Designs and Decompositions

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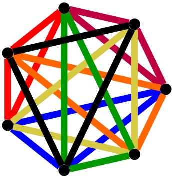
Designs and hypergraph decompositions

r-graph $=$ r-uniform hypergraph

Definition

An F-decomposition of an r-graph G is a set of edge-disjoint copies of F covering all edges of G
(also called an (n, q, r)-Steiner system if $G = K_n^{(r)}$ and $F = K_q^{(r)}$).

$(7, 3, 2)$-Steiner system $=$ triangle decomposition of $K_7^{(2)}$



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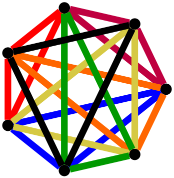

A set of distinct copies of $K_q^{(r)}$ in G such that every edge of G is covered exactly λ times is a (q,r,λ)-design of G
(also called an (n,q,r,λ)-design if $G = K_n^{(r)}$).

It's the year 1853...

Jakob Steiner

For which n does a triple system of order n exist?

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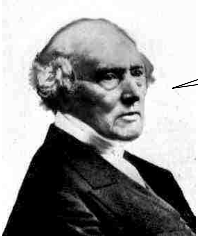
6 years earlier...

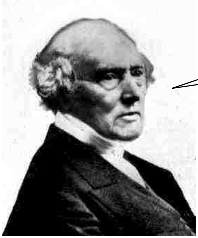
Theorem (Kirkman, 1847)

A triple system of order n exists if and only if $n \equiv 1, 3 \mod 6$.



Arrh! It should read **Kirkman system.**

Thomas Kirkman



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Wesley Woolhouse

Julius Plücker

EXCUSE ME!

Divisibility conditions

Question

When does G have an F-decomposition?

If G has a triangle decomposition, then

(a) the number of edges of G is divisible by 3,

(b) every vertex has even degree.

Call G triangle divisible if (a) and (b) are satisfied.

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Divisibility conditions can be generalised for arbitrary q, r, λ, in which case we say that G is (q, r, λ)-divisible (or $K_q^{(r)}$-divisible if $\lambda = 1$).

Previous results for graphs

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Theorem (Wilson 1975)

For n large, every F-divisible K_n has an F-decomposition.

Previous results for hypergraphs

(n, q, r, λ)-**design** $=$ set of distinct copies of $K_q^{(r)}$ in $K_n^{(r)}$ such that every edge of $K_n^{(r)}$ is covered exactly λ times

Theorem (Teirlinck 1987)

For every r, there exist infinitely many nontrivial $(n, r+1, r, \lambda)$-designs, where $\lambda = (r+1)!^{r+1}$.

Theorem (Kuperberg, Lovett and Peled 2013^+)

There exists an absolute constant C such that whenever $q \geq Cr$ there are infinitely many nontrivial (n, q, r, λ)-designs (for some (large) λ).

Question: What about decompositions, i.e. case $\lambda = 1$?

Relaxation: aim for an 'approximate decomposition'
(i.e. an almost perfect packing of edge disjoint $K_q^{(r)}$)

Conjecture (Erdős and Hanani, 1963)

There exists a $K_q^{(r)}$-packing in $K_n^{(r)}$ covering all but $o(n^r)$ of the edges of $K_n^{(r)}$ (as $n \to \infty$).

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Theorem (Rödl, 1985)

The conjecture is true.

Proof: 'Rödl nibble' or 'semirandom method'
(also very important ingredient in our proof)

Previous results

Theorem (Keevash 2014^+)

For any fixed q, r, λ, there exist (n, q, r, λ)-designs.
More precisely, if $n \gg q, \lambda$ and $K_n^{(r)}$ is (q, r, λ)-divisible, then there exists an (n, q, r, λ)-design.

- can actually replace $K_n^{(r)}$ by any dense quasirandom r-graph
- proof is based on algebraic and probabilistic arguments.

We generalize this beyond the quasi-random setting,
using combinatorial and probabilistic arguments.

Minimum degree version

from now on restrict to case $\lambda = 1$, results also extend to $\lambda > 1$
$\delta_{r-1}(G)$= minimum degree of an $(r-1)$-tuple of vertices

Theorem (Glock, Kühn, Lo, Osthus 2016^{+})

For all $q > r \geq 2$, there exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let

$$c^{\diamond}_{q,r} := \frac{r!}{3 \cdot 14^r q^{2r}}.$$

If G is an n-vertex r-graph with $\delta_{r-1}(G) \geq (1 - c^{\diamond}_{q,r})n$, then G has a $K_q^{(r)}$-decomposition whenever it is $K_q^{(r)}$-divisible.

The decomposition threshold

Previous result leads to notion of decomposition threshold $\delta_{q,r}$:

Definition

Let $\delta_{q,r}$ be the smallest $\delta \in [0,1]$ satisfying the following: for all large enough n, every $K_q^{(r)}$-divisible r-graph G on n vertices with $\delta(G) \geq (\delta + o(1))n$ has a $K_q^{(r)}$-decomposition.

- Keevash $\Rightarrow \delta_{q,r} < 1$
- GKLO $\Rightarrow \delta_{q,r} \leq 1 - c^{\diamond}_{q,r} \approx 1 - q^{-2r}$.
- Lower bound construction: $\delta_{q,r} \geq 1 - c_r q^{-r+1} \log q \approx 1 - q^{-r+1}$.

graph case $r = 2$ has received much attention – see later

Main result: supercomplexes

Previous result follows from our main result on designs in 'supercomplexes'.

Theorem (Glock, Kühn, Lo, Osthus 2016$^+$)

If $n \gg q, \lambda$ and G is a (q, r, λ)-divisible supercomplex on n vertices, then G has a (q, r, λ)-design.
($+$ generalisation to dense quasirandom r-graphs)

The conditions of being a supercomplex depend mainly on the distribution of q-cliques, which should be 'random-like'.

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Examples of supercomplexes

- complete r-graphs
- quasirandom r-graphs, in particular 'typical' r-graphs
- k-partite graphs where $k \geq q + 6$

Existence of F-designs for arbitrary F

so far: considered designs/decompositions into cliques
What about decompositions into arbitrary hypergraphs F?

F-decomposition= decomposition of edge set of G into copies of F

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F-decomposition $=$ decomposition of edge set of G into copies of F

Theorem (Glock, Kühn, Lo, Osthus 2017$^+$)

Suppose F is an r-graph and suppose that $K_n^{(r)}$ is F-divisible, where $n \gg |F|$. Then $K_n^{(r)}$ has an F-decomposition. (+ generalisation to dense quasirandom r-graphs)

- answers question of Keevash
- graph case $r = 2$ is due to Wilson
- can replace $K_n^{(r)}$ by any dense quasirandom r-graph G
- can prove design version with $\lambda > 1$
- effective minimum degree version if F is 'weakly regular'

Application: Graph decompositions and embeddings

Special case:

Theorem (Glock, Kühn, Lo, Osthus 2017^+)

Suppose G is a large quasi-random graph and F is fixed with
(i) $e(F)$ divides $e(G)$;
(ii) hcf$\{$degrees of $F\}$ divides hcf$\{$degrees of $G\}$.
Then G has an F-decomposition.

Theorem (Archdeacon)

If graph G has a decomposition into K_4's, K_5's and K_6's, then G has a self-dual embedding.

Corollary (Glock, Kühn, Lo, Osthus 2017^+)

Almost every graph has a self-dual embedding.

Proof sketch: Absorption

Suppose we seek a $K_q^{(r)}$-decomposition of an r-graph G

iterative absorption approach

Split up the absorbing process into many steps which gradually make leftover smaller and smaller.

$\Rightarrow$ final leftover L has bounded size and lies within prescribed set X
$\Rightarrow$ only boundedly many possibilities $H_1, \ldots, H_s$ for leftover L

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Split up the absorbing process into many steps which gradually make leftover smaller and smaller.

$\Rightarrow$ final leftover L has bounded size and lies within prescribed set X
$\Rightarrow$ only boundedly many possibilities $H_1, \ldots, H_s$ for leftover L
$\Rightarrow$ suffices to find an 'exclusive absorber' A_i for each H_i, i.e.

- $A_i \cup H_i$ has a $K_q^{(r)}$-decomposition
- A_i has a $K_q^{(r)}$-decomposition

Recall:

An exclusive absorber A for a potential leftover graph H satisfies

- $A \cup H$ has a $K_q^{(r)}$-decomposition
- A has a $K_q^{(r)}$-decomposition

We construct exclusive absorbers out of 'transformers'.

Ignore divisibility.

Definition

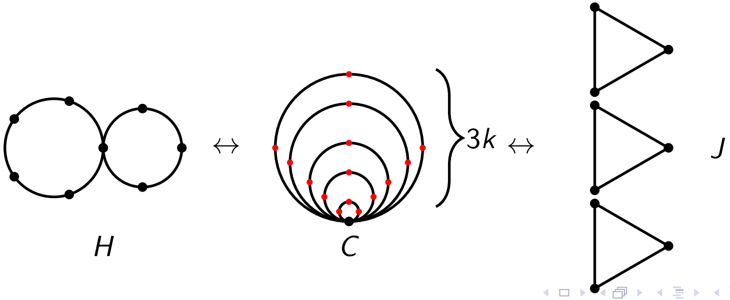
An r-graph T is an (H_1, H_2)-transformer if both $H_1 \cup T$ and $T \cup H_2$ have $K_q^{(r)}$-decompositions.

Aim: transform leftover H_1 step by step into r-graph which is trivially decomposable

The exclusive absorber: general idea

General Idea:

- construct absorber as concatenation of transformers
- show that each H can be transformed into 'canonical graph' C which only depends on $e(H)$
- by transitivity this implies that each H can be transformed into a disjoint union J of $K_q^{(r)}$, which is trivially decomposable

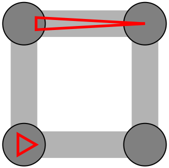


Open question: the decomposition threshold for graphs

Conjecture (Nash-Williams 1970)

Every large K_3-divisible graph G on n vertices with $\delta(G) \geq 3n/4$ has a K_3-decomposition.

Extremal example: blow up each vertex of C_4 to a K_m (m odd and divisible by 3).



Each triangle has at least one edge in one of the four cliques but less than a third of the edges lie inside the cliques.

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- true if $\delta(G) \geq (0.9 + o(1))n$ (Barber, Kühn, Lo, Osthus & Dross)
- showing that $\frac{3n}{4}$ guarantees 'fractional decomposition' or approx. decomposition would suffice
- conjectured threshold for K_q-decompositions: $\frac{qn}{q+1}$, partial results by Barber, Glock, Kühn, Lo, Montgomery, Osthus
- similar questions in partite setting, partial results by BKLMOT (applications to completions of partially filled latin squares)