)^k}$$

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for $f \in E_\ell$, $x \in E_{k\ell}$. It is easily checked that this is well-defined and that the θ_n satisfy all of the properties required to define an m-PD-stratification on $B_{(f)} = \Gamma(D_+(f), \mathcal{O}_{Proj(E)})$, i.e. a left $\hat{D}^{(m)}_{A/R}$-module structure.

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If E is quasi-nilpotent as a left $\hat{D}^{(m)}_{A/R}$-module, the θ_n extend to a morphism

$$\theta : E_{(f)} \to E_{(f)} \underset{A}{\hat{\otimes}} \hat{P}_{A/R,(m)}$$

defining an m-HPD-stratification on $E_{(f)}$, i.e. the $\hat{D}^{(m)}_{A/R}$-module structure is quasi-nilpotent. We could say that the left $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$-module structure of $\mathcal{O}_{Proj(\mathcal{E})}$ is quasi-nilpotent.

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Suppose $\phi : \mathcal{E} \to \mathcal{F}$ is a graded morphism of graded $\mathcal{O}_{\mathcal{X}}$-algebras and let $G(\phi)$ be the complement of $V_+(\phi(\mathcal{E}))$. We denote by

$${}^a\phi : G(\phi) \to Proj(\mathcal{E})$$

the morphism over $\mathcal{X}$ induced by ϕ. If $\mathcal{E}$ and $\mathcal{F}$ both satisfy our earlier assumptions and ϕ horizontal, the induced morphism

$${}^a\phi^*\mathcal{O}_{Proj(\mathcal{E})} \to \mathcal{O}_{G(\phi)}$$

is also horizontal.

Admissible Blowups

Suppose now $I \subset \mathcal{O}_{\mathcal{X}}$ is an open ideal. We can apply the *Proj* construction to the graded $\mathcal{O}_{\mathcal{X}}$-algebra

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For $I' \subseteq I$ the evident morphism $\phi : \mathcal{B}_{I'} \to \mathcal{B}_I$ induces a morphism $G(\phi) \to \mathcal{X}_{I'}$ for some open $G(\phi) \subseteq \mathcal{X}_I$.

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Suppose now $\mathcal{X} \to \mathcal{S}$ is quasi-smooth and $\mathcal{S}$ has an m-PD-structure. If $I \subset \mathcal{O}_{\mathcal{X}}$ is m-bilateralising, $\mathcal{B}_I$ has a quasi-nilpotent left $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-module structure.

For any $x \in I$ we denote by $\bar{x}$ the element x viewed as a homogenous elements of B_I of degree 1.

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Suppose now $\mathfrak{c} \subset \mathcal{S}$ is an ideal such that $\mathfrak{c}\mathcal{O}_{\mathcal{X}} \subseteq I$. The formal scheme

$$\mathcal{X}_{I,\mathfrak{c}} = \mathcal{X}_I \setminus V_+(\overline{\mathfrak{c}\mathcal{O}_{\mathcal{X}}}).$$

is the largest open subscheme of $\mathcal{X}_I$ such that the ideal $I\mathcal{O}_{\mathcal{X}_I}$ is locally generated by a local section of $\mathfrak{c}$. In the affine case, if $\mathfrak{c} = (c_1, \dots, c_r)$ then the $D_+(\bar{c}_i)$ for $1 \leq i \leq r$ cover $\mathcal{X}_{I,\mathfrak{c}}$.

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When $\mathcal{X}$ is affine we denote by $\mathcal{X}[I]_{\mathfrak{c}} \subset \mathcal{X}_{I,\mathfrak{c}}$ the closed formal subscheme whose ideal is the ideal of $\mathfrak{c}$-torsion elements, i.e. local sections x such that $\mathfrak{c}^k x = 0$ for some $k > 0$. When $I \subset \mathcal{O}_{\mathcal{X}}$ is m-bilateralising, the left $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$-module structure on $\mathcal{O}_{\mathcal{X}_I}$ induces similar structures on $\mathcal{O}_{\mathcal{X}_{I,\mathfrak{c}}}$ and $\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}$.

When $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathfrak{c} = (c)$ is principal, $\mathcal{X}[I]_{\mathfrak{c}} = D_+(c)$ is affine. If we set

$$C = A[T_f,\ f \in I]/(cT_f - f,\ f \in I)$$

and let $\tilde{C}$ be the quotient of C by its c-torsion subring, the affine ring of $\mathcal{X}[I]_{\mathfrak{c}}$ is the completion of $\tilde{C}$ in the $J\tilde{C}$-adic topology.

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Suppose now $\mathcal{S} = \mathrm{Spf}(\mathcal{V})$ for some completed discrete valuation ring of mixed characteristic $p > 0$, $(\mathfrak{a}, \mathfrak{b}, \gamma)$ is $((\pi), (p), [\ ])$ where $(\pi) \subset \mathcal{V}$ is the maximal ideal and $[\ ]$ are the canonical divided powers of (p).

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Suppose $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of adic formal $\mathcal{S}$-schemes. If $I \subseteq \mathcal{O}_{\mathcal{X}}$ and $I' \subseteq \mathcal{O}_{\mathcal{X}'}$ are open ideals such that $I'\mathcal{O}_{\mathcal{X}} \subseteq I$, there is a unique morphism $f_{\mathfrak{c}} : \mathcal{X}[I]_{\mathfrak{c}} \to \mathcal{X}'[I']_{\mathfrak{c}}$ over f. Since $\mathfrak{c}\mathcal{O}_{\mathcal{X}'} \subseteq I'$ this follows from the functoriality of blowups. We can apply this in the case $\mathcal{X}' = \mathcal{X}$, so that when $I' \subseteq I$ there is a natural $\mathcal{X}$-morphism $\mathcal{X}[I]_{\mathfrak{c}} \to \mathcal{X}[I']_{\mathfrak{c}}$.

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If $f : \mathcal{Y} \to \mathcal{X}$ is a flat morphism of noetherian formal $\mathcal{S}$-schemes there is a canonical isomorphism

$$\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}[I]_{\mathfrak{c}} \xrightarrow{\sim} \mathcal{Y}[I\mathcal{O}_{\mathcal{Y}}]_{\mathfrak{c}} \tag{1}$$

natural in $f : \mathcal{Y} \to \mathcal{X}$ and transitive for pairs of composable morphisms $f : \mathcal{Y} \to \mathcal{X}$, $g : \mathcal{Z} \to \mathcal{Y}$. This follows from the standard base-change properties of blowups.

An important case when $\mathfrak{c} = \mathfrak{b}$ is the PD-ideal in the m-PD-structure $(\mathfrak{a}, \mathfrak{b}, \gamma)$ of $\mathcal{S}$. Then $(\mathfrak{a}, \mathfrak{b}, \gamma)$ extends uniquely to an m-PD-structure $(\mathfrak{a} + I\mathcal{O}_{\mathcal{X}[I]}, I\mathcal{O}_{\mathcal{X}[I]}, \gamma_{\mathcal{X}})$ on $\mathcal{X}[I]_{\mathfrak{b}}$ itself.

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An important case when $\mathfrak{c} = \mathfrak{b}$ is the PD-ideal in the m-PD-structure $(\mathfrak{a}, \mathfrak{b}, \gamma)$ of $\mathcal{S}$. Then $(\mathfrak{a}, \mathfrak{b}, \gamma)$ extends uniquely to an m-PD-structure $(\mathfrak{a} + I\mathcal{O}_{\mathcal{X}[I]}, I\mathcal{O}_{\mathcal{X}[I]}, \gamma_{\mathcal{X}})$ on $\mathcal{X}[I]_{\mathfrak{b}}$ itself. Since $I\mathcal{O}_{\mathcal{X}[I]}$ is locally principal and locally generated by an element of $\mathfrak{b}$, this can be proven in the same way one shows that a PD-structure on a principal ideal of R extends uniquely to any R-algebra.

If $\mathfrak{j} \subset \mathcal{O}_{\mathcal{S}}$ is an ideal containing $\mathfrak{b}$ then the same goes for $\mathcal{S}$ and $\mathfrak{j}$: the m-PD-structure $(\mathfrak{a}, \mathfrak{b}, \gamma)$ extends uniquely to an m-PD-structure $(\mathfrak{a} + I\mathcal{O}_{\mathcal{S}[\mathfrak{j}]}, I\mathcal{O}_{\mathcal{S}[\mathfrak{j}]}, \gamma_{\mathcal{S}})$ on $\mathcal{S}[\mathfrak{j}]_{\mathfrak{b}}$. Then $\mathcal{X}[I]_{\mathfrak{b}} \to \mathcal{S}[\mathfrak{j}]_{\mathfrak{b}}$ is an m-PD-morphism and the m-PD-structures on $\mathcal{X}[I]_{\mathfrak{b}}$ and $\mathcal{S}[\mathfrak{j}]_{\mathfrak{b}}$ are compatible.

The formal fibration theorem

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$$\begin{tikzcd} & \mathcal{X}' \arrow[d, "f"] \\ X_0 \arrow[ur, "u'"] \arrow[r, "u"'] & \mathcal{X} \end{tikzcd}$$

in which X_0 is a scheme, f is open, surjective and quasi-smooth of formal dimension d, and u, u' are closed immersions. Let $I \subset \mathcal{O}_{\mathcal{X}}$, $I' \subset \mathcal{O}_{\mathcal{X}'}$ be the ideals corresponding to u and u'.

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More generally, suppose we are given a filtered inductive system of diagrams indexed by $\alpha \in S$

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in which the X_α are closed subschemes with the same reduced closed subscheme.

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in which the X_α are closed subschemes with the same reduced closed subscheme. Let I_α, I'_α be the ideals corresponding to u_α and $u_{\alpha'}$. Then for all α, the induced morphism $f_\alpha : \mathcal{X}'[I'_\alpha]_\mathfrak{c} \to \mathcal{X}[I_\alpha]_\mathfrak{c}$ is a d-dimensional affine space bundle and locally on $\mathcal{X}$ there are systems of sections $s_\alpha : \mathcal{X}[I_\alpha]_\mathfrak{c} \to \mathcal{X}'[I'_\alpha]_\mathfrak{c}$ such that

$$\begin{tikzcd} \mathcal{X}'[I'_\alpha] \arrow[r] \arrow[d, shift left, "f_\alpha"] & \mathcal{X}'[I'_\beta] \arrow[d, shift right, "f_\beta"'] \\ \mathcal{X}[I_\alpha] \arrow[r] \arrow[u, bend left, "s_\alpha"] & \mathcal{X}[I_\beta] \arrow[u, bend right, "s_\beta"'] \end{tikzcd}$$

commutes for all $\alpha \leq \beta$.

Suppose now $I \subset \mathcal{O}_{\mathcal{X}}$ is m-bilateralising. If $\pi : \mathcal{X}[I]_{\mathfrak{c}} \to \mathcal{X}$ is the structure map we have seen that $\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}$ has a natural left $\pi^{-1}\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-module structure, and it makes sense to define

$$\hat{\mathcal{D}}^{(m)}_{\mathcal{X}[I]_{\mathfrak{c}}/\mathcal{S}} = \mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}} \underset{\pi^{-1}(\mathcal{O}_{\mathcal{X}})}{\hat{\otimes}} \pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$$

(the notation is merely symbolic since $\mathcal{X}[I]_{\mathfrak{c}} \to \mathcal{S}$ is not quasi-smooth).

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(the notation is merely symbolic since $\mathcal{X}[I]_{\mathfrak{c}} \to \mathcal{S}$ is not quasi-smooth). Since $\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}$ is topologically quasi-nilpotent and its ring structure is compatible with its left $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$-module structure, $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}[I]_{\mathfrak{c}}/\mathcal{S}}$ has a natural ring structure such that the evident maps $\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}} \to \hat{\mathcal{D}}^{(m)}_{\mathcal{X}[I]_{\mathfrak{c}}/\mathcal{S}}$, $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}) \to \hat{\mathcal{D}}^{(m)}_{\mathcal{X}[I]_{\mathfrak{c}}/\mathcal{S}}$ are ring homomorphisms.

A left $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}[I]_{\mathfrak{c}}/\mathcal{S}}$-module is the same as an $\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}$-module with with a compatible left $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$-module structure. We will say it is topologically quasi-nilpotent if it is so as a left $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$-module.

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We can describe topologically quasi-nilpotent left $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}[I]_{\mathfrak{c}}/\mathcal{S}}$-modules in terms of something like an m-PD-stratification. We first recall the sheaf of rings $\mathcal{P}_{\mathcal{X}/\mathcal{S},(m)}(r)$ which has $r+1$ $\mathcal{O}_{\mathcal{X}}$-module structures, which we denote by $p_i^* : \mathcal{O}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)}(r)$, $0 \leq i \leq r$.

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$$p_i^*\mathcal{X}[I]_{\mathfrak{c}}(r)=\pi^{-1}(\mathcal{P}_{\mathcal{X}/\mathcal{S},(m)}(r))\underset{\pi^{-1}(\mathcal{O}_{\mathcal{X}}),p_i^*}{\hat{\otimes}}\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}$$

(again this is just symbolic). As usual when $r=1$ we omit (r).

The fact that $\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}$ is a topologically quasi-nilpotent left $\pi^{-1}(\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}})$-module amounts to the existence of an isomorphism

$$\chi[I]_{\mathfrak{c}} : p_1^*\mathcal{X}[I]_{\mathfrak{c}} \xrightarrow{\sim} p_0^*\mathcal{X}[I]_{\mathfrak{c}}$$

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Then for any J-adically complete $\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}$-module M, a topologically quasi-nilpotent left $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}[I]_{\mathfrak{c}}/\mathcal{S}}$-module structure on M is equivalent to an $\chi[I]_{\mathfrak{c}}$-semilinear isomorphism

$$\chi : p_1^*\mathcal{X}[I]_{\mathfrak{c}} \underset{\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}}{\hat{\otimes}} M \xrightarrow{\sim} p_0^*\mathcal{X}[I]_{\mathfrak{c}} \underset{\mathcal{O}_{\mathcal{X}[I]_{\mathfrak{c}}}}{\hat{\otimes}} M.$$

Recall the setup: $\mathcal{X} \to \mathcal{S}$ is quasi-smooth, $(\mathfrak{a}, \mathfrak{b}, \gamma)$ is a m-structure on $\mathcal{S}$ and $I \subset \mathcal{O}_{\mathcal{X}}$ is m-bilateralising.

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$$\begin{tikzcd}
 & \mathcal{X}' \arrow[d, "f"] & \mathcal{X}'[I']_{\mathfrak{b}} \arrow[l] \arrow[d, "f_{\mathfrak{b}}"] \\
X_0 \arrow[ur, "u'"] \arrow[r, "u"'] & \mathcal{X} & \mathcal{X}[I]_{\mathfrak{b}} \arrow[l]
\end{tikzcd}$$

in which f is quasi-smooth, open and surjective, and u, u' are closed immersions. Suppose furthermore that the ideals I, I' associated to u, u' are m-bilateralising.

Theorem

The functor

$$f_{\mathfrak{b}}^{*} : \mathrm{CM}^{(m)}(\mathcal{X}[I]_{\mathfrak{b}}) \to \mathrm{CM}^{(m)}(\mathcal{X}'[I']_{\mathfrak{b}})$$

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The assumptions on f allow us to invoke the formal fibration theorem, so locally the induced map $f_{\mathfrak{b}}$ on tubes has a section s. Since the categories $\mathrm{CM}^{(m)}(\mathcal{X}[I]_{\mathfrak{b}})$, $\mathrm{CM}^{(m)}(\mathcal{X}'[I']_{\mathfrak{b}})$ are of local nature we may assume s exists globally.

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There is a similar result for the categories $\mathrm{CM}^{(m)}(\mathcal{X}[I])_{\mathbb{Q}}$, $\mathrm{CM}^{(m)}(\mathcal{X}'[I'])_{\mathbb{Q}}$ of objects up to isogeny.

Isocrystals on $\mathcal{X}/\mathcal{S}$

For the moment let $\mathcal{S}$ be any locally noetherian scheme and $\mathcal{X}$, $\mathcal{Y}$ locally noetherian formal $\mathcal{S}$-schemes. We will say that an $\mathcal{S}$-morphism $f : \mathcal{X} \to \mathcal{Y}$ is *bounded* if it is quasi-compact, and locally on $\mathcal{X}$ and $\mathcal{Y}$ there is a quasi-smooth formal $\mathcal{S}$-scheme $\mathcal{P}$ such that f factors

$$\mathcal{X} \xrightarrow{i} \mathcal{P} \times_{\mathcal{S}} \mathcal{Y} \xrightarrow{p_2} \mathcal{Y}$$

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Suppose now X is a bounded $\mathcal{S}$-scheme (not formal). We will assume for simplicity that the structure morphism factors globally as $X \to \mathcal{P} \to \mathcal{S}$ with $\mathcal{P} \to \mathcal{S}$ quasi-smooth; the general case can be handled by simplicial techniques.

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Let I be the ideal of the closed immersion $X \to \mathcal{P}$. If $\hat{\mathcal{P}}$ is the I-adic completion of $\mathcal{P}$, $\hat{\mathcal{P}} \to \mathcal{S}$ is also quasi-smooth. Replacing $\mathcal{P}$ by $\hat{\mathcal{P}}$ we may assume that I is an ideal of definition of $\mathcal{P}$, and then $X \to \mathcal{P}$ is a homeomorphism.

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$$I_n = I^{(p^{n+1})} + p\mathcal{O}_{\mathcal{P}}.$$

For $n \geq m$, I_n is n-bilateralising. For $n' \geq n$ let

$$i_{n'n} : \mathcal{X}[I_n]_{\mathfrak{b}} \to \mathcal{X}[I_{n'}]_{\mathfrak{b}}$$

be the natural morphism. We denote by X_n the closed subscheme of $\mathcal{P}$ defined by I_n.

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- Objects are systems $(M_n, f_{nn'})$ where M_n for $n \geq m$ is an object of $\mathrm{CM}^{(n)}(\mathcal{P}[I_n])_{\mathbb{Q}}$ and for $n' \geq n$ the

$$f_{nn'} : i^*_{n'n} M_{n'} \xrightarrow{\sim} M_n$$

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are a transitive system of isomorphisms horizontal for the left $\hat{\mathcal{D}}^{(n)}_{\mathcal{X}[I_n]/\mathcal{S}}$-module structures.
- A morphism $(M_n, f_{nn'}) \to (N_n, g_{nn'})$ is a system of horizontal morphisms $M_n \to N_n$ compatible with the $f_{nn'}$ and $g_{nn'}$.

If

$$\begin{tikzcd} & \mathcal{P}' \arrow[d, "f"] \\ X \arrow[ur, "u'"] \arrow[r, "u"'] & \mathcal{P} \end{tikzcd}$$

is commutative there is an evident functor $f^* : \mathrm{Isoc}(X/\mathcal{S}, \mathcal{P}) \to \mathrm{Isoc}(X/\mathcal{S}, \mathcal{P}')$. Since u and u' are homeomorphisms so is f, which is thus surjective and open.

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This allows us to extend the construction to the case when the factorization $X \to \mathcal{P} \to \mathcal{S}$ exists only locally. We denote the resulting category by $\mathrm{Isoc}(X/\mathcal{S})$. It is of local nature on X and functorial in $X \to \mathcal{S}$.

Suppose finally that $\mathcal{V}$ is a complete discrete valuation ring of mixed characteristic p and $\mathcal{S} = \mathsf{Spf}(\mathcal{V})$ and X is of finite type over the residue field of $\mathcal{V}$. The construction of $\mathrm{Isoc}(X/\mathcal{S})$ as above is equivalent to the category of the same name constructed in [DModsAdic]. It follows that $\mathrm{Isoc}(X/\mathcal{S})$ is equivalent to the category of convergent isocrystals on $\mathcal{X}$.

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Thank you.