Zeros and critical points of monochromatic random waves

Joint works with B.Hanin and P.Sarnak

07-19-2018

Yaiza Canzani

Results on $\mathbb{S}^n$ and $\mathbb{T}^n$

- $\dfrac{\#\{\text{critical points of } \Psi_\lambda\}}{\lambda^n}$

- $\dfrac{\mathcal{H}^{n-1}(Z_{\Psi_\lambda})}{\lambda}$

- $\dfrac{\#\left\{\text{components of } Z_{\Psi_\lambda}\right\}}{\lambda^n}$

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- $\dfrac{\#\{\text{critical points of } \Psi_\lambda\}}{\lambda^n} \xrightarrow{p} A_n$

  - $\mathbb{S}^2$ Nicolaescu '10
  Cammarota-Marinucci-Wigman '14
  Cammarota-Wigman '15

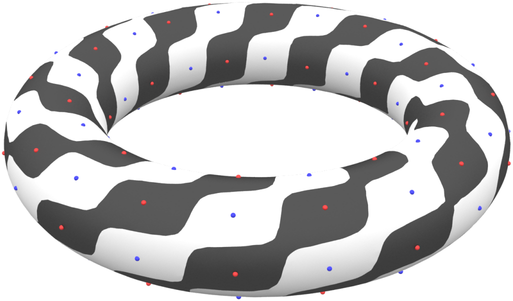
- $\dfrac{\mathcal{H}^{n-1}(Z_{\Psi_\lambda})}{\lambda} \xrightarrow{p} B_n$

  - $\mathbb{S}^2$ Neuheisel '00, Wigman '09, '10
  - $\mathbb{T}^2$ Rudnick-Wigman '07

- $\dfrac{\#\left\{\text{components of } Z_{\Psi_\lambda}\right\}}{\lambda^n} \xrightarrow{\mathbb{E}} C_n$

  - $\mathbb{S}^n, \mathbb{T}^n$ Nazarov-Sodin '07 ,'16

The topics of this talk:



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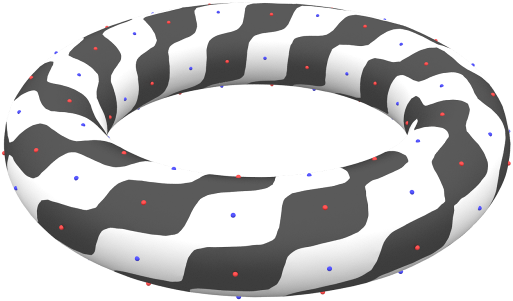

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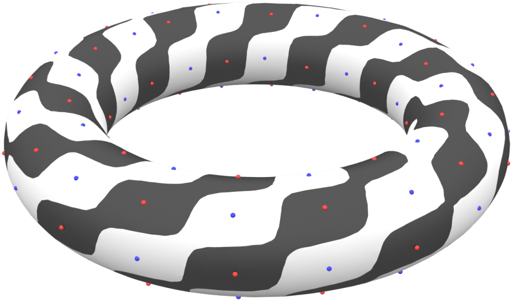

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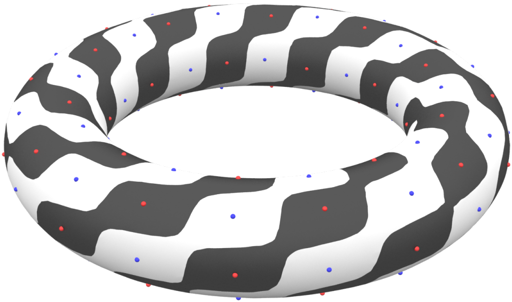

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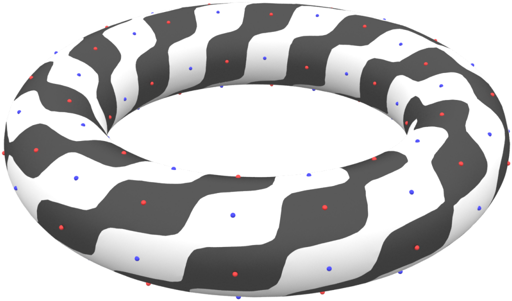

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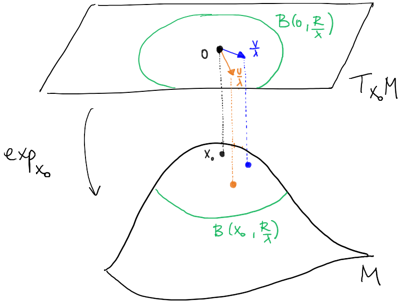

Universal behavior

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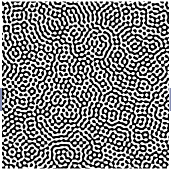

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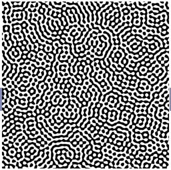

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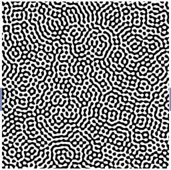

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Characterized by

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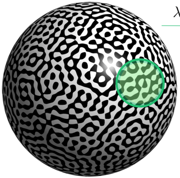

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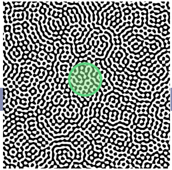

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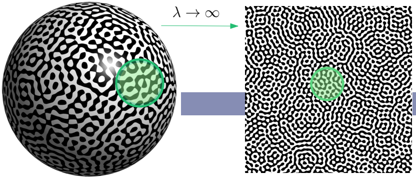

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Theorem (C-Hanin '15, '16)

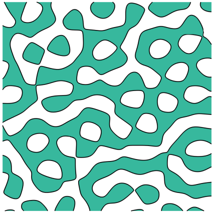
Let $x_0 \in M$. If measure {geodesic loops closing at x_0} $= 0$, then

$$\lim_{\lambda\to\infty} Cov_{\Psi_\lambda^{x_0}}(u,v) = Cov_{\Psi_\infty}(u,v),$$

uniformly in $u, v \in B(0,R)$ in the C^∞-topology. In particular,

$$\Psi_\lambda^{x_0}(u) \xrightarrow{d} \Psi_\infty(u).$$

Zero sets in $\frac{1}{\lambda}$ scales



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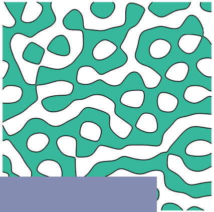

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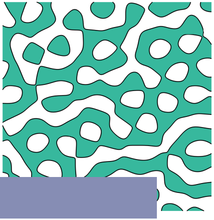

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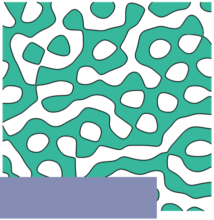

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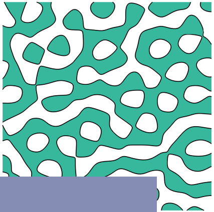

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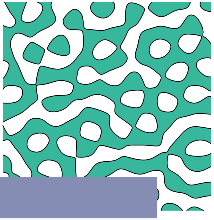

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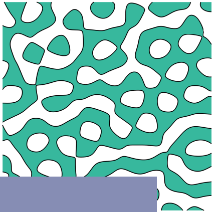

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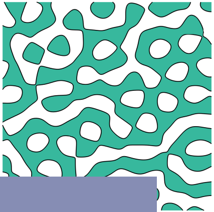

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Same is true *for Euler characteristic, Betti numbers, and topologies of components.*

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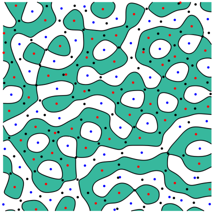
Critical points in $\frac{1}{\lambda}$ scales



$$\mathrm{Crit}_{\Psi_\lambda^{x_0}} := \frac{1}{\mathrm{Vol}(B_R)} \sum_{\substack{\nabla \Psi_\lambda^{x_0}(u)=0 \\ u \in B_R}} \delta_u$$

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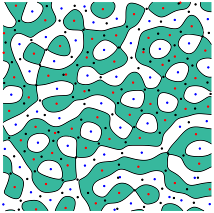

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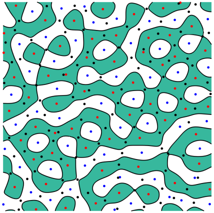

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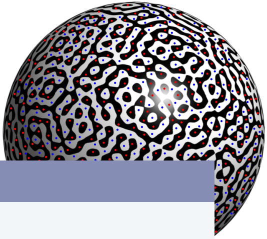
Theorem (C-Hanin '17)

Let $x_0 \in M$ with measure$\{$geodesic loops closing at $x_0\} = 0$. For every $m \in \mathbb{N}$

$$\lim_{\lambda \to \infty} \mathbb{E}\left[\mathrm{Crit}_{\Psi_\lambda^{x_0}}\right]^m = \mathbb{E}\left[\mathrm{Crit}_{\Psi_\infty}\right]^m$$

provided the limit is finite, which is true for $m = 1, 2$.

Global statistics



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If measure{geodesic loops closing at x}$= 0$ for a.e $x \in M$, then

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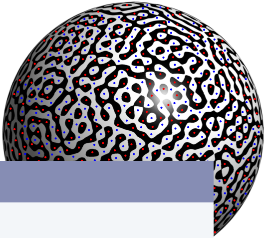

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If measure{geodesics joining x, y} $= 0$ for a.e. $x, y \in M$, then

$$Var\left[\frac{\#\{\text{critical points of } \Psi_\lambda\}}{\lambda^n}\right] = O\left(\lambda^{-\frac{n-1}{2}}\right)$$

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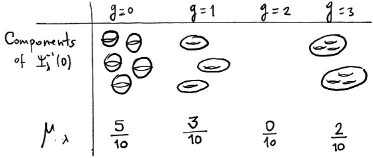
Distribution of diffeomorphism types



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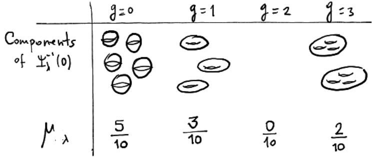

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$$\mathcal{H}(n-1) = \left\{ \begin{matrix} \text{compact manifold, dim=n-1, no bdry,} \\ \text{smooth, can be embedded in } \mathbb{R}^n \end{matrix} \right\} \Big/ \textbf{diffeos}$$

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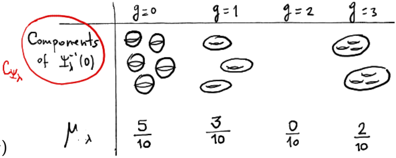

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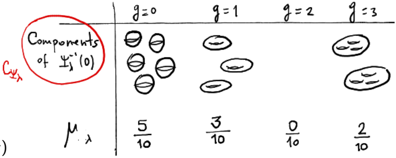

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Theorem (Sarnak-Wigman '13 combined with C-Hanin '15**)**

Let (M, g) *be s.t.* measure{ geodesic loops at x } $= 0$ *for a.e.* $x \in M$. *Then,*
$\exists\ \mu_\infty : \mathcal{H}(n-1) \to \mathbb{R}$ *probability measure so that*

$$\mu_\lambda \xrightarrow{\lambda \to \infty} \mu_\infty .$$

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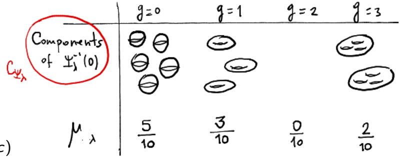

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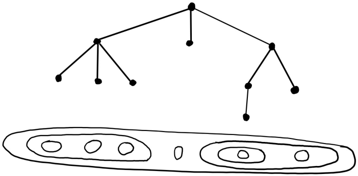
Let (M, g) *be s.t. measure{ geodesic loops at* x *}* $= 0$ *for a.e.* $x \in M$*. Then,*
$\exists\ \mu_\infty : \mathcal{H}(n-1) \to \mathbb{R}$ *probability measure so that*

$$\mu_\lambda \xrightarrow{\lambda \to \infty} \mu_\infty.$$

Theorem (C-Sarnak '15)

$$supp(\mu_\infty) = \mathcal{H}(n-1).$$

Distribution of nestings



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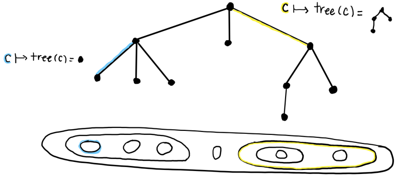

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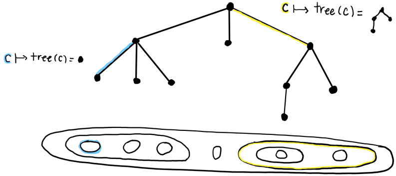

$\nu_\lambda : \mathcal{T} \to [0,1]$ $\qquad$ $\nu_\lambda = \dfrac{1}{|C_{\Psi_\lambda}|} \sum_{c \in C_{\Psi_\lambda}} \delta_{\mathrm{tree}(c)}$

$\mathcal{T}$ = space of finite rooted trees

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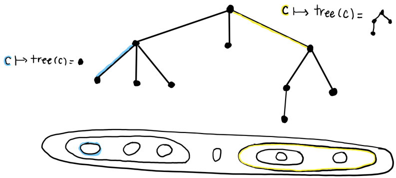

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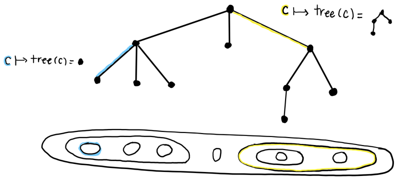

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Theorem (C-Sarnak '16 case $n = 2$ done by Sarnak-Wigman '13 **)**

$$supp(\nu_\infty) = \mathcal{T}.$$

$supp(\nu_\infty) = \mathcal{T}$: Given $T \in \mathcal{T}$ find $\Psi : \mathbb{R}^n \to \mathbb{R}$ so that $\begin{cases} -\Delta_{\mathbb{R}^n}\Psi = \Psi, \\ Z_\Psi \text{ contains a copy of } T. \end{cases}$

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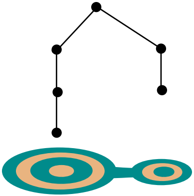

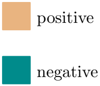

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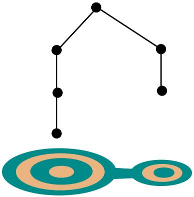
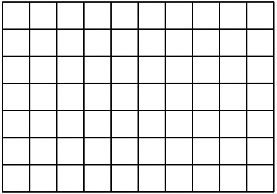
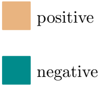
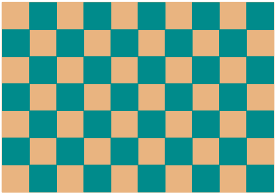






$\sin(x)\sin(y)$



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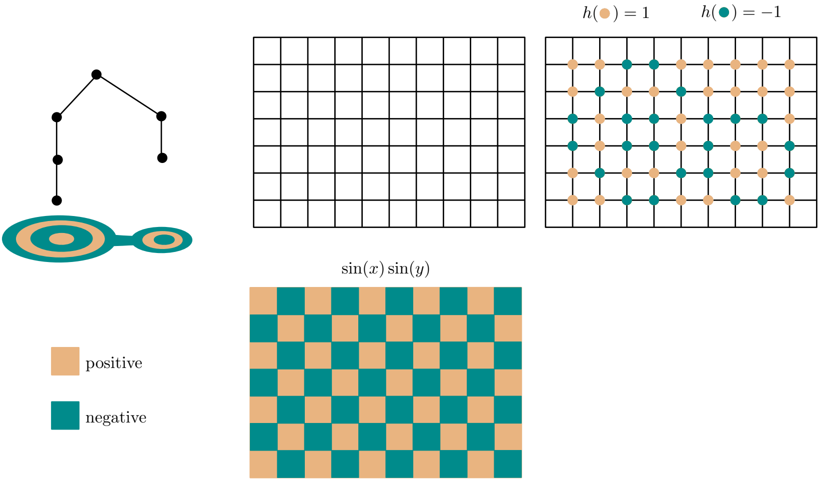

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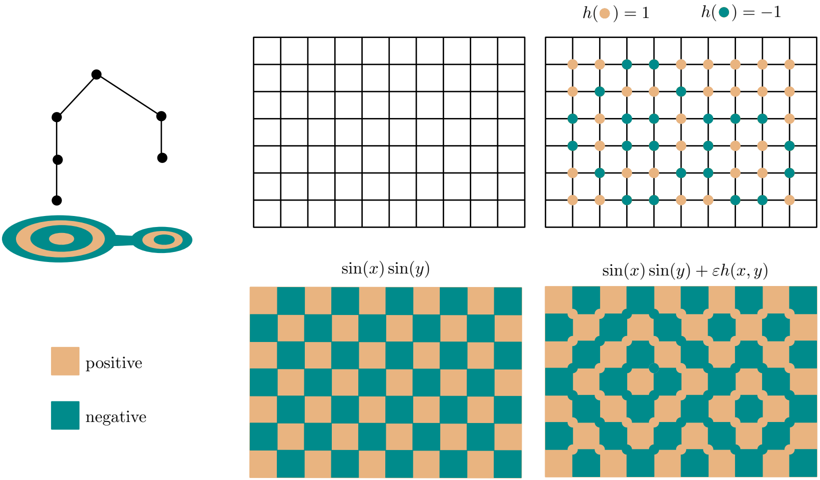

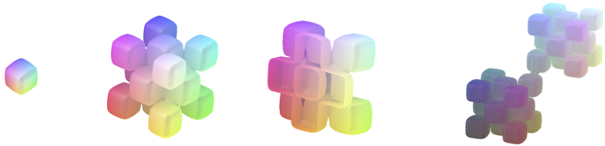

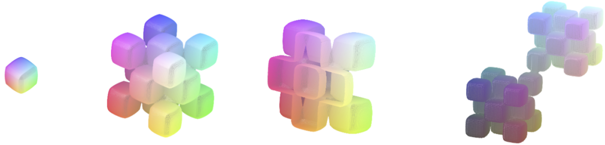

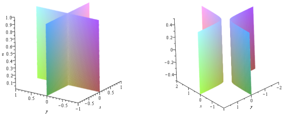

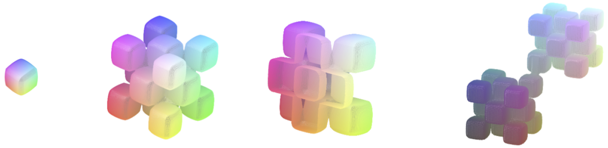

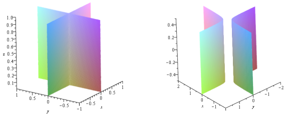

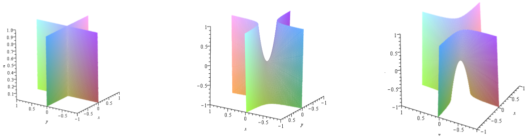

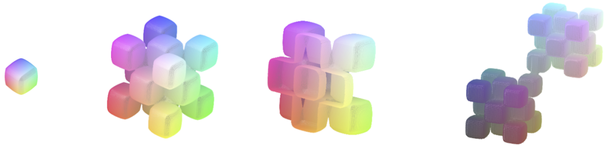

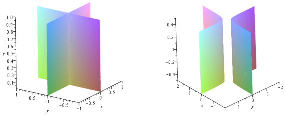

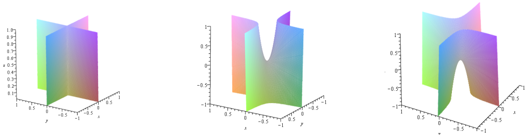

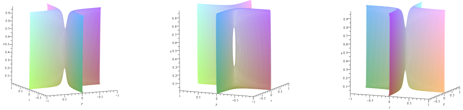

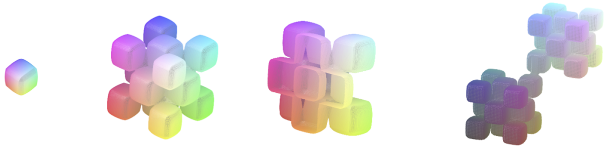

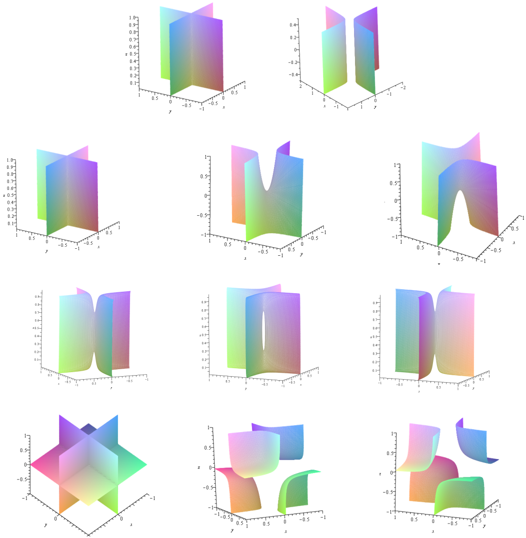

Thank you!