From Monge Transport to Skorokhod Embeddings

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Monge Transport and Skorokhod Embedding

Move snow from μ

to ν, optimally!

$$\inf_{T_\sharp \mu = \nu} \int c(y, T(y))\mu(dy)$$

Given $W_0 \sim \mu$

find a stopping-time with $W_\tau \sim \nu$

as a barrier, $\tau = \inf\{t; W_t \in R\}$.

Tour of Dynamic Problems

1. Fixed end-time: $c^L(y,z) = \inf_{\gamma} \int_0^1 \left\{ L(t,\gamma,\dot{\gamma})dt; \gamma(0) = y, \gamma(1) = z \right\}$.

   E.g. $L(t,x,v) = \frac{1}{2}|v|^2 \quad \Rightarrow \quad c^L(y,z) = \frac{1}{2}|z-y|^2$.

   Benamou-Brenier, Bernard-Buffoni, Fathi-Figalli

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2. $\ast$ Free end-time: $c_L(y,z) = \inf_{\gamma,\tau} \int_0^\tau \left\{ L(t,\gamma,\dot\gamma)dt; \gamma(0) = y, \gamma(\tau) = z \right\}$.

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   Mikame-Theullen, Gentil-Léonard-Ripani (Fixed end-time)

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5. $\ast$ Skorokhod Embedding (without control):

   $$\inf_{\tau} \Big\{ E\Big[\int_0^\tau L(t,W_t)dt \Big] ; \quad W_0 \sim \mu, \ W_\tau \sim \nu \Big\}.$$

   Literature in probability and finance; Beiglböck-Cox-Heusmann

Problems We Consider

1. Classical Existence and Uniqueness of Transport.
   Gangbo-McCann, Kantorovich, Sudakov, Evans-Gangbo for Monge

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5. Understanding Regularity; Viscosity Solutions / Weak Solutions.
   Everyone here to name a few

Part 1: Free End-Time and Eulerian formulation

Optimal Transport:

$$V(\mu, \nu) := \inf_{T_\sharp \mu = \nu} \int c_L\big(y, T(y)\big) d\mu(y)$$

built from dynamic optimization

$$c_L(y, z) := \inf_{\tau, \gamma(\cdot)} \left\{ \int_0^\tau L\big(t, \gamma(t), \dot{\gamma}(t)\big) dt; \ \gamma(0) = y, \ \gamma(\tau) = z \right\}.$$

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Eulerian Formulation:

$$\textit{Thm}: \quad V(\mu,\nu) = E(\mu,\nu) := \inf_{\eta,\rho} \int_0^\infty \int_{T\mathbb{R}^n} L(t,x,v)\eta(t,dx,dv)dt$$

Phase-Space Density $\eta : \mathbb{R}^+ \to \mathcal{M}(T\mathbb{R}^n)$, $\eta \geqslant 0$, $\int_{T_x\mathbb{R}^n} \eta(0,x,dv) = \mu(x)$

Stopping-Measure $\rho \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^n)$, $\rho \geqslant 0$, $\int_0^\infty \rho(dt,x) = \nu(x)$

$$\rho(t,x) + \partial_t \int_{T_x\mathbb{R}^n} \eta(t,x,dv) + \nabla_x \cdot \int_{T_x\mathbb{R}^n} v\ \eta(t,x,dv) = 0$$

Duality Review: Optimal Transport for Fixed End-Time

The general Kantorovich dual problem is

$$W(\mu,\nu) := \sup\Big\{\int \psi d\nu - \int \phi d\mu; \quad \psi(z) - \phi(y) \leqslant c^L(y,z)\Big\}$$

Optimizer satisfies $\big(c^L(y,z) = \inf_\gamma \int_0^1 L(t,\gamma,\dot\gamma)dt; \gamma(0) = y, \gamma(1) = z\big)$

$$\phi(y) = \sup_z \{\psi(z) - c^L(y,z)\} = \sup_\gamma \Big\{\psi\big(\gamma(1)\big) - \int_0^1 L(t,\gamma,\dot\gamma)dt\Big\}.$$

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Bernard-Buffoni, Fathi-Figalli: $\phi = J_\psi(0,\cdot)$,
Let $H(t,x,p) = \sup_v\{p\cdot v - L(t,x,v)\}$

$$\partial_t J_\psi(t,x) + H\big(t,x,\nabla J_\psi(t,x)\big) = 0, \quad J_\psi(1,\cdot) = \psi.$$

Hamiltonian flow, $p(t) = \nabla J_\psi(t,\gamma(t))$:

$$\dot\gamma(t) = D_p H\big(t,\gamma(t),p(t)\big), \quad \dot p(t) = -D_x H\big(t,\gamma(t),p(t)\big)$$

Eulerian Duality: Free End-Time

The dual problem via Eulerian formulation:

$$Thm' : W(\mu,\nu) = D(\mu,\nu) := \sup_{(J,\psi)\in N(L)} \int_{\mathbb{R}^n} \psi(z)\nu(dz) - \int_{\mathbb{R}^n} J(0,y)\mu(dy),$$

$(J,\psi) \in N(L)$ satisfy

$$\psi(x) - J(t,x) \leqslant 0$$
$$\partial_t J(t,x) + v \cdot \nabla J(t,x) - L(t,x,v) \leqslant 0.$$

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Thm": Optimal J_ψ satisfies the Hamilton-Jacobi-Bellman inequality:

$$\max \left\{ \begin{array}{c} \psi(t,x) - J_\psi(t,x), \\ \partial_t J_\psi(t,x) + H\big(t,x,\nabla J_\psi(t,x)\big) \end{array} \right\} = 0.$$

$H(t,x,p) = \sup_v \{p \cdot v - L(t,x,v)\}$ (proof by Perron's method)

Dynamic programming shows equivalence:

$$\phi(y) = \sup_z \{\psi(z) - c_L(y,z)\} = \sup_{\gamma,\tau} \Big\{\psi\big(\gamma(\tau)\big) - \int_0^\tau L(t,\gamma,\dot\gamma)dt\Big\}.$$

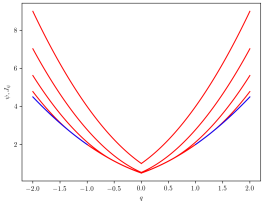
Monotonicity by Viscosity Solution Methods

Proposition

A. $t \mapsto L$ *increasing* $\Rightarrow$ $t \mapsto J_\psi$ *decreasing.*
B. $t \mapsto L$ *decreasing* $\Rightarrow$ $t \mapsto J_\psi$ *increasing.*

Let $s(x) = \inf\{t; J_\psi(t,x) = \psi(x)\}$ *(for A., for B. use* $\sup$*);*
We have the transversality condition:

$$H\big(s(x), x, \nabla\psi(x)\big) = 0.$$



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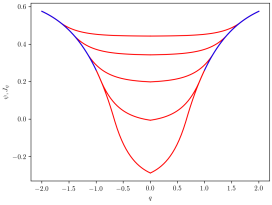

B.

Hitting-Time

With A. or B., and good H, μ, ν:

Theorem

Unique optimal $\pi^* \in \Pi(\mu,\nu)$ *given by* $T(x) := \gamma^x(\tau^x)$.
Pontryagin transversality

$$H(\tau^x, T(x), \nabla\psi(T(x)) = 0, \quad \text{so } \tau^x = s(T(x)).$$

Hamiltonian flow for $t < \tau^x$, $p(\tau^x) = \nabla\psi(T(x))$*:*

$$\dot{\gamma}(t) = D_p H\big(t, \gamma(t), p(t)\big), \quad \dot{p}(t) = -D_x H\big(t, \gamma(t), p(t)\big)$$

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Optimal attainment of Eulerian problem with $\operatorname{supp}\rho \in \{(s(x),x)\}$.

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Optimal attainment of Eulerian problem with $\operatorname{supp} \rho \in \{(s(x), x)\}$.
If $J \in C^1$ then $\hat{\eta} = \int_{T_x\mathbb{R}^n} d\eta$ is determined by (for A.)

$$\partial_t \hat{\eta} + \nabla_x \cdot D_p H(\nabla J)\hat{\eta} = 0, \quad t < s(x)$$

Unresolved: Is η uniquely determined with J Lipschitz?

Suppose

$$L(t,x,v) := \left\{ \begin{array}{ll} g'(t), & |v| \leqslant 1 \\ \infty, & |v| > 1 \end{array} \right\}$$

with $g(0) = 0$ and $g'(t) \geqslant 0$.

If g is convex or concave, characteristics are straight lines, cost is

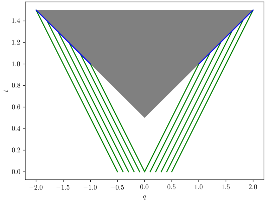
$$c_L(y,z) = g(|z-y|).$$

Gangbo-McCann for Monge Map

A. $\Leftrightarrow$ g convex

B. $\Leftrightarrow$ g concave

A.



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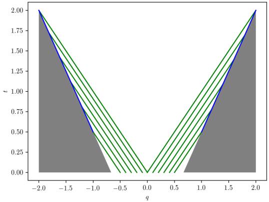

Part 2: Skorokhod Embeddings

Uncontrolled diffusion
$W_0 \sim \mu$

Stopping-time to transport
$W_\tau \sim \nu$.

28 constructions in 1-D
Azéma-Yor, Perkins, Root, Rost ...

Our problem is (multi-D):

$$V(\mu, \nu) := \inf_{\tau} E\left[\int_0^\tau L(t, W_t)dt;\ W_0 \sim \mu,\ W_\tau \sim \nu\right].$$

(Real applications in mathematical finance; options pricing)

Hobson, Obłój, Henry-Labordere ...

Eulerian Formulation and Duality

No optimal transportation cost!

- Eulerian Formulation:

$$E(\mu,\nu) := \inf_{(\eta,\rho)\in\Gamma(\mu,\nu)} \int_0^\infty \int_{\mathbb{R}^n} L(t,x)\eta(t,x)dxdt,$$

$(\eta,\rho)\in\Gamma(\mu,\nu)$ if $\eta \geqslant 0$, $\rho \geqslant 0$, and (weakly)

$$\rho + \partial_t \eta = \frac{1}{2}\Delta\eta, \quad \eta(0,\cdot) = \mu, \; \int_0^\infty \rho(dt,\cdot) = \nu.$$

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- Dual Problem (via Eulerian Formulation):

$$Thm : V(\mu,\nu) = E(\mu,\nu) = D(\mu,\nu) := \sup_{(J,\psi)\in\Upsilon(L)} \int \psi d\nu - \int J(0,\cdot)d\mu,$$

$(J,\psi) \in \Upsilon(L)$ if

$$\psi(x) - J(t,x) \leqslant 0$$

$$\partial_t J(t,x) + \frac{1}{2}\Delta J(t,x) \leqslant L(t,x)$$

Dual Attainment

Maximize:

$$\int \psi d\nu - \int J_\psi(0,\cdot) d\mu; \qquad \max \left\{ \begin{array}{r} \psi - J_\psi, \\ \partial_t J_\psi + \frac{1}{2}\Delta J_\psi - L \end{array} \right\} = 0.$$

Remaining degrees of freedom for (ψ, J_ψ)

- Subtract a positive function from ψ. (Let $\psi = \inf_{t\geqslant 0} J_\psi$.)
- Subtract a subharmonic function from ψ and J_ψ. (Harder to handle. This shows that $D(\mu,\nu) = \infty$ unless $\mu \leqslant_{SH} \nu$.)

$$\int h \, d\mu \leqslant \int h \, d\nu \quad \forall \; h \text{ s.t. } \Delta h \geqslant 0$$

Theorem (in progress)

Under suitable assumptions the dual problem is attained at regular (ψ, J_ψ).

Complementary Slackness

- To verify optimality:
  $(\eta, \rho) \in \Gamma(\mu, \nu)$ and $(\psi, J_\psi) \in \Upsilon(L)$ are optimal if and only if

$$\int_0^\infty \int_{\mathbb{R}^n} L(t,x)\eta(t,x)dxdt = \int_{\mathbb{R}^n} \psi d\nu - \int_{\mathbb{R}^n} J(0,\cdot)d\mu.$$

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- If (ψ,J_ψ) and (η,ρ) are optimal and regular then:

$$\psi(x)=J_\psi(t,x),\ \ \rho \text{ a.e. } (t,x),$$

and

$$\partial_t J_\psi(t,x)+\frac{1}{2}\Delta J_\psi(t,x)=L(t,x),\ \ \eta \text{ a.e. } (t,x).$$

Root/Rost Embeddings

Proposition

A. $t \mapsto L$ *is increasing* $\Rightarrow$ $t \mapsto J_\psi$ *is decreasing.*
B. $t \mapsto L$ *is decreasing* $\Rightarrow$ $t \mapsto J_\psi$ *is increasing.*
$s(x) = \inf\{t; J_\psi(t,x) = \psi(x)\}$ *(For B. use* $\sup$*). We can choose* ψ *to solve*

$$\frac{1}{2}\Delta\psi(x) = L\big(s(x), x\big).$$

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- By complementary slackness, $\eta(t,\cdot) \in H_0^1(\{x;\ t < s(x)\})$, is unique given s and A. If $s \in C^1$, $\nu(x) = \nabla s(x) \cdot \nabla\eta(s(x), x).$

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- Rigidity Theorem: If s is optimal for increasing L, s is optimal for any increasing $\tilde{L}$ (For B. $t > s(x)$ and $-\nu$).

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- Rigidity Theorem: If s is optimal for increasing L, s is optimal for any increasing $\tilde{L}$ (For B. $t > s(x)$ and $-\nu$).
- Martingale Duality of Beiglböck-Cox-Huesmann: Maximize $\int \psi d\nu - E[M_0]$, $W_0 \sim \mu$, M_t martingale,

$$\psi(W_t) - M_t \leqslant 0 \text{ a.s.} \quad M_t = J_\psi(t, W_t) + E\left[\int_t^\tau L(s, W_s)ds\right]$$

Stochastic Transportation

General stochastic control with free end-time:

$$V(\mu, \nu) := \inf_{\tau, A(\cdot)} E\left[\int_0^\tau L(t, X_t, A_t)dt\right]$$

with $dX_t = f(X_t, A_t)dt + \sigma(X_t, A_t)dW_t,\ X_0 \sim \mu,\ X_\tau \sim \nu$. Eulerian form:

$$\rho + \partial_t \int_{\mathbb{A}} d\eta + \nabla \cdot \int_{\mathbb{A}} f\ d\eta = \frac{1}{2}\nabla^2 \cdot \int_{\mathbb{A}} \sigma^2 d\eta.$$

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$$\rho + \partial_t \int_{\mathbb{A}} d\eta + \nabla \cdot \int_{\mathbb{A}} f\ d\eta = \frac{1}{2}\nabla^2 \cdot \int_{\mathbb{A}} \sigma^2 d\eta.$$

Dual problem is

$$W(\mu,\nu) := \sup_{\psi, J}\left\{\int \psi d\nu - \int J(0,\cdot)d\mu\right\}$$

subject to

$$\psi(x) - J(t,x) \leqslant 0$$

$$\partial_t J + f(\cdot, A)\cdot\nabla J + \frac{1}{2}\sigma^2(\cdot, A)\cdot\nabla^2 J \leqslant L(\cdot,\cdot,A).$$

Many interesting questions!