KP theory, planar bipartite networks in the disk and rational degenerations of M-curves

Simonetta Abenda (UniBo)
&
Petr G. Grinevich (LITP,RAS)

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Goal: Connect totally non–negative Grassmannians to $\mathtt{M}$–curves through finite–gap KP theory

$$\mathrm{KP-II\ equation:}\ \ (-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

Two relevant classes of solutions:

• Real regular multiline KP solitons which are in natural correspondence with totally non–negative Grassmannians [Chakravarthy-Kodama; Kodama-Williams];

• Real regular finite–gap KP solutions parametrized by degree g real regular non–special divisors on genus g $\mathtt{M}$–curves [Dubrovin-Natanzon]

Novikov: Soliton solutions are obtained from regular finite gap ones in the so called solitonic limit (= some cycles degenerate to double points)+ real regular soliton solutions should be obtainable from degeneration of real regular finite–gap solutions

Krichever: Finite–gap theory goes through also for degenerate solutions (ex. solitons) on reducible curves

Postnikov: Parametrization via planar bipartite networks in the disk of positroid cells (= Gelfand-Serganova stratum + positivity) of totally non–negative Grassmannians

• Problem 1: Start from soliton data in totally non–negative Grassmannians and canonically associate rational degenerations of M–curves and real regular divisors to such data :

◇ [AG - CMP 2018]: We construct real and regular divisors on rational degenerations of smooth genus g M–curves for any soliton data in $Gr^{\mathrm{TP}}(k,n)$, with $g = k(n-k)$ minimal using classical total positivity;

◇ [AG - Arxiv Dec. 2017]: To any planar bipartite directed trivalent perfect graph $\mathcal{G}$ in the disk with $g+1$ faces in Postnikov class representing a given $|D|$–dimensional positroid cell in $Gr^{\mathrm{TNN}}(k,n)$ we associate the rational degeneration of a genus g M–curve, Γ, and locally parametrize the cell with degree g non special real and regular divisors on Γ.

◇ [AG - Arxiv May 2018]: $g = |D|$ if $\mathcal{N}$ is the Le–network + explicit relation to construction in [AG- CMP 2018].

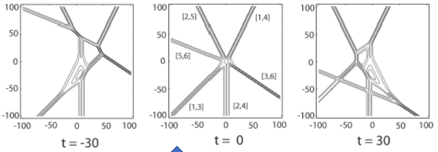
◇ [AG- Arxiv Dec 2017]: Effect of Postnikov moves and reductions (which transform networks preserving the point in $Gr^{\mathrm{TNN}}(k,n)$) on curves and divisors.

• Problem 2: Reconstruct soliton data in the Grassmannian from real and regular divisors on reducible rational curves is at an early stage.

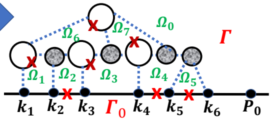
In [A-JGP2017]: Start from Γ, a rational degeneration of a hyperelliptic curve of genus $n-1$ canonically associated to soliton data in $Gr^{TP}(1,n)$ in our construction and identify soliton data in $Gr^{\mathrm{TNN}}(k,n)$, $k>1$, parametrized by real and regular KP-II divisors on Γ. This special family of $(n-k,k)$–line solitons naturally linked to the finite Toda lattice.

Multiline soliton solutions in totally non–negative Grassmannians

u(x,y,t) is a (3,3)-line soliton KP-II solution belonging to a 7-dim positroid cell of $Gr^{TNN}(3,6)$: k=3, n=6 π= (4,5,1,2,6,3)



Γ is reducible and the rational degeneration of a smooth genus 7 M-curve



Chakravarthy+Kodama, Kodama+Williams:

Asymptotic behavior in (x,y) at t fixed + tropical limit of soliton web of u(x,y,t) via Postnikov combinatorial characterization of Gr^{TNN}(k,n), the totally non-negative part of Gr(k,n), connection to cluster algebras of Fomin-Zelevinsky

A-Grinevich :

u(x,y,t) is obtained in the solitonic limit from real quasi-periodic KP-II solutions. Combining Krichever and Dubrovin + Natanzon, we construct reducible real curves $\Gamma(G)$ which are rational degenerations of M-curves using G Postnikov planar bipartite graphs in the disk to represent the cell in Gr^{TNN}(k,n) + real regular divisors satisfying Sato constraint

- Start from the soliton data: n phases $\boxed{\mathcal{K} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_n\}}$

a real $k \times n$ matrix, $\boxed{A = (A^i_j)}$

- Take $\boxed{f^{(i)}(x,y,t) = \sum_{j=1}^{n} A^i_j \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t), \; i \in [k],}$

- Take their Wronskian:

$$\boxed{\tau(\mathbf{t}) = \mathrm{Wr}_{\mathrm{x}}(f^{(1)}, \ldots, f^{(k)}) = \sum_{1\le j_1<\cdots<j_k\le n} \Delta(j_1,\ldots,j_k)(A) E(j_1,\ldots,j_k; x,y,t),}$$

$\Delta(j_1,\ldots,j_k)(A)$ is the maximal minor of the matrix A associated to the columns $j_1 < \cdots < j_k$

$E(j_1,\ldots,j_k; x,y,t) = \prod_{1\le l<s\le k}(\kappa_{j_s} - \kappa_{j_l}) \prod_{l=1}^{k} \exp(\kappa_{j_l} x + \kappa_{j_l}^2 y + \kappa_{j_l}^3 t)$

- Obtain a KP $(n-k,k)$–soliton solution: $\boxed{u(x,y,t) = 2\partial_x^2 log(\tau(x,y,t))}$.

Total positivity and Sato's Grassmannian reduction for the KP multi-line solitons

• The KP solution $u(x, y, t)$ is the same if we recombine linearly the k rows of the matrix A.

That is the soliton datum is the point in the finite dimensional real Grassmannian $Gr(k, n)$ represented by the matrix A.

• In the denominator of u, we have a linear combination of exponential functions with real coefficients $\Delta(j_1, \ldots, j_k)(A) \prod_{1 \le l < s \le k} (\kappa_{j_s} - \kappa_{j_l})$.

The solution u is bounded for real (x, y, t) if and only if all the minors $\Delta(j_1, \ldots, j_k)(A) \ge 0$, *i.e.* $[A]$ is in the totally non–negative part of the Grassmannian, $Gr^{\text{TNN}}(k, n) \equiv GL_k^+ \backslash Mat_{k,n}^{\text{TNN}}$ [Kodama-Williams 2013];

• The functions $f_i(x, y, t)$, $i \in [k]$, are a basis of solutions to the linear ODE

$$\boxed{\mathfrak{D} f^{(i)} = 0}, \qquad \mathfrak{D} = \partial_x^k - w_1(x, y, t)\partial_x^{k-1} - \ldots - w_k(x, y, t)$$

We may express such differential operator $\mathfrak{D} = W \partial_x^k$ through a Dressing operator W satisfying Sato equations.

$\boxed{\mathfrak{D} f^{(i)} = 0}$, $\quad \mathfrak{D} = \partial_x^k - w_1(\mathbf{t})\partial_x^{k-1} - \ldots - w_k(\mathbf{t}) = W\partial_x^k$

$W = 1 - w_1\partial_x^{-1} - w_2\partial_x^{-2} - \cdots - w_k\partial_x^{-k}$ is a Dressing operator!

$L = W\partial_x W^{-1}$, satisfies the KP hierarchy

$$\begin{cases} L\Psi(\lambda,\mathbf{t}) = \lambda\Psi(\lambda,\mathbf{t}), \\ \partial_{t_l}\Psi(\lambda,\mathbf{t}) = B_l\Psi(\lambda,\mathbf{t}), \qquad l \geq 1. \end{cases}$$

Lax operator: $\boxed{L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots}$,

KP–solution: $\boxed{u(x,y,t) = u_2 = \partial_x w_1}$

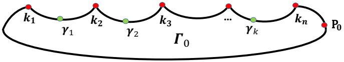
KP–wave function: $\boxed{\Psi(\lambda; x, y, t, \ldots) = W\Psi^{(0)}(\lambda; x, y, t, \ldots)}$
with

$$\Psi^{(0)}(\lambda; x, y, t, \ldots) = \exp(\lambda x + \lambda^2 y + \lambda^3 t + \cdots).$$

Soliton data: $(\mathcal{K}, [A]) \quad \mapsto \quad$ Sato algebraic geometric data: $(\Gamma_0, P_0, \zeta; \mathcal{D}_S^{(k)})$

Γ_0 copy of $\mathbb{CP}^1$, ζ such that $\zeta^{-1}(P_0) = 0$ and $\zeta(\kappa_1) < \zeta(\kappa_2) < \cdots < \zeta(\kappa_n)$.

Sato divisor $\mathcal{D}_S^{(k)} = \{\gamma_j,\ j \in [k]\}$: $\gamma_j^k - \mathfrak{w}_1(\vec{t}_0)\gamma_j^{k-1} - \cdots - \mathfrak{w}_{k-1}(\vec{t}_0)\gamma_j - \mathfrak{w}_k(\vec{t}_0) = 0$



[Malanyuk 1991]: $\gamma_j \in [\kappa_1, \kappa_n]$, $j \in [k]$ and for a.a. $\vec{t}_0$ γ_j are distinct.

Normalized Sato wave function $\hat{\psi}(P, \vec{t}) = \frac{\mathfrak{D}\phi^{(0)}(P;\vec{t})}{\mathfrak{D}\phi^{(0)}(P;\vec{t}_0)} = \frac{\psi^{(0)}(P;\vec{t})}{\psi^{(0)}(P;\vec{t}_0)}$, $\forall P \in \Gamma_0 \backslash \{P_0\}$

By definition $(\hat{\psi}_0(P, \vec{t})) + \mathcal{D}_{S,\Gamma_0} \geq 0$, for all $\vec{t}$.

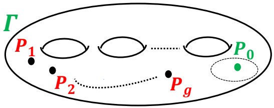
Incompleteness of Sato algebraic–geometric data:

Fix $1 \leq k < n$, $\vec{t}_0$, $\kappa_1 < \cdots < \kappa_n$ and the spectral data $(\Gamma_0 \backslash \{P_0\}, \mathcal{D}_{S,\Gamma_0})$. Then it is impossible to identify uniquely the point $[A] \in Gr^{\mathrm{TNN}}(k, n)$ corresponding to such spectral data since for generic soliton data

$$\deg\,(\mathcal{D}_{S,\Gamma_0}) = k < k(n-k) = \dim\,(Gr^{\mathrm{TNN}}(k,n))$$

Krichever approach to degenerate finite–gap solutions: construct reducible curve Γ such that Γ_0 is a component and extend Sato wavefunction from Γ_0 to Γ

Algebraic geometric data:
(Γ, P_0, ζ) $\qquad \zeta^{-1}(P_0)=0$

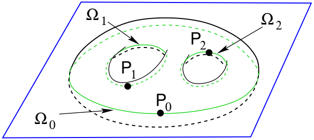


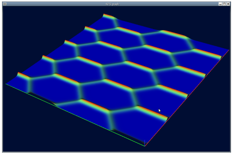
Families of regular quasi–periodic solutions $\boxed{u(\mathbf{t})}$ on Γ, non–singular genus g algebraic curve, are parametrized by non special divisors $\boxed{\mathcal{D} = (P_1, \ldots, P_g)}$:
There exists a unique normalized KP wave–function $\Psi(P, \mathbf{t})$, meromorphic on $\Gamma \backslash \{P_0\}$, with poles in $\mathcal{D}$ and asymptotics at P_0

$$\Psi(\zeta, \vec{t}) = \Big(1 - \frac{w_1(\vec{t})}{\zeta} + O(\zeta^{-2})\Big) e^{\zeta x + \zeta^2 y + \zeta^3 t + \cdots} \qquad (\zeta \to \infty).$$

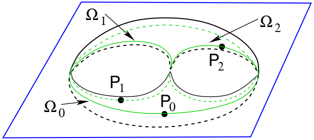
$$u(\vec{t}) = 2\partial_x^2 \log \Theta(xU^{(1)} + yU^{(2)} + tU^{(3)}) + c_1$$

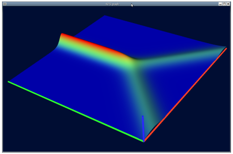
Dubrovin–Natanzon: Smooth, $\boxed{\text{real}}$ (quasi–)periodic $u(x,y,t)$ correspond to $\boxed{\text{real and regular divisors on smooth M–curves}}$: Γ possesses an antiholomorphic involution which fixes the maximum number $g+1$ of ovals, $\Omega_0,\dots,\Omega_g$; $P_0 \in \Omega_0$ (infinite oval) and $P_j \in \Omega_j$, $j=1,\dots,g$ (finite ovals).





Real smooth bounded solitons may be obtained from regular real quasi–periodic solutions in the rational degeneration of such curves (some cycles shrink to double points). Example: a real and regular divisor for soliton data in $Gr^{\mathrm{TP}}(1,3)$ when Γ is a rational degeneration of a genus 2 hyperelliptic curve





◇ A matroid $\mathcal{M}$ of rank k on the set $[n]$ is a non empty collection of k–element subsets in $[n]= \{1, \ldots, n\}$ that satisfy the exchange axiom:

$\forall I, J \in \mathcal{M}$ and $\forall i \in I$ $\exists j \in J$ s.t. $(I \backslash \{i\}) \cup \{j\} \in \mathcal{M}$.

◇ An element in $[A] \in Gr(k, n)$ represented by a matrix A gives a matroid $\mathcal{M}_{[A]} = \{I \,:\, \Delta_I(A) \neq 0\}$ since exchange axiom follows from Grassmann–Plücker relations.

◇ $Gr(k, n)$ has a subdivion in matroid strata (Gelfand–Serganova)

$S_{\mathcal{M}} = \{[A] \in Gr(k, n) \,:\, \mathcal{M}_{[A]} = \mathcal{M}\}$ labelled by matroids $\mathcal{M}$.

which is a finer subdivision than the decomposition into Schubert cells.

◇ Totally non–negative Grassmann cell (positroid cell) $S_{\mathcal{M}}^{\text{TNN}} = S_{\mathcal{M}} \cap Gr^{\text{TNN}}(k, n)$:

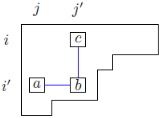
$S_{\mathcal{M}}^{\text{TNN}} = \{[A] \in Gr^{\text{TNN}}(k, n) \,:\, \Delta_I(A) > 0 \text{ for } I \in \mathcal{M}, \text{ and } \Delta_I(A) = 0 \text{ for } I \notin \mathcal{M}\}$.

Natural question: when $S_{\mathcal{M}}^{\text{TNN}} \neq \emptyset$?

Representation of $S_{\mathcal{M}}^{\text{TNN}}$ via Le–diagrams [Postnikov 2006]

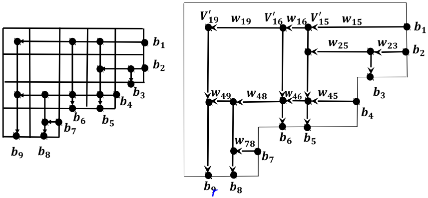
Postnikov introduces Le–diagrams and constructs a bijection between $Gr^{\text{TNN}}(k,n)$ and $\{$ Le–tableaux $\}$.

For a partition λ, a Le–diagram D of shape λ is a filling of the corresponding Young diagram with 0's and 1's following the rule: for any 3 boxes (i',j), (i,j'), (i',j'), with $i < i'$, $j < j'$, if $a, c \neq 0$ then $b \neq 0$:



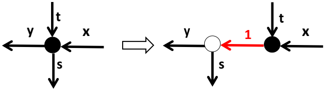
To a Le–diagram associate a Le–graph: draw a hook for each box with a dot (two lines going to the right and down from the dotted box). The Le–property means that every box of the Young diagram located at the intersection of two lines contains a dot.

w_{19}	0	w_{16}	w_{15}	0
0	0	0	w_{25}	w_{23}
w_{49}	w_{48}	w_{46}	w_{45}	
0	w_{78}			



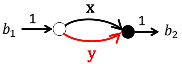
$$A^r_j = (-1)^{\sigma_{irj}} \sum_{P:i_r \mapsto j} w(P) \qquad \Delta_J(A) = \sum_{Q:I\mapsto J} \prod_{i=1}^{r} w(Q_i) \text{ (Lindström Lemma)}$$

- Any oriented planar network in the disk associated to a point $[A] \in \mathcal{S}_{\mathcal{M}}^{\mathrm{TNN}} \subset Gr^{\mathrm{TNN}}(k,n)$ may be transformed to an directed planar bipartite trivalent perfect graph in the disk:

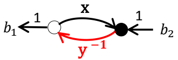


Black (white) vertex has exactly one outgoing (incoming) edge

- Change of base in the matroid $\mathcal{M}$ induces well defined change of orientation in the network N in which boundary sources/sinks corresponding to initial base transform to boundary sinks/sources for new base.

$N=$ 

$A(N)=(1,\mathrm{x}+\mathrm{y})$

$N'=$ 

$A(N')=((\mathrm{x}+\mathrm{y})^{-1},1)$

Equivalent networks [Postnikov 2006]

• Two networks are equivalent if they may be transformed one into the other via a sequence of moves and reductions:

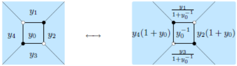


FIGURE 12.1. Square move

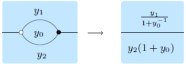


FIGURE 12.4. Parallel edge reduction

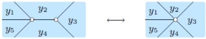


FIGURE 12.2. Unicolored edge contraction

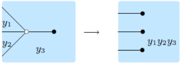


FIGURE 12.5. Leaf reduction

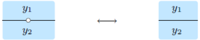


FIGURE 12.3. Middle vertex insertion/removal

FIGURE 12.6. Dipole reduction

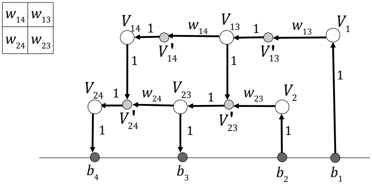
• To any directed graph there is associated a positroid (matroid + positivity) obtained considering all possible orientations

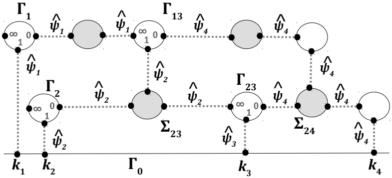
• Two networks are move-reduction equivalent if and only if they belong to the same positroid cell

• Le–networks are reduced network and provide minimal parametrization of the positroid cells

From planar bipartite trivalent directed graphs in the disk to rational degenerations of M–curves

$\mathcal{G}$	Γ
Boundary of disk	Copy of $\mathbb{CP}^1$ denoted Γ_0
Boundary vertex b_l	Marked point κ_l on Γ_0
Internal black vertex V'_s	Copy of $\mathbb{CP}^1$ denoted Σ_s
Internal white vertex V_l	Copy of $\mathbb{CP}^1$ denoted Γ_l
Internal Edge	Double point
Face	Oval



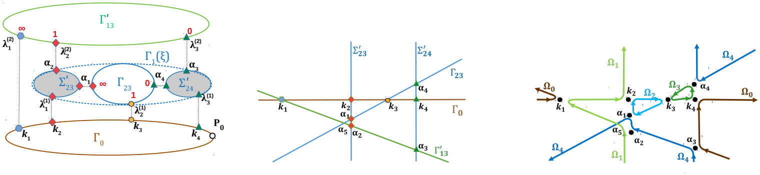


The universal curve Γ representing a cell in $Gr^{\mathrm{TNN}}(k,n)$ [AG-2017 +AG-2018]

For any fixed graph $\mathcal{G}$ representing a positroid cell $\mathcal{S} = \mathcal{S}^{\mathrm{TNN}}_{\mathcal{M}}$ and for any $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$ the above construction provides an **universal** curve $\Gamma = \Gamma(\mathcal{S};\mathcal{G})$ for the whole positroid cell and such that:

1. Γ possesses $g+1$ ovals which we label Ω_s, $s \in [0,g]$;
2. Γ is the rational degeneration of a regular $\mathtt{M}$–curve of genus g.

In particular, if $\mathcal{G}$ is the Le–graph then $g = |D|$, the dimension of the positroid cell.



The curve $\Gamma(\xi)$ in [AG-CMP 2018] is a rational desingularization of the curve for the Le–graph in [AG- ArXiv May 2018] which reduces the number of rational components at the price of adding a parameter ξ to rule the position of the double points.

The construction of the KP divisor for soliton data $[A] \in \mathcal{S}_{\mathcal{M}}^{\text{TNN}}$ on Γ [AG-2017+2018]

Key ideas:

1. Associate to each edge e of the directed network $\mathcal{N}$ an edge vector E_e so that Sato constraints are satisfied;
2. Use edge vectors to rule the values of the dressed edge wave function at the edges $e \in \mathcal{N}$ (=double points on Γ) $\implies$ the Baker-Akhiezer function on Γ automatically takes equal values at double points;
3. Use linear relations at vertices to compute the position of the KP divisor

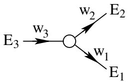
- The j–th component of E_e: $(E_e)_j = \sum\limits_{\mathcal{P}: e \to b_j} (-1)^{\text{wind}(\mathcal{P}) + \text{int}(\mathcal{P})} w(\mathcal{P})$.

◇ Explicit expressions for components of edge vectors on any network (modification of Postnikov and Talaska): the edge vector components are rational in weights with subtraction free denominators;

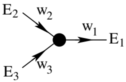
◇ Complete control of change of edge vectors and KP edge wave function w.r.t.:

- orientation of the network (correspond to changes of coordinates on the components of the curve);
- gauge ray direction (the way by which we assign sign to edge vectors' components);
- weight gauge (there is not a unique way to assign weights on $\mathcal{N}$!)
- vertex-edge gauge

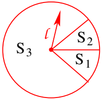
Linear relations at vertices fix position of divisor points on corresponding components [AG-2017 +AG-2018]

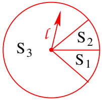


S_1: $E_3 = w_3(E_2 - E_1)$
S_2: $E_3 = w_3(E_1 - E_2)$
S_3: $E_3 = w_3(E_2 + E_1)$



S_1: $E_3 = +w_3\ E_1$ $\quad E_2 = -w_2\ E_1$
S_2: $E_3 = -w_3\ E_1$ $\quad E_2 = +w_2\ E_1$
S_3: $E_3 = +w_3\ E_1$ $\quad E_2 = +w_2\ E_1$





- Linear relations at internal vertices analogous to momentum-elicity conservation conditions in the planar limit of $N = 4$–SYM theory (see Arkani-Ahmed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka [2016]);
- Linear relations at bivalent vertices + trivalent black vertices $\Longrightarrow$ extend the normalized edge wave function to a function constant w.r.t. the spectral parameter on corresponding rational component of Γ.
- Linear relations at white trivalent vertices rule the position of the KP divisor in the ovals.
- Edge vectors are real $\Longrightarrow$ Edge wave function real for real KP times $\Longrightarrow$ KP divisor belongs to the union of the ovals.

The network divisor number assigned to V_l is the coordinate of the divisor point on component Γ_l [AG-2017 +AG-2018]

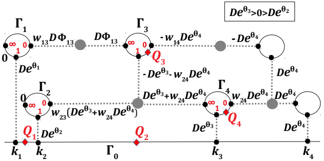
E_e edge vector at edge e

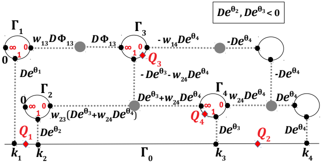
Vacuum wave function $\Phi^{vac}_{e,\mathcal{O},\mathfrak{l}}(\vec{t}) = \sum_{j=1}^{n} (E_e)_j \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t + \cdots)$

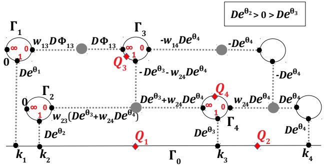
Dressed wave function $\Phi^{dr}_{e,\mathcal{O},\mathfrak{l}}(\vec{t}) = \sum_{j=1}^{n} (E_e)_j \mathfrak{D} \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t + \cdots)$

Network dressed divisor number at trivalent white vertex V_l:

$$\gamma_{\mathrm{dr},V_l} = \frac{(-1)^{\mathrm{wind}(e_3,e_1)}\Phi^{dr}_{e_1,\mathcal{O},\mathfrak{l}}(\vec{t}_0)}{(-1)^{\mathrm{wind}(e_3,e_1)}\Phi^{dr}_{e_1,\mathcal{O},\mathfrak{l}}(\vec{t}_0) + (-1)^{\mathrm{wind}(e_3,e_2)}\Phi^{dr}_{e_2,\mathcal{O},\mathfrak{l}}(\vec{t}_0)},$$







Reality and regularity of KP divisor = exactly one divisor point in each finite oval

[AG-CMP2018] : Proof for soliton data in $Gr^{\mathrm{TP}}(k,n)$ in two steps:

⋄ Use total positivity in classical sense to control position of an auxiliary vacuum divisor;
⋄ Dressing acts on divisor as shift.

[AG -Arxiv May 2018] : Proof for soliton data in $Gr^{\mathrm{TNN}}(k,n)$ and Le–network case in two steps:

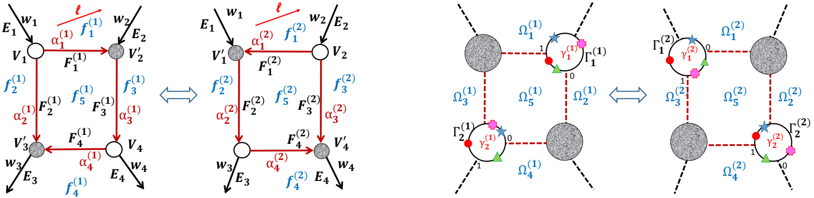
⋄ Combinatorial proof to control position of an auxiliary vacuum divisor;
⋄ Dressing acts on divisor as shift.
It is possible to adapt the combinatorial proof to directly prove that the KP divisor satisfies the reality and regularity properties.

[AG -Arxiv Dec 2017] : Combinatorial proof for soliton data in $Gr^{\mathrm{TNN}}(k,n)$ and general planar bipartite networks

Invariance of the KP divisor + Rules for Postnikov moves and reductions [AG- Arxiv 2017]

The KP divisor position in the oval is invariant w.r.t. changes of the orientation of the graph and of the gauges for ray direction, weights, edge-vertex. Indeed the value of the normalized KP edge wave function (= value of the the wave function at double points) is independent from the graph orientation and the gauges.

Explicit transformations of edge vectors w.r.t. Postnikov moves and reductions. These transformation change the network representing $[A]$, therefore in our construction they transform in a controlled way both the reducible rational curve and the KP divisor.



Soliton lattices of KP-II and desingularization of spectral curves in $Gr^{\mathrm{TP}}(2,4)$ [AG-2018 Proc.St.]

Reducible plane curve $P_0(\lambda,\mu)=0$, with

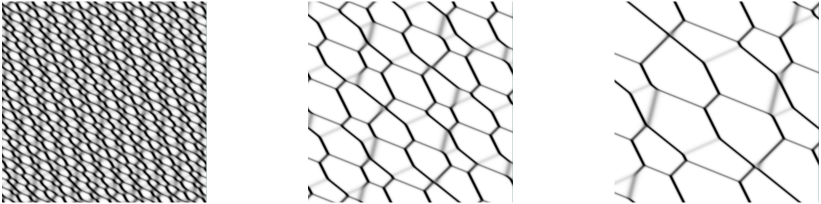
$$P_0(\lambda,\mu)=\mu\cdot(\mu-(\lambda-\kappa_1))\cdot(\mu+(\lambda-\kappa_2))\cdot(\mu-(\lambda-\kappa_3))\cdot(\mu+(\lambda-\kappa_4)).$$

Genus 4 $\mathtt{M}$–curve after desingularization:

$$\Gamma(\varepsilon)\ :\qquad P(\lambda,\mu)=P_0(\lambda,\mu)+\varepsilon(\beta^2-\mu^2)=0,\qquad 0<\varepsilon\ll 1,$$

where

$$\beta=\frac{\kappa_4-\kappa_1}{4}+\frac{1}{4}\max\{\kappa_2-\kappa_1,\kappa_3-\kappa_2,\kappa_4-\kappa_3\}.$$



$$\kappa_1=-1.5,\quad \kappa_2=-0.75,\quad \kappa_3=0.5,\quad \kappa_4=2.$$

Level plots for the KP-II finite gap solutions for $\epsilon=10^{-2}$ [left], $\epsilon=10^{-10}$ [center] and $\epsilon=10^{-18}$ [right]. The horizontal axis is $-60\le x\le 60$, the vertical axis is $0\le y\le 120$, $t=0$. The white color corresponds to lowest values of u, the dark color corresponds to the highest values of u.