The infrared problems in QED. Some topics of current research.

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(Non-)existence of wave operators

Theorem

Let $H = H_0 + V(x)$, *where* $H_0 = -\frac{1}{2}\Delta$, $V(x) = \frac{\mathrm{e}^{-\mu|x|}}{|x|+1}$, $\mu \geq 0$. *Then the wave operator*

$$W^{\mathrm{out}} := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}.$$

1. *exists for* $\mu > 0$. *(Short-range potential).*
2. *does not exist for* $\mu = 0$. *(Long-range potential).*

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The infrared problem is the breakdown of conventional scattering theory due to slow decay of the interaction potential with distance.

Curing the infrared problem in Quantum Mechanics

Dollard prescription:

1. $H = H_0 + V(x)$
2. $H_{\mathrm{as}}(t) := H_0 + V(-\mathrm{i}\nabla_x t)$
3. $U_{\mathrm{as}}(t) := \mathrm{e}^{-\mathrm{i}\int_0^t H_{\mathrm{as}}(\tau)d\tau}$

Theorem

Let $V(x) = \frac{1}{|x|+1}$. Then:

1. $W^{\mathrm{out}} = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}$ *does not exist.*
2. $W_{\mathrm{D}}^{\mathrm{out}} := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}\int_0^t H_{\mathrm{as}}(\tau)d\tau}$ *exists.*

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Outline

Nelson model with many atoms/electrons

Definition

The Nelson model with many atoms/electrons is given by:

(1) Hilbert space $\mathcal{H} = \Gamma(L^2(\mathbb{R}^3, dp)_{\mathrm{at/el}}) \otimes \Gamma(L^2(\mathbb{R}^3, dk)_{\mathrm{ph}})$.

(2) Hamiltonian $H = (H_{\mathrm{at/el}} \otimes 1) + (1 \otimes H_{\mathrm{ph}}) + V$, where

(a) $H_{\mathrm{at/el}} = \int dp\, \frac{p^2}{2}\, c_p^* c_p$,

(b) $H_{\mathrm{ph}} = \int dk\, |k| a_k^* a_k$,

(c) $V = g \int dp\, dk\, \frac{\tilde{\rho}(k)}{\sqrt{2|k|}} (c_{p+k}^* c_p \otimes a_k + c_p^* c_{p+k} \otimes a_k^*)$.

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1. For $\tilde{\rho}(0) = 0$ we call the massive particle an atom.
2. For $\tilde{\rho}(0) \neq 0$ we call the massive particle an electron.

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(3) Momentum operator: $P = \int dp\, p\, c_p^* c_p + \int dk\, k\, a_k^* a_k$.

Nelson model with N atoms/electrons

Definition

The Nelson model with N atoms/electron is given by:

(1) Hilbert space $\mathcal{H}^{(N)} = L^2_{\mathrm{s/a}}(\mathbb{R}^{3N}, dx)_{\mathrm{at/el}} \otimes \Gamma(L^2(\mathbb{R}^3, dk)_{\mathrm{ph}})$.

(2) Hamiltonian $H^{(N)} = H^{(N)}_{\mathrm{at/el}} + H_{\mathrm{ph}} + V$, where

(a) $H^{(N)}_{\mathrm{at/el}} = -\frac{1}{2}\sum_{i=1}^{N} \Delta_{x_i}$,

(b) $H_{\mathrm{ph}} = \int dk\, |k| a^*_k a_k$,

(c) $V = g \sum_{i=1}^{N} \int dk\, \frac{\tilde{\rho}(k)}{\sqrt{2|k|}} (e^{-\mathrm{i}kx_i} a^*_k + e^{\mathrm{i}kx_i} a_k)$.

(3) Momentum operator: $P^{(N)} = \sum_{i=1}^{N}(-\mathrm{i}\nabla_{x_i}) + \int\, dk\, k\, a^*_k a_k$.

Nelson model with one atom/electron

Definition

The Nelson model with one atom/electron is given by:

(1) Hilbert space $\mathcal{H}^{(1)} = L^2(\mathbb{R}^3, dp)_{\mathrm{at/el}} \otimes \Gamma(L^2(\mathbb{R}^3, dk)_{\mathrm{ph}})$.

(2) Hamiltonian $H^{(1)} = H^{(1)}_{\mathrm{at/el}} + H_{\mathrm{ph}} + V(x)$, where

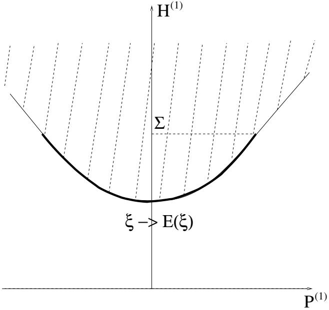
(a) $H^{(1)}_{\mathrm{at/el}} = -\frac{1}{2}\Delta_x$,

(b) $H_{\mathrm{ph}} = \int dk\, |k| a_k^* a_k$,

(c) $V(x) = g \int dk\, \frac{\tilde{\rho}(k)}{\sqrt{2|k|}}(e^{-\mathrm{i}kx} a_k^* + e^{\mathrm{i}kx} a_k)$.

(3) Momentum operator: $P^{(1)} = -\mathrm{i}\nabla_x + \int\, dk\, k\, a_k^* a_k$.

Spectral properties



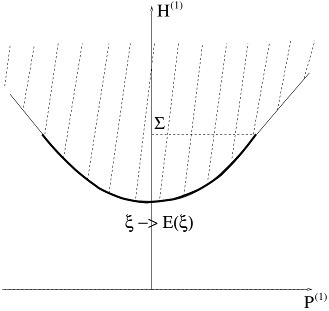
Theorem (Fröhlich 73... Abdesselam-Hasler 10)

There exist $\Sigma > \inf \sigma(H^{(1)})$ *and* $g > 0$ *s.t. for* $E(\xi) \leq \Sigma$*.*

(a) $|\nabla E(\xi)| < 1$,

(b) $\xi \to \nabla E(\xi)$ *is invertible.*

Neutral particle (atom)



Suppose that $\tilde{\rho}(0) = 0$ i.e. the massive particle is an atom.
Then, (generically),

$$\mathcal{H}_{\mathrm{sp}} := \{\text{Spectral subspace of the lower boundary}\} \neq \{0\}$$

Asymptotic creation operators of photons

Definition

For $h \in C_0^\infty(\mathbb{R}^3)$ we define

$$a_t^*(h) := \mathrm{e}^{\mathrm{i}Ht} a^*(e^{-\mathrm{i}|k|t} h)\mathrm{e}^{-\mathrm{i}Ht},$$

which is called (the approximating sequence of) the asymptotic creation operator of a photon.

Scattering states of one atom and photons

Theorem (Hoegh-Krohn 69...Griesemer-Zenk 09)

For any $h_i \in C_0^\infty(\mathbb{R}^3)$ *and* $\Psi \in \mathcal{H}_{\mathrm{sp}}$ *there exist scattering states*

$$\Psi^{\mathrm{out}} = \lim_{t\to\infty} a_t^*(h_1)\ldots a_t^*(h_n)\Psi$$

and span a subspace naturally isomorphic to $\Gamma(L^2(\mathbb{R}^3, dk)_{\mathrm{ph}}) \otimes \mathcal{H}_{\mathrm{sp}}$.

Renormalized creation operators of atoms

1. Since $H^{(1)}$ commutes with $P^{(1)}$, we can diagonalize:

$$H^{(1)} = \Pi^* \int^{\oplus} d\xi \, H^{(1)}(\xi) \, \Pi, \quad P^{(1)} = \Pi^* \int^{\oplus} d\xi \, \xi \, \Pi,$$

where $H^{(1)}(\xi)$ are operators on $\Gamma(L^2(\mathbb{R}^3, dk))$.

2. Let $\psi_\xi \in \Gamma(L^2(\mathbb{R}^3, dk))$ be ground-states of $H^{(1)}(\xi)$ i.e.

$$H^{(1)}(\xi)\psi_\xi = E(\xi)\psi_\xi.$$

3. Let us define the renormalized creation operators of atoms:

$$\hat{c}^*(h) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int d\xi \int_{\mathbb{R}^{3n}} dk \, h(\xi) \psi_\xi^{(n)}(k_1, \ldots, k_n) a^*_{k_1} \ldots a^*_{k_n} c^*_{\xi - \underline{k}},$$

where $\{\psi_\xi^{(n)}\}_{n \geq 0}$ are components of ψ_ξ and $h \in C_0^\infty(\mathbb{R}^3)$.

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2. Let $\psi_\xi \in \Gamma(L^2(\mathbb{R}^3))$ be ground-states of $H^{(1)}(\xi)$ i.e.

$$H^{(1)}(\xi)\psi_\xi = E(\xi)\psi_\xi.$$

3. The renormalized creation operators of atoms satisfy:

$$\hat{c}^*(h)\Omega \in \mathcal{H}_{\rm sp}$$

Asymptotic creation operators of atoms

Definition

For $h \in C_0^\infty(\mathbb{R}^3)$ we define

$$\hat{c}_t^*(h) := \mathrm{e}^{\mathrm{i}Ht} \hat{c}^*(\mathrm{e}^{-\mathrm{i}Et} h) \mathrm{e}^{-\mathrm{i}Ht},$$

which is called (the approximating sequence of) the asymptotic creation operator of an atom.

Scattering states of two atoms

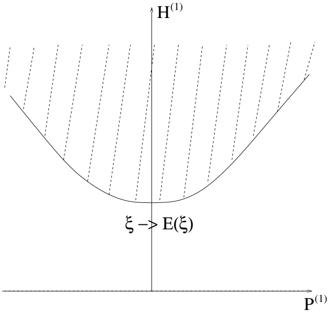
Theorem (Pizzo-W.D.)

For $h_1, h_2 \in C_0^\infty(\mathbb{R}^3)$ with disjoint supports the limits

$$\Psi^{\rm out} := \lim_{t\to\infty} \hat{c}_t^*(h_1)\hat{c}_t^*(h_2)\Omega$$

exist and span a subspace naturally isomorphic to $\mathcal{H}_{\rm sp} \otimes_{\rm s/a} \mathcal{H}_{\rm sp}$.

Charged particle (electron)



Theorem (Fröhlich 74...Hasler-Herbst 07)

$\mathcal{H}_{\rm sp} = \{Spectral\ subspace\ of\ the\ lower\ boundary\} = \{0\}$ *for* $\tilde{\rho}(0) \neq 0$, $g \neq 0$.

Remark: Electron is an infraparticle.

Dollard's formalism

Let us construct a scattering state describing N electrons with velocities $\underline{v} = (v_1, \dots, v_N)$ following the Dollard's prescription:

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3. $H_{\mathrm{as},\underline{v}}(t) := \hat{H}_0 + V_{\mathrm{as},\underline{v}}(t)$

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4. $U_{\mathrm{as},\underline{v}}(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_0 t}\, T \mathrm{e}^{-\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}\hat{H}_0\tau} V_{\mathrm{as},\underline{v}}(\tau) \mathrm{e}^{-\mathrm{i}\hat{H}_0\tau} d\tau}$

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Candidate scattering states approximants have the form:

$$\Psi_{(N)}(t) := \sum_{\underline{v}} e^{\mathrm{i}Ht} U_{\mathrm{as},\underline{v}}(t) \prod_{i=1}^{N} c^*(h_{v_i})\Omega$$

Electron scattering states

After (heuristic) rearrangements [W.D. *Nucl.Phys.B*, 2017]:

$$\Psi_{(N)}(t) = \sum_{\underline{v}} \left(\prod_{i=1}^{N} W_t(v_i) \right) \prod_{i=1}^{N} \hat{c}_t^*(h_{v_i})\Omega$$

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$$G_v(k) := g \frac{\tilde{\rho}(k)}{\sqrt{2|k|}} \frac{1}{|k| - kv}$$

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1. $W_t(v)$ involves the function G_v which is not square-integrable. It requires a regularization.

Electron scattering states

After (heuristic) rearrangements [W.D. *Nucl.Phys.B*, 2017]:

$$\Psi_{(N),\sigma}(t) = \sum_{\underline{v}} \left(\prod_{i=1}^{N} W_{\sigma,t}(v_i) \right) \prod_{i=1}^{N} \hat{c}_t^*(h_{v_i})\Omega$$

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Electron scattering states

For $\sigma_t \to 0$ as $t \to \infty$:

Theorem (Pizzo 05)

The following one-electron scattering state exist and are non-zero:

$$\Psi^{\mathrm{out}}_{(N=1)} = \lim_{t\to\infty} \sum_{v} W_{\sigma_t,t}(v)\hat{c}^*_{\sigma_t,t}(h_v)\Omega.$$

Conjecture (Pizzo-W.D.)

The following two-electron scattering states exist and are non-zero:

$$\Psi^{\mathrm{out}}_{(N=2)} = \lim_{t\to\infty} \sum_{v_1,v_2} W_{\sigma_t,t}(v_1)W_{\sigma_t,t}(v_2)\hat{c}^*_{\sigma_t,t}(h_{v_1})\hat{c}^*_{\sigma_t,t}(h_{v_2})\Omega.$$

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Relativistic QFT

Definition

A relativistic QFT is given by:

(1) A net of local algebras $\mathbb{R}^4 \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset B(\mathcal{H})$ s.t.

(a) If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.

(b) If $\mathcal{O}_1 \times \mathcal{O}_2$ then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$.

(2) A Hamiltonian H and momentum operators P s.t.

(a) Joint spectrum of H and P is in the closed future lightcone.

(b) If $A \in \mathcal{A}(\mathcal{O})$ then

$$A(t,x) := e^{\mathrm{i}(Ht-Px)} A e^{-\mathrm{i}(Ht-Px)} \in \mathcal{A}(\mathcal{O} + (t,x)).$$

Relativistic QED

Definition (Fredenhagen-Hertel 81, Bostelmann 04)

A quadratic form ϕ is a pointlike field of a relativistic QFT, if there exist:

(a) $A_r \in \mathcal{A}(\mathcal{O}_r)$, *where $\mathcal{O}_r$ is the ball of radius r centered at zero,*

(b) $k > 0$,

s.t. $\|(1+H)^{-k}(\phi - A_r)(1+H)^{-k}\| \underset{r\to 0}{\to} 0.$

Definition

- Relativistic QED is a QFT whose pointlike fields include the Faraday tensor F and a conserved current j which satisfy the Maxwell equations: $dF = 0$, $d * F = j$.
- The electric charge exists and is given (formally) by $Q = \int dx\, j^0(x)$.

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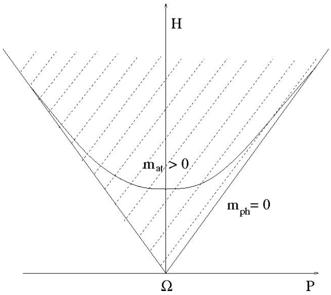
Vacuum representation of QED



We assume:

(a) Existence of the vacuum vector Ω. (We set $\mathcal{H}_0 = \mathbb{C}\Omega$).

(b) Non-triviality of

$$\mathcal{H}_{\mathrm{sp}} = \mathbf{1}_{\{m_{\mathrm{ph}}^2\}}(H^2 - P^2)\mathcal{H}_0^{\perp} \oplus \mathbf{1}_{\{m_{\mathrm{at}}^2\}}(H^2 - P^2)\mathcal{H}.$$

(c) Hölder cont. of the spectrum of $(H^2 - P^2)$ near $\{m_{\mathrm{ph}}^2, m_{\mathrm{at}}^2\}$.

Asymptotic creation operators

Definition

(a) Free dynamics: $\hat{h}_t(x) := \int \frac{dk}{(2\pi)^3} \mathrm{e}^{-\mathrm{i}\omega(k)t+\mathrm{i}kx} \hat{h}(k)$,
$\omega(k) = \sqrt{k^2 + m^2}$.

(b) Interacting dynymics: $A^*(t,x) = \mathrm{e}^{\mathrm{i}(Ht-Px)} A^* \mathrm{e}^{-\mathrm{i}(Ht-Px)}$,
$A^* \in \mathcal{A}(\mathcal{O})$.

(c) LSZ creation operator: $A^*_t(\hat{h}) := \int dx \, \hat{h}_t(x) A^*(t,x)$.

(d) HR creation operator: $A^*_T(\hat{h}) := \frac{1}{\ln|T|} \int_T^{T+\ln|T|} dt \, A^*_t(\hat{h})$.

Remark: $h := \lim_{T\to\infty} A^*_T(\hat{h})\Omega$ exists and is a single-particle state.

Scattering states of atoms and photons

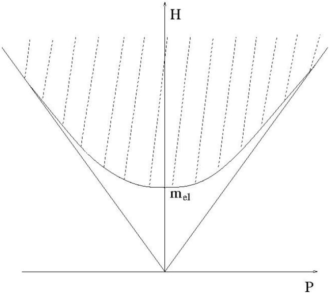
Theorem (W.D. 05, Herdegen 12, Herdegen-Duch 14, Duell 16)

Suppose the particles $h_i = \lim_{T\to\infty} A^*_{i,T}(\hat{h}_i)\Omega$ *have disjoint velocity supports, separated from zero. Then there exist the scattering states*

$$\Psi^{\mathrm{out}} = \lim_{T\to\infty} A^*_{1,T}(\hat{h}_1)\dots A^*_{n,T}(\hat{h}_n)\Omega.$$

Such states span a subspace naturally isomorphic to $\Gamma(\mathcal{H}_{\mathrm{sp}})$.

Charged representations



Theorem (Buchholz 86)

$\mathcal{H}_{\rm sp} := \mathbf{1}_{\{m_{\rm el}^2\}}(H^2 - P^2)\mathcal{H} = \{0\}$ *in charged representations of QED.*

Remark: Electron is an infraparticle.

Charged representations

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1. In non-relativistic QED we could construct scattering states in this situation starting from the Dollard prescription.
2. In the relativistic setting this strategy does not seem feasible.
3. But there is a different strategy [Buchholz-Roberts 13]: Consider representations in which the flux f does not exist. 'Infravacuum representations'.
4. In such representations one can hope for $\mathcal{H}_{\mathrm{sp}} \neq \{0\}$.

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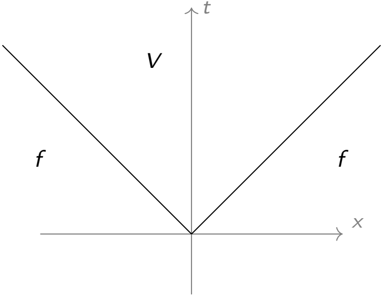

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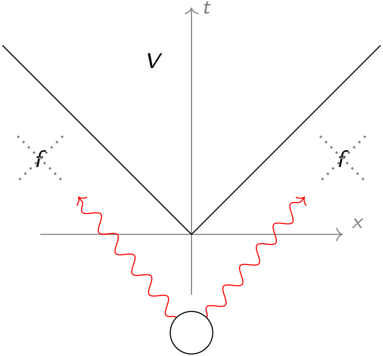

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2. Such radiation does not enter the future lightcone V.

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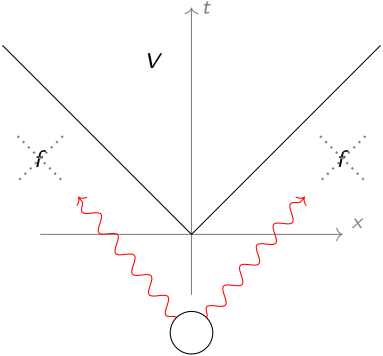

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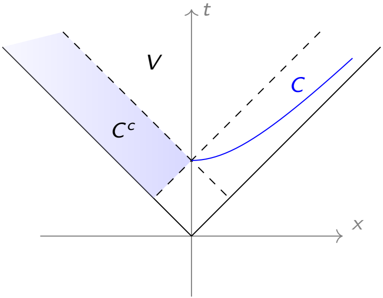
Hyperbolic geometry



1. V : future lightcone.
2. C : hyperbolic cone in V.
3. C^c : causal complement of C in V.
4. $\mathcal{C} := C^{cc}$: hypercone.
5. $\mathcal{F}$: family of admissible hyperbolic cones.

Hypercone localized representations

Definition (Buchholz-Roberts 13)

Let $\mathcal{A}$ be the algebra of observables in the vacuum representation.

We say that a (covariant, positive energy) representation $\pi : \mathcal{A} \to B(\mathcal{H}_\pi)$ is hypercone localized if for any $\mathsf{C} \in \mathcal{F}$

$$\pi \restriction \mathcal{A}(\mathsf{C}^c) \simeq \mathrm{id} \restriction \mathcal{A}(\mathsf{C}^c).$$

Scattering states of one electron and photons

Let $(\pi(\mathcal{A}), H_\pi, P_\pi)$ be a hypercone localized representation π, containing massive particles (electrons). That is

$$\mathcal{H}_{\pi,\mathrm{sp}} := \mathbf{1}_{\{m_{\mathrm{el}}^2\}}(H_\pi^2 - P_\pi^2)\mathcal{H}_\pi \neq 0.$$

Theorem (Alazzawi-W.D. 15)

There exist scattering states of one electron and n-photons:

$$\Psi^{\mathrm{out}} := \lim_{T\to\infty} A_{1,T}^*(\hat{h}_1)\dots A_{n,T}^*(\hat{h}_n)\Psi_{\mathrm{el}}, \quad \Psi_{\mathrm{el}} \in \mathcal{H}_{\pi,\mathrm{sp}}, \quad A_i \in \pi(\mathcal{A}).$$

They span a subspace naturally isomorphic to $\Gamma(\mathcal{H}_{\mathrm{sp}}) \otimes \mathcal{H}_{\pi,\mathrm{sp}}$.

Summary of scattering theory

Scattering states of:	NRQED	RQED
one atom and photons	🟢	🟢
many atoms and photons	🟢	🟢
one electron and photons	🟢	🟢
electron and atom	🟢	🔴
many electrons and photons	🔴	🔴

🔴 - not understood

🟢 - partially understood

Sectors

1. $\mathcal{A}$ - C^*-algebra.
2. $P_{\mathcal{A}}$ - pure states.
3. $\mathrm{In}\,\mathcal{A} \subset \mathrm{Aut}\,\mathcal{A}$ - inner automorphisms.
4. $X := P_{\mathcal{A}}/\mathrm{In}\,\mathcal{A}$ - sectors.

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Question: Can this be done without locality?

(Second) conjugate classes

1. $G \subset \mathrm{Aut}\,\mathcal{A}$.
2. $X \times G \ni (x, g) \mapsto x \cdot g \in X$ - group action on X.

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2. For $x \in X$ set $G^a_{x,x_0} := \{ g \in G \,|\, x = x_0 \cdot a \cdot g \}$.
3. $\overline{[x]}^a := \{ x_0 \cdot a \cdot g^{-1} \,|\, g \in G^a_{x,x_0} \}$ is called the conjugate class.
4. $\overline{\overline{[x]}}^a := \{ x_0 \cdot a \cdot (g')^{-1} \,|\, g' \in G^a_{y,x_0},\ y \in \overline{[x]}^a \}$ is called the second conjugate class.

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Claim: second conjugate classes are meaningful candidates for 'charge classes' in the absence of locality.

Main general result

Theorem (Cadamuro-W.D. 18)

Let $R \subset S \subset G$ be subgroups. Suppose that

1. $x_0 \cdot r = x_0$ *for all* $r \in R$.
2. $x_0 \cdot s \neq x_0$ *may hold for some* $s \in S$.
3. $a \cdot S \cdot a^{-1} \subset R$.

Then, $\overline{[x_0 \cdot s]}^a = \overline{[x_0]}^a$ *and* $\overline{\overline{[x_0 \cdot s]}}^a = \overline{\overline{[x_0]}}^a$ *for all* $s \in S$.

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Definition

The relative normalizer of $R \subset S \subset G$ is defined as

$$N_G(R, S) := \{\, g \in G \,|\, g \cdot S \cdot g^{-1} \subset R \,\}.$$

Existence of relative normalizers for $R \subsetneq S$

1. 'Tension': $R \subsetneq S$ vs $N_G(R,S) := \{ g \in G \,|\, g \cdot S \cdot g^{-1} \subset R \}$.
2. Hence relative normalizers are empty for
   - abelian groups,
   - finite groups,
   - finite-dimensional Lie groups (under some assumptions).
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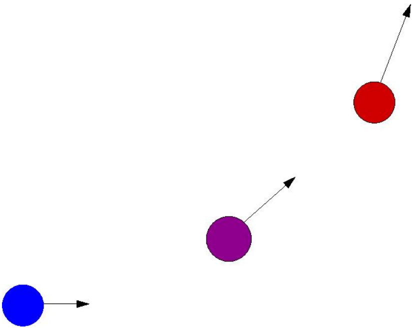

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Thm (Fröhlich 74, Chen-Fröhlich-Pizzo 09, Könenberg-Matte 14)

For any $\xi \in \mathcal{S}$ the following limits exist and define states on $\mathcal{A}$

$$\omega_\xi(A) := \lim_{\sigma\to 0}\langle \psi_{\xi,\sigma}, \pi_0(A)\psi_{\xi,\sigma}\rangle, \qquad A \in \mathcal{A}.$$

The corresponding sectors are mutually disjoint i.e.

$$[\omega_{\xi_1}]_{\mathrm{In}\mathcal{A}} \neq [\omega_{\xi_2}]_{\mathrm{In}\mathcal{A}} \quad \text{for} \quad \xi_1 \neq \xi_2.$$

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Theorem (Cadamuro-W.D. 18)

Let $\hat{T}$ be the KPR infravacuum. Then, for all $\xi_1, \xi_2 \in \mathcal{S}$

$$\overline{[[\omega_{\xi_1}]_{\mathrm{In}\mathcal{A}}]}^{\alpha_{\hat{T}}} = \overline{[[\omega_{\xi_2}]_{\mathrm{In}\mathcal{A}}]}^{\alpha_{\hat{T}}}, \quad \text{and} \quad \overline{\overline{[[\omega_{\xi_1}]_{\mathrm{In}\mathcal{A}}]}}^{\alpha_{\hat{T}}} = \overline{\overline{[[\omega_{\xi_2}]_{\mathrm{In}\mathcal{A}}]}}^{\alpha_{\hat{T}}}.$$

Outlook

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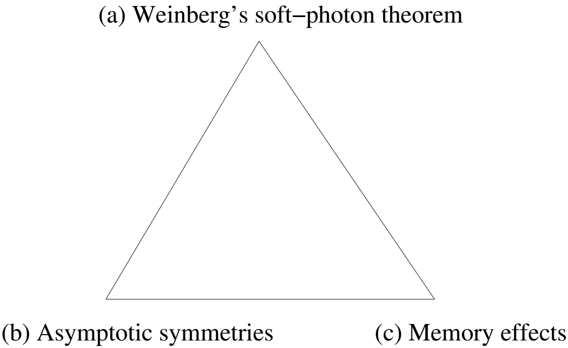
Infravacuum approach:

Infravacuum representations ⟶ Sharp mass of electron ⟶ YFS algorithm

Strominger's infrared triangle



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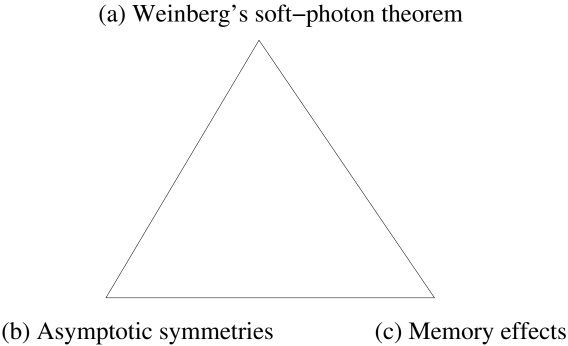

(a) Soft-photon theorem:

$$\langle \mathrm{out}|a_{+}^{\mathrm{out}}(q)S|\mathrm{in}\rangle = e\Bigg[\sum_{k=1}^{m}\frac{Q_k^{\mathrm{out}}p_k^{\mathrm{out}}\cdot\varepsilon^{+}}{p_k^{\mathrm{out}}\cdot q} - \sum_{k=1}^{n}\frac{Q_k^{\mathrm{in}}p_k^{\mathrm{in}}\cdot\varepsilon^{+}}{p_k^{\mathrm{in}}\cdot q}\Bigg]\langle \mathrm{out}|S|\mathrm{in}\rangle + O(q^0)$$

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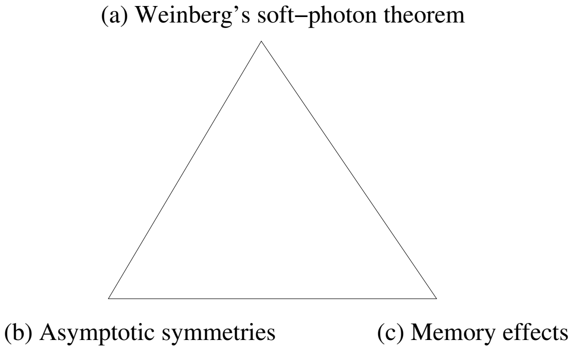

(b) Asymptotic symmetries:

$$f(n) := \lim_{r\to\infty} r^2 n^i F^{0i}(nr).$$