Optimization and Control in Free/Moving Boundary Fluid-Structure Interactions

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BANFF Women in Control: New Trends in Infinite Dimensions

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Outline

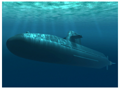
1. FSI: Kinematics and Computational Domain
2. Fluid-Elasticity: PDE Model
3. Well-posedness Analysis
4. Optimization, Sensitivity Analysis, and Control in FSI $\rightarrow$ Shape Analysis
5. Existence of Optimal Controls, Sensitivity and Adjoint Sensitivity Systems, Necessary Optimality Conditions

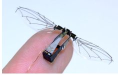
Fluid-Structure Interactions (FSI)

Interaction of some movable and/or deformable structure with

an internal or surrounding fluid flow

- industrial processes, aero-elasticity, and biomechanics




The boundary of the domain is **not known** in advance,

but has to be determined as part of the solution.

- Free boundary: **steady-state** problem.
- Moving boundary: time dependent problems and the position of the boundary is a function of **time and space**.

Fluid-Elasticity Interactions - Kinematics

- Coupling of **incompressible Navier-Stokes equations** with an **elastic solid**.

Motion of the 2 continuous media:

Mass + Momentum Balance

- same for solids and fluids.

Characterize how the media react internally to an exterior action

- behaviors of the 2 types of media diverge.

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Computational Domain: **reference** vs. **current configuration**?

Deformation and Motion

Reference configuration: $\widehat{\Omega} \subset \mathbb{R}^3$ be bounded, open, simply connected set, with smooth boundary, filled by a continuum media.

A **deformation** is a smooth, 1-1 map:

$$\widehat{\phi} : \widehat{\Omega} \to \Omega, \quad \widehat{x} \to x = \widehat{\phi}(\widehat{x})$$

- Ω: current configuration.
- $\widehat{\eta}(\widehat{x}) = \widehat{\phi}(\widehat{x}) - \widehat{x}$: displacement of the material point $\widehat{x}$.

A **motion** is a smooth map:

$$\widehat{\varphi} : \widehat{\Omega} \times \mathbb{R}^+ \to \Omega(t), \quad (\widehat{x}, t) \to x = \widehat{\varphi}(\widehat{x}, t)$$

s.t. for any $t \geq 0, \widehat{\varphi}_t = \widehat{\varphi}(\cdot, t)$ is a deformation.

A motion is a 1-parameter family of deformations.

- $\widehat{\Omega}$ can be arb., or $\widehat{\Omega} = \Omega(0)$.
- $\Omega(t)$: current configuration at time t.

Displacement at time t: $\widehat{\eta}(\widehat{x}, t) = \widehat{\varphi}(\widehat{x}, t) - \widehat{x}$.

Deformation **Gradient**: $\widehat{G} : \widehat{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^{3\times 3}$,

$$\widehat{G}(\widehat{x}, t) = D_{\widehat{x}}\widehat{\phi}(\widehat{x}, t) = \nabla_{\widehat{x}}\widehat{\phi}(\widehat{x}, t).$$

Jacobian of the Deformation:

$$\widehat{J} = \det(\widehat{G}) > 0$$

- measures the variation of the volume due to the deformation: for $\widehat{V} \subset \widehat{\Omega}$, and $V(t) = \{x \in \Omega(t) \mid x = \widehat{\varphi}(\widehat{x}, t), \widehat{x} \in \widehat{V}\}$,

$$|V(t)| = \int_{V(t)} dx = \int_{\widehat{V}} \widehat{J}(\widehat{x}, t) d\widehat{x}$$

Velocity:

$$\widehat{u}(\widehat{x}, t) = \frac{\partial}{\partial t}\widehat{\eta}(\widehat{x}, t) = \frac{\partial}{\partial t}\widehat{\varphi}(\widehat{x}, t)$$

All physical quantities can be defined on the reference or on the current configuration.

Lagrangian, Eulerian, and ALE

Solid: displacements are often relatively small

- computational domain: $\widehat{\Omega}$
- Lagrangian formulation: focus on the material particle $\widehat{x}$ and its evolution

Fluid: displacements are large and usually irrelevant

- we are mostly interested in the velocity field
- Eulerian framework: observe what happens at a given point x in the physical space.

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FSI: Fluid + elasticity + interface conditions between solid and fluid

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To match these two different frameworks:

Arbitrary Lagrangian-Eulerian (ALE) formulation

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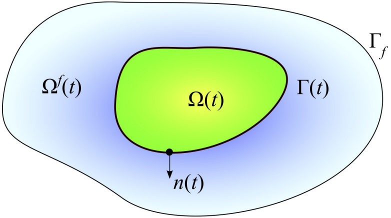
To match these two different frameworks:

Arbitrary Lagrangian-Eulerian (ALE) formulation

- Evolution of the computational domain is not governed by the fluid motion, but has to comply with the evolution of the boundary, which is the result of the coupling with the structural model.

Fluid - Elasticity Interaction: PDE model

- Configuration: the elastic body moves and deforms inside the fluid.



- Elastic body located at time $t \geq 0$ in a domain $\Omega(t) \subset \mathbb{R}^3$ with boundary $\Gamma(t)$.
- The fluid occupies domain $\Omega^f(t) = \mathcal{D} \setminus \bar{\Omega}(t)$, with smooth boundary $\Gamma(t) \cup \Gamma_f$.
- Let $\mathcal{D} \subset \mathbb{R}^3$ be the control volume. $\mathcal{D}$ contains the solid and the fluid at each time $t \geq 0$, i.e. $\mathcal{D} = \Omega(t) \cup \Omega^f(t)$, with smooth boundary $\partial\mathcal{D} = \Gamma_f$.

Navier-Stokes - Eulerian Framework

- Fluid: Newtonian viscous, homogeneous, and incompressible.
- Its behavior is described by its **velocity** w and **pressure** p.
- The viscosity of the fluid is $\nu > 0$, and the fluid strain tensor is given by

$$\varepsilon(w) = \frac{1}{2}[Dw + (Dw)^*],$$

  where Dw is the gradient matrix of w, and $(Dw)^*$ represents the transpose of Dw.
- The fluid state satisfies the following Navier-Stokes equations:

$$\begin{cases} w_t - \nu\,\Delta w + Dw \cdot w + \nabla p = v_1 & \text{on} \quad \Omega^f(t) \\ \text{div } w = 0 & \text{on} \quad \Omega^f(t) \\ w = 0 & \text{on} \quad \Gamma_f \end{cases}$$

Structural Deformation: Lagrangian formulation

- The evolution of the fluid domain $\Omega^f(t)$ is induced by the structural deformation through the common interface $\Gamma(t)$.
- $\mathcal{O} \subset \mathcal{D}$: reference configuration for the solid; $\partial\mathcal{O} = \mathcal{S}$
- $\mathcal{O}^f = \mathcal{D} \setminus \bar{\mathcal{O}}$: reference fluid configuration. T
- $\mathcal{D}$ is described by a smooth, injective map:

$$\varphi : \bar{\mathcal{D}} \times \mathbb{R}^+ \longrightarrow \bar{\mathcal{D}}, \quad (x, t) \mapsto \varphi = \varphi(x, t).$$

- For $x \in \mathcal{O}$, $\varphi(x, t)$: the position at time t of the material point x.
- On $\mathcal{O}^f$, $\varphi(x, t)$ is defined as an arbitrary extension of the restriction of φ to $\mathcal{S}$, which preserves the boundary Γ_f, i.e. $\varphi = I_{\Gamma_f}$ on Γ_f.
- $J(\varphi) > 0$: Jacobian of the deformation $\varphi(t)$

Nonlinear elasticity

- St. Venant - Kirchhoff equations: large displacement, small deformation elasticity. Green-St. Venant nonlinear strain tensor:

$$\sigma(\varphi) = \frac{1}{2}[(D\varphi)^* D\varphi - I].$$

Piola transform of the Cauchy stress tensor:

$$\mathcal{P}(x) = D\varphi(x)[\lambda \mathrm{Tr}[\sigma(\varphi)]I + 2\mu\sigma(\varphi)])$$

- Equilibrium equations for elasticity :

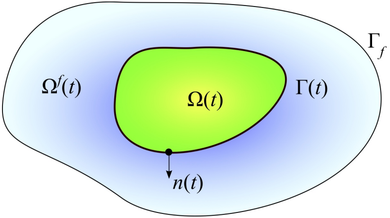
$$\mathrm{J}\rho\partial_{tt}\varphi - \mathrm{Div}\mathcal{P} = J\rho v_2 \quad \text{on } \mathcal{O}$$

On $\Gamma(t) = \varphi(t)(S)$, we have suitable transmission boundary conditions:

$$\begin{cases} w \circ \varphi = \varphi_t & \text{on } \mathcal{S} \\ \mathcal{P}n = J(\varphi)(\sigma(p,w) \circ \varphi)(D\varphi)^{-*}n & \text{on } \mathcal{S}, \end{cases}$$

where $n(t)$ is the unit outer normal vector along $\Gamma(t)$ with respect to $\Omega(t)$, and $\sigma(p,w) = -pI + 2\nu\varepsilon(w)$ is the fluid stress tensor.

FSI - PDE model



$$\begin{cases} w_t - \nu \Delta w + Dw \cdot w + \nabla p = v_1 & \text{on} \ \ \Omega^f(t) \\ \text{div } w = 0 & \text{on} \ \ \Omega^f(t) \\ w = 0 & \text{on} \ \ \Gamma_f \\ J\rho \partial_{tt}\varphi - \text{Div}\mathcal{P} = J\rho v_2 & \text{on} \ \ \mathcal{O} \\ w \circ \varphi = \varphi_t & \text{on} \ \ \mathcal{S} \\ \mathcal{P}n = J(\varphi)(\sigma(p,w) \circ \varphi)(D\varphi)^{-*} n & \text{on} \ \ \mathcal{S} \\ \varphi = I_{\Gamma_f} & \text{on} \ \ \Gamma_f, \end{cases}$$

with IC

$\varphi(\cdot,0) = \varphi^0, \varphi_t(\cdot,0) = \varphi^1, w(\cdot,0) = w^0, p(\cdot,0) = p^0$ on $(\mathcal{O})^2 \times (\mathcal{O}^c)^2$.

Well-posedness Analysis

FSI: parabolic-hyperbolic coupled system

- **regularity gap** of the fluid and structure velocities on the common interface: the traces of the elastic component at the energy level are not defined via the standard trace theory, and this induces a loss of regularity at the boundary of the coupled system.

- Coutand-Shkoller '05-'06: Existence of strong solutions for the case of a linear and then quasi-linear elastic body flowing within a viscous, incompressible fluid, under the assumptions of smooth initial data (i.e., the initial fluid velocity w^0 belongs to H^5, and the initial data for elasticity (φ^0, φ^1) belong to $H^3 \times H^2$). Due to the incompressibility condition of the fluid, uniqueness of solution for the model required higher regularity for the initial data (i.e., $(w^0, \varphi^0, \varphi^1) \in H^7 \times H^5 \times H^4$).
- Kukavica-Tuffaha-Ziane '09-'11, Ignatova-Kukavica-Lasiecka-Tuffaha '12-'14, Raymond-Vanninathan '15 for N-S coupled with linear elasticity/wave equation.
- The authors of [Ignatova-Kukavica-Lasiecka-Tuffaha] also prove global in time well-posedness for small initial data of the Navier–Stokes-elasticity model involving a wave equation with frictional damping, and they show that the energy associated with smooth and sufficiently small solutions of the damped model decay exponentially to zero.
- Canic-Muha '13-'14: dynamical coupling (which is of great interest in the modeling and analysis of the cardiovascular system).
- Grandmont'02, Wick-Wollner'14: steady state NS-St. Venant elasticity equations.

PDE-constrained Optimization Problems governed by FSI

In most of the applications, the ultimate goal is the

$$\begin{cases} \textbf{optimization or optimal control} \text{ of the considered process,} \\ \text{related } \textbf{sensitivity analysis} \text{ (with respect to relevant physical parameters).} \end{cases}$$

- minimize turbulence in the fluid
- optimize fluid velocity or pressure
- optimize the deformation of the structure
- minimize wall shear stresses
- ...

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- Control problems in FSI: most of the literature is focused on the assumption of small but rapid oscillations of the solid body, so that the common **interface may be assumed fixed**:
  Lasiecka and Bucci '05, '10, Lasiecka, Triggiani, and Zhang '11, Lasiecka and Tuffaha, '08-'09, Avalos-Triggiani '08-'12.
- Recently, PDE constrained optimization problems governed by free boundary interactions have been considered, with most research studies mainly addressed in the context of the **numerical analysis of the finite element methods** [Antil-Nochetto-Sodre '14, Richter-Wick '13, Van Der Zee et al '10]

Steady State Navier-Stokes and Elasticity

$$\begin{cases} -\nu\,\Delta w \;+\; Dw\cdot w \;+\; \nabla p \;=\; v|_{\Omega_f} & \text{on} \quad \Omega_f \\ \operatorname{div} w = 0 & \text{on} \quad \Omega_f \\ w = 0 & \text{on} \quad \Gamma := \varphi(\mathcal{S}) \\ -\operatorname{Div}\mathcal{P} = v|_{\Omega_e} & \text{on} \quad \mathcal{O} \\ \mathcal{P}n = J(\varphi)(\sigma(p,w)\circ\varphi)(D\varphi)^{-*}n & \text{on} \quad \mathcal{S} \\ w = 0,\;\; \varphi = I_{\Gamma_f} & \text{on} \quad \Gamma_f \end{cases}$$

[2] P.G. Ciarlet, Mathematical Elasticity Vol. I: Three-dimensional Elasticity, North-Holland Publishing Co., 1988

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Cauchy Stress Tensor $\mathcal{T} : \Omega_e \to \mathbb{S}^3$, $\mathcal{T} = [J^{-1}\mathcal{P}\cdot(D\varphi)^*] \circ \varphi^{-1}$ [2]

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OCP

We consider the optimal control problem:

$$\min J(w,v) = 1/2\|w - w_d\|^2_{L^2(\Omega_f)} + 1/2\|v\|^2_{H^3(\mathcal{D})} \tag{1}$$

subject to

$$\begin{cases} -\nu\,\Delta w + Dw \cdot w + \nabla p = v|_{\Omega_f} & \text{on} \quad \Omega_f \\ \mathrm{div} w = 0 & \text{on} \quad \Omega_f \\ w = 0 & \text{on} \quad \Gamma := \varphi(\mathcal{S}) \\ -\mathrm{Div}\mathcal{T} = v|_{\Omega_e} & \text{on} \quad \Omega_e = \varphi(\mathcal{O}) \\ \mathcal{T}n = \sigma(p,w)n & \text{on} \quad \Gamma \\ w = 0,\ \varphi = I_{\Gamma_f} & \text{on} \quad \Gamma_f\,. \end{cases}$$

- distributed control $v \in H^3(\mathcal{D})$
- $w_d \in L^2(\Omega_f)$ is a desired fluid velocity.

OCP

$$\min J(w,v) = 1/2\|w - w_d\|^2_{L^2(\Omega_f)} + 1/2\|v\|^2_{H^3(\mathcal{D})} \tag{2}$$

subject to

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Goals:

1. **Existence of an optimal control**
   L. Bociu, L. Castle, K. Martin, and D. Toundykov, Optimal Control in a Free Boundary Fluid-Elasticity Interaction, AIMS Proceedings, (2015), 122-131.

2. **First-order necessary conditions of optimality (NOC)**

OCP

$$\min J(w,v) = 1/2\|w - w_d\|^2_{L^2(\Omega_f)} + 1/2\|v\|^2_{H^3(\mathcal{D})} \tag{3}$$

subject to

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Goals:

1. **Existence of an optimal control**
2. **First-order necessary conditions of optimality (NOC)**
    - Compute the gradient of the functional J.
    - Characterization of the optimal control will pave the way for a numerical study of the problem.

Main Challenge

- Lagrangian: $\mathcal{L} = J -$ (weak form of the system)
  - **Not** convex-concave, due to the nonlinearity of the control-to-state map.
  - Min-Max theory **does not apply**, i.e., one can not reduce the cost function gradient to the derivative of the Lagrangian with respect to the control, at its saddle point [Delfour-Zolesio '86]
- Optimality conditions must be derived from **differentiability** arguments on the cost functional J with respect to the control v.
  - **Main challenge**: dependence of the cost integrals in J on the unknown domain Ω_f, which also depends on the control v.

$$\min J(w, v) = 1/2\|w - w_d\|^2_{L^2(\Omega_f)} + 1/2\|v\|^2_{H^3(\mathcal{D})}$$

- **Directional derivative** of J with respect to v in the direction of v': for small parameter $s \geq 0$, consider the perturbed functional $J(v + sv')$ and then calculate the derivative at $s = 0$ of the function $s \to J(v + sv')$.

With the following notation for the s-derivatives at $s = 0$,

$$\varphi' = \frac{\partial}{\partial s}\varphi_s\Big|_{s=0}, \quad U' = \varphi' \circ \varphi^{-1}, \quad w' = \frac{\partial}{\partial s} w_s\Big|_{s=0}, \quad \text{and} \quad p' = \frac{\partial}{\partial s} p_s\Big|_{s=0},$$

we can compute the directional derivative of J as

$$\begin{aligned} \partial J(v; v') &= \lim_{s\to 0} \frac{J(v+sv') - J(v)}{s} = \frac{\partial}{\partial s} J(v+sv')\Big|_{s=0} \\ &= \frac{\partial}{\partial s}\left[\frac{1}{2}\int_{(\Omega_f)_s} |w_s - w_d|^2 + \frac{1}{2}\|v+sv'\|^2_{H^3(\mathcal{D})}\right]\Big|_{s=0} \\ &= \int_{\Omega_f} (w - w_d)\cdot w' + \frac{1}{2}\int_\Gamma |w - w_d|^2 U' \cdot n_f + (v, v')_{H^3(\mathcal{D})} \end{aligned}$$

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&= \int_{\Omega_f} (w - w_d)\cdot w' + \frac{1}{2}\int_{\Gamma} |w - w_d|^2 U' \cdot n_f + (v, v')_{H^3(\mathcal{D})}
\end{aligned}$$

- The challenge of applying optimization tools to free boundary FSI is the proper derivation of the **sensitivity and adjoint sensitivity information with correct balancing conditions on the common interface**.
- As the interaction is a coupling of Eulerian and Lagrangian quantities, sensitivity analysis on the system falls into the framework of **shape analysis**.

Sensitivity System [LB - Zolesio]

$$
\begin{cases}
-\nu \Delta w' + (Dw')\,w + (Dw)\,w' + \nabla p' = v'\big|_{\Omega_f} & \text{in} \quad \Omega_f \\
\mathrm{div} w' = 0 & \text{in} \quad \Omega_f \\
w' + (Dw)U' = 0 & \text{on} \quad \Gamma \\
-\mathrm{Div}\, T(U') = v'\big|_{\Omega_e} & \text{in} \quad \Omega_e \\
T(U') \cdot n = (-p' I + 2\nu \varepsilon(w')) \cdot n + B(U') & \text{on} \quad \Gamma \\
w' = 0, \;\; U' = 0 & \text{on} \quad \Gamma_f
\end{cases}
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w' = 0, \ \ U' = 0 & \text{on} \quad \Gamma_f
\end{cases}
$$

$$
\Theta = D\varphi \circ \varphi^{-1} \quad \overline{DU'} := \Theta^*(DU')\Theta,
$$

$$
T(U') := (DU')\mathcal{T} + \frac{1}{\det \Theta}\Theta \cdot \{\lambda \text{Tr}(\overline{DU'})I + \mu[\overline{DU'} + (\overline{DU'})^*]\}\Theta^*,
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Sensitivity System [LB - Zolesio]

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Sensitivity System [LB - Zolesio]

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\operatorname{div} w' = 0 & \text{in} \quad \Omega_f \\
w' + \text{(Dw)U'} = 0 & \text{on} \quad \Gamma \\
-\operatorname{Div} \text{T(U')} = v'\big|_{\Omega_e} & \text{in} \quad \Omega_e \\
T(U')\cdot n = (-p'I + 2\nu\varepsilon(w'))\cdot n + \text{B(U')} & \text{on} \quad \Gamma \\
w' = 0, \;\; U' = 0 & \text{on} \quad \Gamma_f
\end{cases}
$$

$$
\Theta = D\varphi \circ \varphi^{-1} \quad \overline{DU'} := \Theta^*(DU')\Theta,
$$

$$
T(U') := \text{(DU')T} + \underbrace{\frac{1}{\det \Theta}\Theta \cdot \underbrace{\{\lambda \mathrm{Tr}(\overline{DU'})I + \mu[\overline{DU'} + (\overline{DU'})^*]\}}_{\Sigma(U')}\Theta^*}_{\tilde{\Sigma}(U')},
$$

Sensitivity System [LB - Zolesio]

$$
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$$B(U') = (\mathcal{T} + pI - 2\nu\varepsilon(w))\cdot \overbrace{[(D_\Gamma U')^* n + (\mathbf{D^2 b_{\Omega_e}})U'_\Gamma]}^{\nabla_\Gamma \langle U', n\rangle} + (D\mathcal{T})U'\cdot n$$

$$+ \,\text{div}(U')\mathcal{T}\cdot n - \mathcal{T}\cdot(DU')^*\cdot n -$$

$$- \langle U', n\rangle(-\text{Div}_\Gamma(\mathcal{T}) + [\partial_\nu pI - 2\nu\partial_\nu\varepsilon(w)]\cdot n).$$

Notation

- $(D\mathbf{f})_{ij} = \partial_j f_i \in \mathbb{M}^3$ is the gradient matrix at $a \in X$ of any vector field $f = (f_i) : X \subset \mathbb{R}^3 \to \mathbb{R}^3$.
- $\text{div } f = \partial_i f_i \in \mathbb{R}$ is the divergence of $f : X \subset \mathbb{R}^3 \to \mathbb{R}^3$.
- $\text{Div } T = \partial_j T_{ij} \in \mathbb{R}^3$ is the divergence of any second-order tensor field $T = (T_{ij}) : X \subset \mathbb{R}^3 \to \mathbb{M}^3$.
- $A^* =$ transpose of A, for any $A \in \mathbb{M}^3$.
- $d_\Omega(x) = \begin{cases} \inf_{y\in\Omega} |y - x| & \Omega \neq \emptyset \\ \infty & \Omega = \emptyset \end{cases}$ is the distance function
- $b_\Omega(x) = d_\Omega(x) - d_{\Omega^c}(x)$, $\forall x \in \mathbb{R}^n$ is the **oriented distance fn.** from x to Ω, for any $\Omega \subset \mathbb{R}^n$.
- $H = \Delta b_\Omega = \text{Tr}(D^2 b_\Omega)$ is the additive **curvature** of $\Gamma = \partial\Omega$. [3]

[3] M.C. Delfour and J.P. Zolesio, Shapes and Geometries: Analysis, Differential Calculus and Optimization, SIAM 2001.

Sensitivity System [LB - Zolesio]

$$\begin{cases} -\nu\Delta w' + (Dw')w + (Dw)w' + \nabla p' = v'\big|_{\Omega_f} & \text{in} \quad \Omega_f \\ \mathrm{div} w' = 0 & \text{in} \quad \Omega_f \\ w' + (Dw)U' = 0 & \text{on} \quad \Gamma \\ -\mathrm{Div}\, T(U') = v'\big|_{\Omega_e} & \text{in} \quad \Omega_e \\ T(U')\cdot n = (-p'I + 2\nu\varepsilon(w'))\cdot n + \mathrm{B(U')} & \text{on} \quad \Gamma \\ w' = 0, \;\; U' = 0 & \text{on} \quad \Gamma_f \end{cases}$$

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$$- \langle U', n\rangle(-\mathrm{Div}_\Gamma(\mathcal{T}) + [\partial_\nu pI - 2\nu\partial_\nu\varepsilon(w)]\cdot n).$$

$$= B_1\cdot\nabla_\Gamma\langle U', n\rangle - \langle U', n\rangle B_2 + (D\mathcal{T})U'\cdot n + \mathrm{div}(U')\mathcal{T}\cdot n - \mathcal{T}\cdot(DU')^*\cdot n$$

Connection to Shape Analysis

- As $v_s = v + sv'$, the **geometry of the problem** moves with the flow of a vector field that depends on the deformation φ_s.
  - The perturbation Γ_s of the boundary is built by the flow of the vector field $V(s,x) = \frac{\partial}{\partial s}\varphi_s \circ \varphi_s^{-1}$, i.e.,

$$\Gamma_s = T_s(V)(S), \text{where} \quad T_s(V) : \Omega_e \to (\Omega_e)_s, \; T_s(V) = \varphi_s \circ \varphi^{-1}.$$

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- (φ', w', p'): 'shape' derivatives with respect to the speed V, which is a vector field that depends on φ_s and is not given a priori.
  - Standard theory on shape derivatives: the domain is perturbed by an a priori given vector field and then the speed method is applied.
  - s-derivatives: **'pseudo-shape derivatives'**, in the sense that much of the theory of shape calculus remains applicable.

Goal: find the gradient of J at v: $J'(v; v')$

$$\partial J(v; v') = \int_{\Omega_f} (w - w_d) \cdot w' + \frac{1}{2} \int_\Gamma |w - w_d|^2 U' \cdot n_f + (v, v')_{H^3(\mathcal{D})}$$

- Sensitivity system provides the characterization for (U', w', p'):

$$\begin{cases} -\nu \Delta w' + (Dw')w + (Dw)w' + \nabla p' = v'\big|_{\Omega_f} & \text{in} \quad \Omega_f \\ \text{div} w' = 0 & \text{in} \quad \Omega_f \\ w' + (Dw)U' = 0 & \text{on} \quad \Gamma \\ -\text{Div}\, T(U') = v'\big|_{\Omega_e} & \text{in} \quad \Omega_e \\ T(U') \cdot n = (-p'I + 2\nu\varepsilon(w')) \cdot n + B(U') & \text{on} \quad \Gamma \\ w' = 0, \;\; U' = 0 & \text{on} \quad \Gamma_f \end{cases}$$

- v' does not appear in the chain rule computation, since it is hidden in the sensitivity equations for w', p', and U'.

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- v' does not appear in the chain rule computation, since it is hidden in the sensitivity equations for w', p', and U'.
- **Idea:** Introduce a suitable **adjoint problem** that eliminates the s-derivatives and provides an explicit representation for $J'(v; v')$.

Theorem (LB - Martin '16)

For the optimal control problem:

$$\min J(w,v) = 1/2\|w - w_d\|^2_{L^2(\Omega_f)} + 1/2\|v\|^2_{H^3(\mathcal{D})},$$

subject to FSI, the gradient of the cost functional is given by

$$J'(v;v') = (v',v)_{\mathcal{D}} + (v'|_{\Omega_f}, Q) + (v'|_{\Omega_e}, R),$$

where Q, P, and R solve the following adjoint sensitivity problem:

$$\begin{cases} -\nu\Delta Q + (Dw)^* Q - (DQ)w + \nabla P = w - w_d & \Omega_f \\ div(Q) = 0 & \Omega_f \\ -Div\bar{\mathcal{T}}'(R) = 0 & \Omega_e \\ Q = R & \Gamma \\ \bar{\mathcal{T}}'(R)n + (Dw)^*\sigma(P,Q)n + div_\Gamma[B_1 R]n - (D\mathcal{T}^\Delta \cdot n)^* R & \\ -H\langle \mathcal{T}n, R\rangle n + \nabla_\Gamma \langle \mathcal{T}n, R\rangle & \\ -Div_\Gamma(n \otimes \mathcal{T}R) + \langle B_2, R\rangle n = \frac{1}{2}|w - w_d|^2 n_f & \Gamma \\ Q = 0 & \Gamma_f \end{cases} \tag{4}$$

Matching of Normal Stress Tensors

$$\bar{\mathcal{T}}'(R)n+(Dw)^*\sigma(P,Q)n+\mathrm{div}_\Gamma[B_1R]n-(D\mathcal{T}^\Delta\cdot n)^*R-H\langle\mathcal{T}n,R\rangle n+\nabla_\Gamma\langle\mathcal{T}n,R\rangle$$
$$-\mathrm{Div}_\Gamma(n\otimes\mathcal{T}R)+\langle B_2,R\rangle n=\frac{1}{2}|w-w_d|^2n_f$$

- $B_1=\mathcal{T}+pI-2\nu\varepsilon(w)$ and $B_2=-\mathrm{Div}_\Gamma(\mathcal{T})+[\partial_\nu pI-2\nu\partial_\nu\varepsilon(w)]\cdot n$
- $(D\mathcal{T}^\Delta\cdot\vec{f})_{ik}:=\partial_k\mathcal{T}_{ij}f_j$

Matching of Normal Stress Tensors

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- $B_1 = \mathcal{T} + pI - 2\nu\varepsilon(w)$ and $B_2 = -\mathrm{Div}_\Gamma(\mathcal{T}) + [\partial_\nu pI - 2\nu\partial_\nu\varepsilon(w)] \cdot n$
- $(D\mathcal{T}^\Delta \cdot \vec{f})_{ik} := \partial_k \mathcal{T}_{ij} f_j$

  $D\mathcal{T}$ is defined as $(D\mathcal{T}.\vec{e})_{ij} = (\partial_k \mathcal{T}_{ij})e_k$. With the above notation, we can IBP

$$\begin{aligned} \int_{\tilde{\Gamma}_c} \langle \{(D\mathcal{T})\gamma\} \cdot n_e, R\rangle &= \int_{\tilde{\Gamma}_c} (\partial_k \mathcal{T}_{ij}\gamma_k)(n_e)_j R_i \\ = \int_{\tilde{\Gamma}_c} \gamma_k(\partial_k \mathcal{T}_{ij}(n_e)_j R_i) &= \int_{\tilde{\Gamma}_c} \langle \gamma, (D\mathcal{T}^\Delta \cdot n_e)^* R\rangle. \end{aligned}$$

Matching of Normal Stress Tensors

$$\bar{\mathcal{T}}'(R)n+(Dw)^*\sigma(P,Q)n+\mathrm{div}_\Gamma[B_1R]n-(D\mathcal{T}^\Delta\cdot n)^*R-H\langle\mathcal{T}n,R\rangle n+\nabla_\Gamma\langle\mathcal{T}n,R\rangle$$
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$$\begin{aligned}\int_{\tilde{\Gamma}_c}\langle\{(D\mathcal{T})\gamma\}\cdot n_e,R\rangle &= \int_{\tilde{\Gamma}_c}(\partial_k\mathcal{T}_{ij}\gamma_k)(n_e)_jR_i\\ =\int_{\tilde{\Gamma}_c}\gamma_k(\partial_k\mathcal{T}_{ij}(n_e)_jR_i) &= \int_{\tilde{\Gamma}_c}\langle\gamma,(D\mathcal{T}^\Delta\cdot n_e)^*R\rangle.\end{aligned}$$
-

$$\tilde{\mathcal{B}}(R)=\mathrm{div}_\Gamma[B_1R]n-(D\mathcal{T}^\Delta\cdot n)^*R-H\langle\mathcal{T}n,R\rangle n+\nabla_\Gamma\langle\mathcal{T}n,R\rangle$$
$$-\mathrm{Div}_\Gamma(n\otimes\mathcal{T}R)+\langle B_2,R\rangle n$$

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Current Work

- Well-posedness analysis for sensitivity and adjoint sensitivity system (LB and K. Martin)
- Optimizing the fluid pressure in a moving boundary fluid-wave interaction with distributed control (LB, L. Castle, and I. Lasiecka)

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THANK YOU !