The Salesman, the Postman and

(Delta-) Matroids

Improved Tours for some Fundamental Instances

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Your future for 80 minutes

1. The main tools

Ratios in the vector form (Goemans, Carr, Vempala),

fundamental vertices (Boyd Carr), (Delta)-matroids (Bouchet, Cunningham)

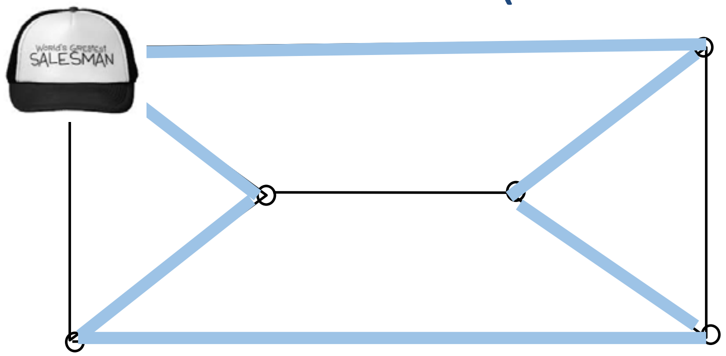
2. Improved Tours

for fundamental vertices, path TSP, graph TSP

3. Challenges

The Salesman and the Postman

The (Travelling) Salesman



Nodes = Cities

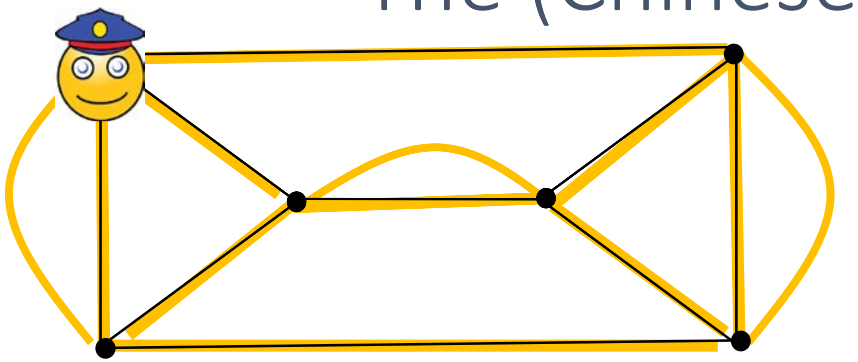
Do all the cities

and come back !

NP-hard

(Karp, 1972)

The (Chinese) Postman



Edges = streets

Do all the streets

and come back !

In P

(Edmonds,Johnson 1973)

1. The main tools

$K_n = (V_n, E_n)$ complete graph on n nodes

tour in G=(V,E): multisubgraph of G, connected on V, all degrees even.

T_n := convex hull of multiplicity vectors of tours in K_n

S_n := subtour elimination

$\{ x \in IR^{E_n} : x \geq 0, x(C) \geq 2$ on cuts, = on stars $\}$

S_n *majorates the spanning tree polytope*

$S_n/2$ *majorates the parity correction polyhedron* :

.

minimizing on it we get an upper bound for parity correction,
that is, for the price of *converting a connected subgraph into a tour*

Notation $x \in IR^E$ conv comb of $\mathcal{F}$
$x = E[\mathcal{F}]$

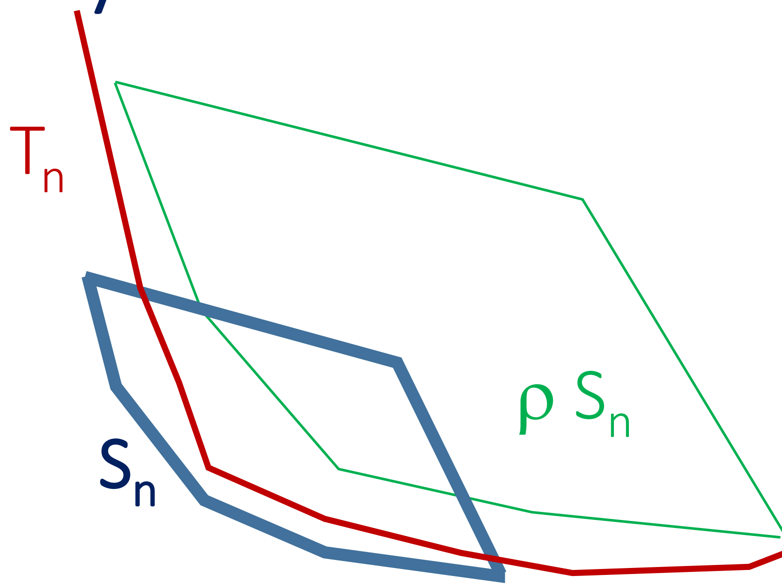
$E[\mathcal{F}_1 + \mathcal{F}_2] = E[\mathcal{F}_1] + E[\mathcal{F}_2]$

1.1 Vector form of the integrality ratio

Theorem : Goemans (1995), Carr, Vempala (2004)

$\text{OPT}(c) \leq \rho \, \text{LP}(c) \;\; \forall c \quad \Leftrightarrow \quad \rho \, S_n \subseteq T_n$

integrality ratio



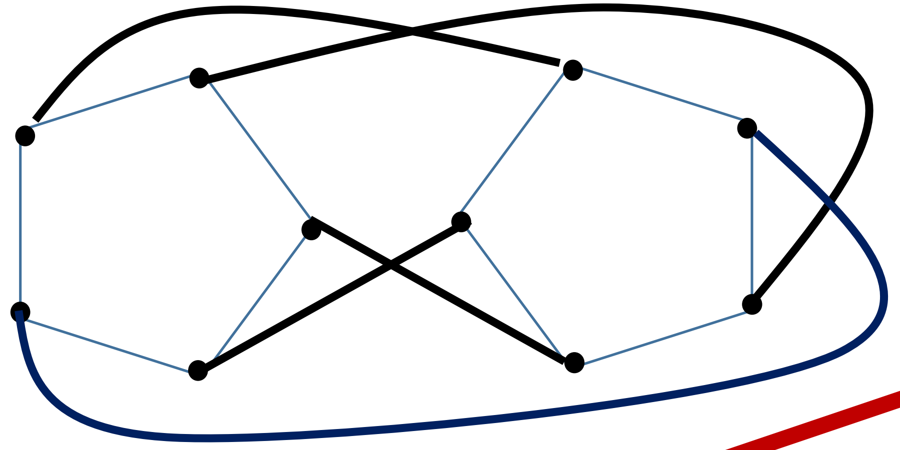
For $\rho = 1$ minmax theorem for all weights $\leftrightarrow$ polyhedral description

Why would we use the more difficult vector form ?

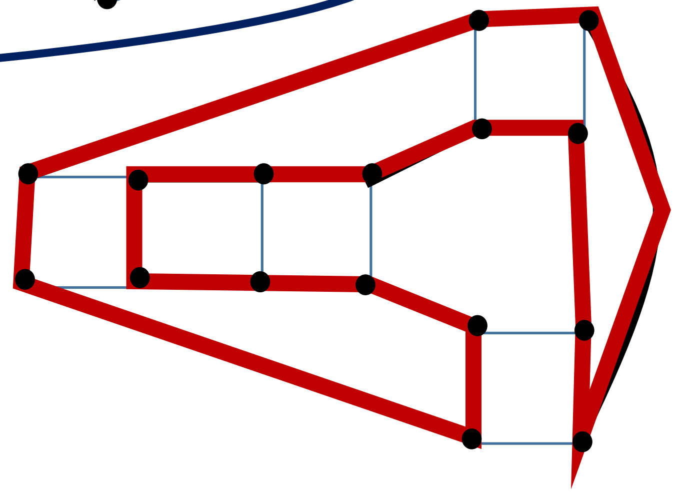
- The objective function disappears,
- Ugly case-checkings may be captured by the convex combination

Examples of $\frac{1}{2}$ - integer points in S_n

Generalized Prism :



square :



3-edge connected cubic : **M** + C

subcubic: $e \in M \rightarrow$ path

½ -edges

1-edges

Conjecture: (Schalekamp, Williamson, van Zuylen) $\frac{1}{2}$ - integer have the largest ratio

Assertions with the vector form

Algorithm: **Christofides-Serdyukov** (1976)

Theorem : Wolsey, Cunningham, Shmoys-Williamson 1980-90 $\frac{3}{2} S_n \subseteq T_n$

Proof: $x \in S_n$, $x = E[\mathcal{F}]$; $\frac{x}{2} =$ E[par. corr.] ; $E[\mathcal{F}+\text{par. corr}] = \frac{3x}{2}$

Conjecture (4/3) : $\frac{4}{3} S_n \subseteq T_n$

Conjecture (S. 2015) : Cubic 3-edge-connected $\frac{8}{9} \underline{1} \in T_n$

Proof from (4/3) : $\frac{2}{3} \underline{1} \in S_n$, so $\frac{4}{3}\frac{2}{3} \underline{1} \in T_n$

Already: $\frac{3}{2}\frac{2}{3} \underline{1} = \underline{1} \in T_n$

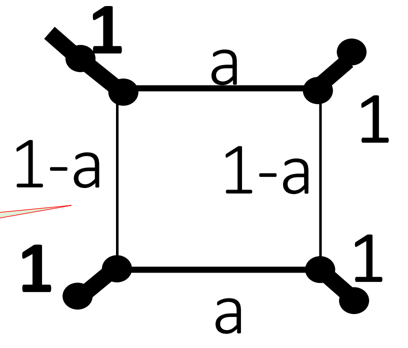
1.2 Fundamental families of points

Carr, Ravi 98 :

fundamental class = particular family of points to which the best ratio is reduced

square graphs (G,M) : M perfect matching

E(G) \ M is partitioned into square components.



3-edge-connected

Part of a Boyd-Carr point

Theorem : Boyd-Carr (2011) Points with square support are fundamental for the TSP

Proof :

1.3 Matroids and Delta-Matroids

$D = (S, \mathcal{D})$, $\emptyset \neq \mathcal{D} \subseteq \mathcal{P}(S)$ is a *delta-matroid* if

$D_1, D_2 \in \mathcal{D}$, $j \in D_1 \Delta D_2$ $\exists\, k \in D_1 \Delta D_2$:

$$D_1 \,\Delta\, \{j,k\} \in \mathcal{D}$$

Bouchet (1988)

$M = (S, \mathcal{B})$, $\mathcal{B} \subseteq \mathcal{P}(S)$ is a *matroid,* and $\mathcal{B}$ is the set of its bases if

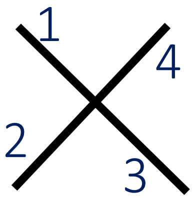
(i) M is a delta-matroid

(ii) All elements of $\mathcal{B}$ have the same size.

Examples of Delta-matroids (Bouchet 1988)

vertex-sets covered by matchings

'sets of *bitranstions*' of Eulerian trails



The 3 bitransitions:

{1 , 4} , {2 , 3}

{1 , 2} , {4 , 3}

{1 , 3} , {2 , 4} **FORBIDDEN**

Choose one of the 2 bitransitions of each node as `reference': S is the set of refs.
Represent each Eulerian trail with its subset of bitransitions D in S.

Thm (Bouchet): (S, D) , where , D is the set of all such sets D, *is a delta-matroid*.

Greedy Algorithm
Bouchet, Cunningham, 1992

$D = (S, \mathcal{D})$ delta-matroid, membership oracle, $c \in IR^n$

$|c_1| \geq |c_2| \geq \ldots \geq |c_n|$. Consider the elements in this order.

If ≥ 0 and possible, fix to 1, if ≤ 0 and possible fix to 0

Theorem (Bouchet, Cunningham): Greedy Algorithm determines the optimum of $\mathcal{D}$ and this characterizes delta-matroids.

conv($\mathcal{D}$) : ±1-0 constraints , « bisubmodular » right hand side.

Summarizing the tools

Remember the vector form of the integrality ratio

Boyd-Carr points and square graphs are fundamental

Half-integer points are nontrivial challenges, and possible intermediaries for the main goal

Main Goal: Improve the approximation ratio for the TSP and the st path TSP

2. Improved tours

2.1 In fundamental and 3-edge-connected cubic graphs

2.2 What blocks the s-t path TSP

2.3 Matroid Intersection for the graph TSP. Corresponding bound for uniform covers !

Guess the answer to the following problems

In square graphs,

1. what is the complexity of HAM ?

2. approx ratio for min weight Hamiltonian cycle containing all 1-edges ?

3. can Christofides-Serdyukov's 3/2 be improved ?

4. Is there a $< \underline{1}$ uniform cover ?

5. Is there a better ratio for $\frac{1}{2}$-integer vertices ?

All this in general in 3-edge-connected cubic graphs ? And what is this good for ?

$\mathcal{NP}$-hard
$\mathcal{P}$

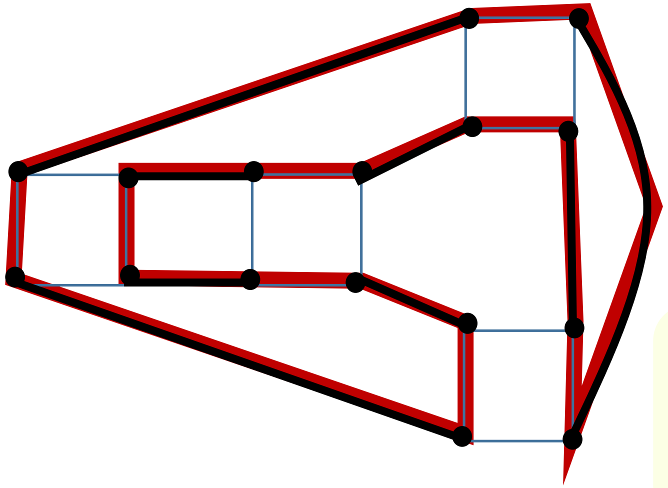
2
3/2
1

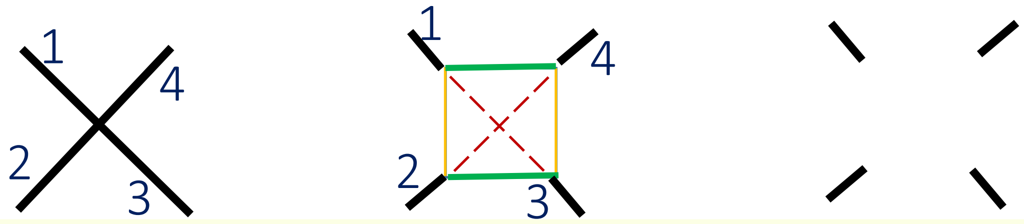
N
Y

Hamiltonicity

M perfect matching, E(G) \ M squares

Theorem : A square graph (G,M) has a Hamiltonian cycle containing M





3 *bitransitions*

(1,3), (2,4) forbidden

If not connected,

either or connects.

Proof =Kotzig's (1968):Eulerian trails in 4-regular graphs with forbidden bitransitions.

Directly: `Blow' squares into nodes s.t. the allowed 2 bitransitions are the 2 matchings

Greedy algorithm for Hamiltonian cycles ?

M perfect matching, E(G) \ M squares

INPUT : (G, M) edge-weighted square graph, c : E(G) --> IR

TASK : Minimize c(H), H Hamiltonian cycle containing M.

1. **∀ square C**, compute w_C , the absolute value of the difference of the two p.m. of C, and order the squares in decreasing order of w_C .
2. In this order, choose the minimum of the two possible values if both keep connectivity (there is always at least one by 'Kotzig's theorem').

Theorem : This algorithm determines the min weight Hamiltonian cycle containing the 1-edges in polynomial time.

Proof Straightforward from Bouchet and Cunningham + what we learnt ...

Could a conjecture on
- uniform covers in
- cubic graphs
be more generally useful ?

Tours

Conjectures (S. 2015)

(s,t)- paths

Cubic 3-edge-connected $\frac{8}{9}\,\underline{1} \in T_n$

G/{s,t} Cubic 3-edge-connected $\underline{1} \in T_n$

For (s,t)-paths **it became a theorem !** The narrow cuts of $\frac{2}{3}\,\underline{1}$ are disjoint ! _

Anke : Let us delete the unique edges of trees in narrow cuts !

Analysis : For all narrow cuts Q, $x^Q := \Pr(|\mathcal{F} \cap Q| = 1)$; this is what you spare in each narrow cut, and you spoil only half of it for parity completion. **Free reconn!**

How to prove good uniform covers or ratios?

Having a **good** $\in T_n$, for instance χ_H , where H is **a Hamiltonian cycle**, look for a **"not-bad"** $\in T_n$ which is **small on H** and maybe larger on the rest.

This does happen sometimes : 2 "compatible Euler trails" of Geelen, Iwata, Murota)

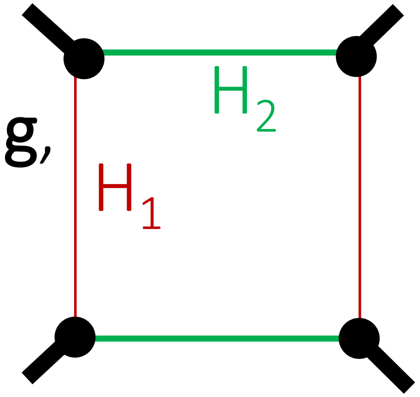
Example : Let G square, 2 Ham cycles with a common **perfect matching**, otherwise disjoint : $1 \ \frac{1}{2} \ \frac{1}{2} \ \in T_n$

$\frac{1}{2} \ 1 \ \frac{1}{2} \ \in S_n$

$\frac{1}{2} \ \frac{1}{2} \ 1 \ \in S_n$

$\frac{1}{2} \ \frac{3}{4} \ \frac{3}{4} \ \in S_n$

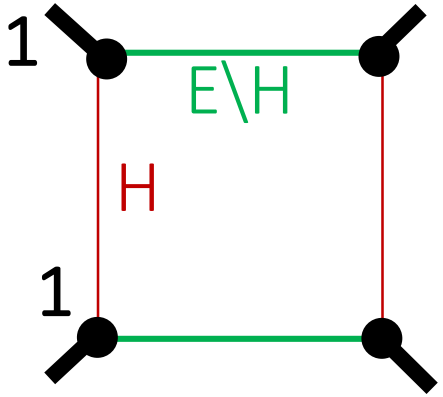
$\frac{3}{4} \ \frac{9}{8} \ \frac{9}{8} \ \in T_n$



With coefficients $\frac{3}{7}$ $\frac{4}{7}$

We get $\frac{6}{7} \ \frac{6}{7} \ \frac{6}{7} \ \in T_n$

Uniform covers for 3-edge-connected cubic



Theorem : If Hamiltonian, for instance square $\frac{6}{7}\underline{1} \in T_n$

Proof : $(1\ 1\ 0) \in T_n$, $(\frac{1}{2}\ \frac{1}{2}\ 1) \in S_n$, $(\frac{3}{4}\ \frac{3}{4}\ \frac{3}{2}) \in T_n$

Can we improve the $\underline{1}$ uniform cover for 3-edge-connected cubic graphs in general ?

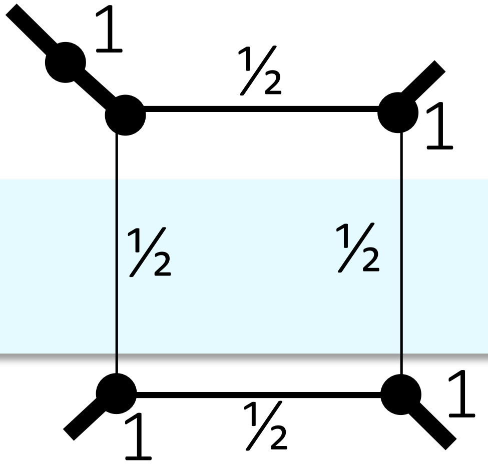
Theorem (Haddadan, Newman, Ravi 2017) : $\frac{18}{19}\underline{1} \in T_n$

Proof : $(\frac{4}{5}, 1) \in T_n$, and as before $\frac{3}{2}(1\ \frac{1}{2}) = (\frac{3}{2}\ \frac{3}{4}) \in T_n$

Not only breaking the general $\underline{1}$ bound, but the cause of « good » vector is *not Hamiltonicity*

Half Integer Boyd-Carr points

Theorem : For x ½ -integer, square,



$\frac{10}{7}x \in T_n$

Proof: $\rho x = \lambda \chi_H + (1-\lambda)\, y$ so we can look for y in the form

$y = (\alpha+1)x - \beta\chi_H$, $\alpha \geq 0$, $\beta \geq 0$, that is, $y = x + \alpha x - \beta\chi_H$, where

$y' = \alpha x - \beta\chi_H$ is a parity correction for every tree composing x.

Claim : $\exists\, \mathcal{F}$, $E[\mathcal{F}] = x$ so that $\frac{2}{3}x - \frac{1}{6}\chi_H$ is a parity correction for all $F \in \mathcal{F}$

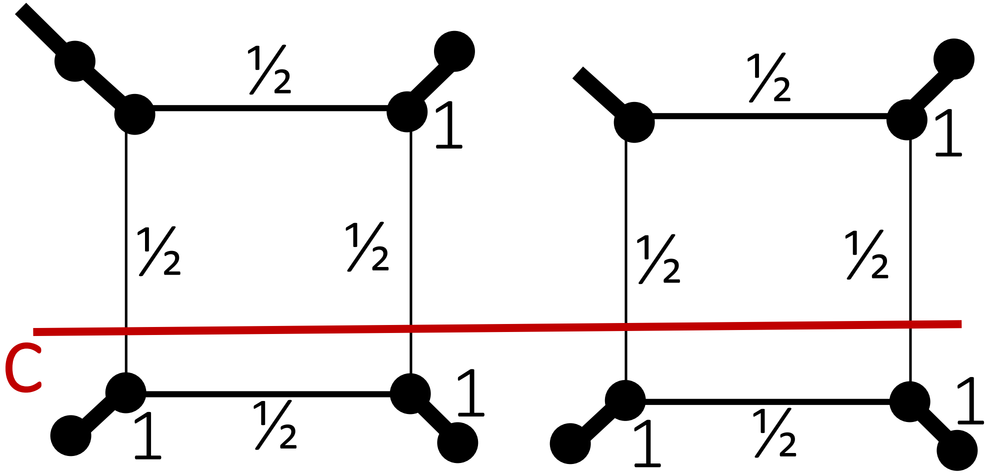
≥ 0 because $H \subseteq \mathrm{supp}\, x$, actually for each $e \in H$ on e we have $\geq \frac{1}{6}$

Concluding with matroid intersection

- so we are done if $|H \cap C| \geq 6$

- If $|H \cap C| = 2$ then $x(C) \geq 2$ makes us safe

- If $|H \cap C| = 4$: $\frac{2}{3}x(C) - \frac{1}{6}\chi_H(C) = \frac{4}{3} - \frac{2}{3} = \mathbf{\frac{2}{3} < 1}$**,**
  bad but there is a patch (Jack Patch) :



Theorem (Edmonds): M_1 = (S, B_1) **,** **M_2 =** (S, B_2) be two matroids on S
If $x \in$ conv (B_1) **, $x \in$ conv** (B_2) **=> $x \in$ conv** $(B_1 \cap B_2)$

Therefore, $\exists\, \mathcal{F}$ so that such a **bad** C never needs parity correction ! **Q.E.D.**

Do these results imply anything for general graphs ?

2.2 (s,t) Path TSP

What prevents us from reaching 3/2 ?

Gottschalk-Vygen's result **with matroid intersection**

(Schalekamp, van Zuylen, Traub, S.)

Doubling is crazy …

The rest of 2.2 and 2.3 had to be cancelled in lack of time

3. Challenges

Further study fundamental graphs?

G 3-edge-connected cubic. Is $\frac{8}{9}\underline{1}$ a convex combination of tours ?

Challenges for $\frac{1}{2}$-integer vertices (prism, fundamental, cubic),

Carr, Vempala fundamental graphs ?

The sufficiency of considering $\frac{1}{2}$-integer vertices

For the {s,t}-path TSP how to reconnect in a more refined way than doubling the spanning trees ?