A variational approach to the Crystalline Mean Curvature Flow

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A classical geometric evolution

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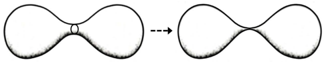

Figure: An example of pinching singularity (Grayson '89).

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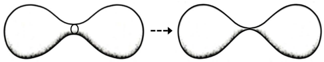

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Question: How to define a global-in-time solution? How to define a solution starting from irregular initial sets?

The level set approach

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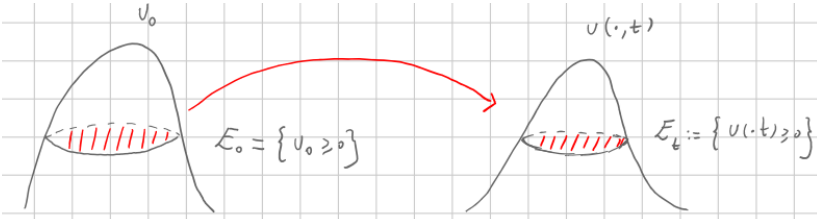

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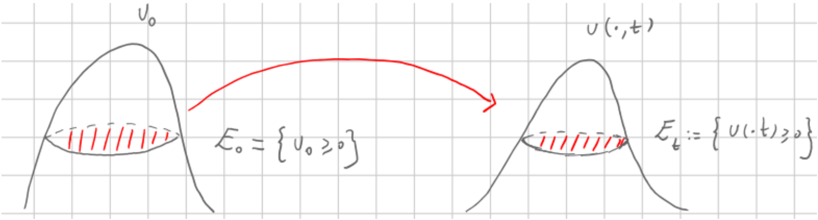

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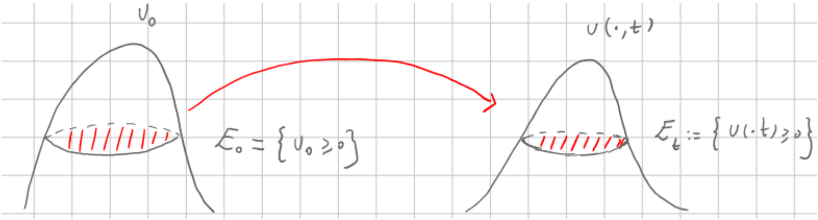

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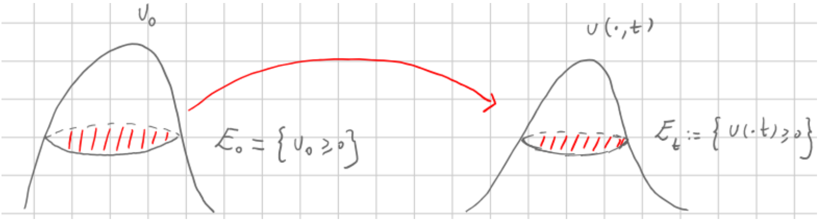

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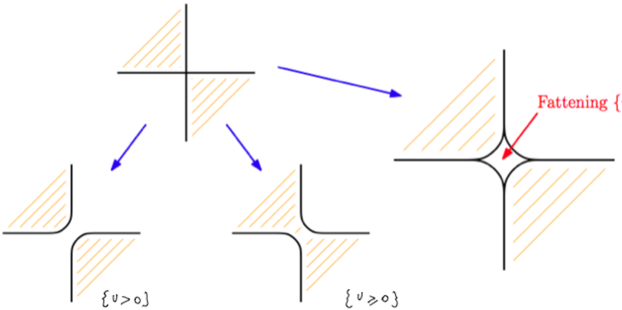
- Global existence and uniqueness for (LS) by Evans-Spruck (1991) and Chen-Giga-Goto (1991) with the machinery of viscosity solutions.

Non uniqueness by fattening

If one fixes the level set, uniqueness can only hold up to fattening:



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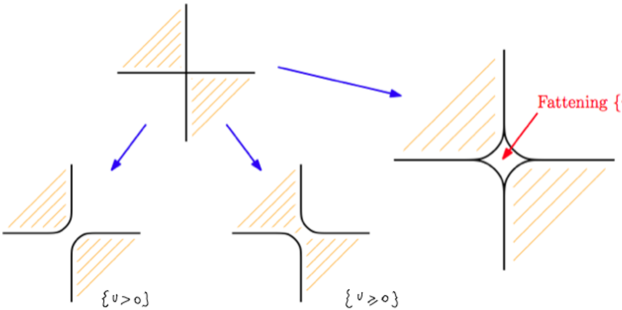

Generic Uniqueness : For all but countably many s, no fattening occurs and the evolution E_s is unique.

The minimizing movements approach

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- F. Almgren, J. E. Taylor, and L.-H. Wang, SIAM J. Control Optim. (1993)
- S. Luckhaus and T. Sturzenhecker, Calc. Var. Partial Differential Equations (1995)

Anisotropic Flows

Consider a norm ϕ and the corresponding anisotropic perimeter

$$P_\phi(E) = \int_{\partial E} \phi(\nu^E)\, d\mathcal{H}^{d-1}$$

The curvature κ_ϕ^E is the the first variation of P_ϕ. If ϕ is smooth, then $\kappa_\phi^E = \mathrm{div}_\tau \left(\nabla\phi(\nu^E)\right)$

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- If ϕ is smooth we apply classical theory
- If ϕ is non-smooth (e.g. crystalline), then the Cahn-Hoffmann field $\nabla\phi(\nu^E)$ and hence κ_ϕ^E are not well defined in a classical way

The crystalline case

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- The curvature becomes nonlocal!

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- The case $d \geq 3$: investigated by many authors, only partial results were available prior to ours:
  - Convex initial data: Bellettini, Caselles, Chambolle & Novaga (2008)
  - Polyhedral initial data: Giga, Gurtin & Matias (1998)
  - the well-posedness and the validity of a comparison principle in the general case has been a long-standing open problem as well as the uniqueness of the crystalline flat flow

Latest developments

Chambolle-M.-Ponsiglione 2016

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- Our result holds for the "natural" mobility $m = \phi$

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- Analogously, setting $d^c(\cdot, t) := \mathrm{dist}(\cdot, E^c(t))$, we have

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Our new weak formulation of $V = -\phi(\nu^{E(t)})\kappa_\phi^{E(t)}$

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$$\partial_t d \geq \mathrm{div} z \qquad \text{in } \mathbb{R}^N \times (0, T^*) \setminus E$$

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- Comparison Principle: exploits the distributional formulation
- Existence: via minimizing movements

Comparison

Let $F(0) \subset E(0)$ and let $\Delta > 0$ be the distance of their boundaries. Let F be a weak superflow, and E a weak subflow.

Claim: We want to prove that $\Delta(t) \geq \Delta$

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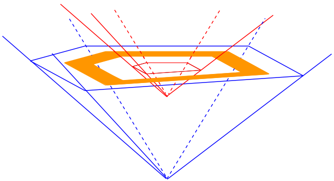

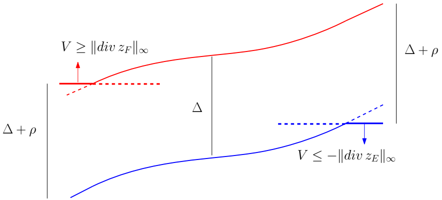

Parabolic maximum principle: In a strip $S \subset\subset F \setminus E$, we want to prove that $\Delta(t) \geq \Delta$ (at least for short time).

Distances are "rigid": $\Delta(t) \geq \Delta$ everywhere

Iteration: $\Delta(t) \geq \Delta$ for all times (before T^*).

Existence and uniqueness up to fattening

Theorem (Chambolle-M.-Ponsiglione, CPAM 2016)

Let ϕ be any anisotropy and u^0 be a uniformly continuous function in $\mathbb{R}^N$. Then, for all but countably many $s \in \mathbb{R}$ there exists a unique weak solution E_s of $V = -\phi(\nu^{E(t)})\kappa_\phi^{E(t)}$ with initial datum $E_s^0 := \{u^0 \geq s\}$. Moreover, such a solution is the limit of the minimizing movements scheme.

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After our preprint appeared, Giga-Pozar (preprint 2016): viscosity approach in three-dimensions for

$$V = -m(\nu^{E(t)})(\kappa_\phi^{E(t)} + 1)\,,$$

for bounded initial sets and when ϕ is purely crystalline.

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Chambolle-M.-Novaga-Ponsiglione 2017

The techniques of Chambolle-M.-Ponsiglione can be pushed to treat $V = -m(\nu^{E(t)})(\kappa_\phi^{E(t)} + g(x,t))$, *when* m *is* ϕ*-regular and* g *is bounded forcing term with spatial Lipschitz continuity*

General mobilities

Theorem (Chambolle-M.-Novaga-Ponsiglione 2017)

For any ϕ and m there exists a unique level set flow u with initial datum u^0 corresponding to $V = -m(\nu^{E(t)})(\kappa_\phi^{E(t)} + g(x,t))$.

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For any ϕ and m there exists a unique level set flow u with initial datum u^0 corresponding to $V = -m(\nu^{E(t)})(\kappa_\phi^{E(t)} + g(x,t))$. Moreover, for all but countably many $s \in \mathbb{R}$, the set flow $t \mapsto \{x : u(t,x) \geq s\}$ is the unique limit of the ATW scheme with initial set $\{u^0 \geq s\}$. Finally, the flow obeys the the comparison principle

General mobilities

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For any ϕ and m there exists a unique level set flow u with initial datum u^0 corresponding to $V = -m(\nu^{E(t)})(\kappa_\phi^{E(t)} + g(x,t))$. Moreover, for all but countably many $s \in \mathbb{R}$, the set flow $t \mapsto \{x : u(t,x) \geq s\}$ is the unique limit of the ATW scheme with initial set $\{u^0 \geq s\}$. Finally, the flow obeys the the comparison principle

- Idea: Let $m_n \to m$, where m_n is ϕ-regular. Then, by delicate stability estimates on the ATW scheme one can show that the corresponding level set flows $\{u_n\}$ admit a unique limit.

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Shortly after our preprint appeared, Giga-Pozar (preprint 2017): viscosity approach in N-dimensions for $V = -m(\nu^{E(t)})(\kappa_\phi^{E(t)} + 1)$, for bounded initial sets and when ϕ is purely crystalline.

Thank you for your attention!