“Dice”-sion Making under Uncertainty: When Can a Random Decision Reduce Risk?

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Facility Location under Uncertainty
Facility Location under Uncertainty

0.45 vs. 0.59
μ = 0.52
Facility Location under Uncertainty

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0.04 vs. 1.79
μ = 0.92
Facility Location under Uncertainty

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-0.05 vs. 3.50
μ = 1.73
Facility Location under Uncertainty

0.45 vs. 0.59
\( \mu = 0.52 \)

0.04 vs. 1.79
\( \mu = 0.92 \)

-0.05 vs. 3.50
\( \mu = 1.73 \)

-0.10 vs. 5.39
\( \mu = 2.65 \)
Facility Location under Uncertainty

Where should be build so as to maximize our expected profits?

- 0.45 vs. 0.59, \( \mu = 0.52 \)
- 0.04 vs. 1.79, \( \mu = 0.92 \)
- -0.05 vs. 3.50, \( \mu = 1.73 \)
- -0.10 vs. 5.39, \( \mu = 2.65 \)
- -0.51 vs. 6.62, \( \mu = 3.06 \)
Where should be build so as to maximize our expected profits?

-0.51 vs. 6.62
μ = 3.06

0.45 vs. 0.59
μ = 0.52

0.04 vs. 1.79
μ = 0.92

-0.05 vs. 3.50
μ = 1.73

-0.10 vs. 5.39
μ = 2.65
Risk Averse Decision-Making
Risk Averse Decision-Making

increasing means
Risk Averse Decision-Making

increasing spreads

increasing means
To account for the profit variation:
use mean semi-variance $\mathbb{E}[X] - \mathbb{E}[\mathbb{E}[X] - X]^2_+$.
Facility Location under Uncertainty

Where should be build so as to maximize the mean semi-variance?

- United Kingdom: 0.45 vs. 0.59, M/SV = 0.52
- Spain: 0.04 vs. 1.79, M/SV = 0.53
- Italy: -0.05 vs. 3.50, M/SV = 0.15
- France: -0.10 vs. 5.39, M/SV = -1.13
- Germany: -0.51 vs. 6.62, M/SV = -3.30
Facility Location under Uncertainty

Where should be build so as to maximize the mean semi-variance?

0.45 vs. 0.59  
M/SV = 0.52

0.04 vs. 1.79  
M/SV = 0.53

-0.05 vs. 3.50  
M/SV = 0.15

-0.10 vs. 5.39  
M/SV = -1.13

-0.51 vs. 6.62  
M/SV = -3.30
Facility Location under Uncertainty

Randomized decisions can reduce the risk:

M/SV = 0.59

0.45 vs. 0.59
M/SV = 0.52

0.04 vs. 1.79
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-0.05 vs. 3.50
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M/SV = -1.13

-0.51 vs. 6.62
M/SV = -3.30
Facility Location under Uncertainty

Randomized decisions can reduce the risk: 81% vs. 19%

M/SV = 0.69

0.45 vs. 0.59
M/SV = 0.52

0.04 vs. 1.79
M/SV = 0.53

-0.05 vs. 3.50
M/SV = 0.15

-0.10 vs. 5.39
M/SV = -1.13

-0.51 vs. 6.62
M/SV = -3.30
Ambiguity Averse Decision-Making

In practice, the probabilities for the profit scenarios may only be partially known:

Distributionally robust optimization: Optimize a risk measure over worst distribution in ambiguity set
Ambiguity Averse Decision-Making

Assume we want to maximize expected profits under the worst probability distribution in the ambiguity set.
Agenda

1. Motivation

2. Randomization under Distributional Ambiguity
   - Background
   - Problem Setup
   - The Power of Randomization

3. Discussion
Ambiguous Probability Spaces

We model uncertainty via an *ambiguous probability space*:

\[(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\]
Ambiguous Probability Spaces

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$$(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$$

$\Omega_0$ is the sample space:
Ambiguous Probability Spaces

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\[(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\]

\(\Omega_0\) is the sample space:

\(\mathcal{F}_0\) is the \(\sigma\)-algebra of events:

\(\emptyset, \bullet, \bullet, \bullet, \bullet, \bullet, \), ..., \(2^6 = 64\) sets
We model uncertainty via an *ambiguous* probability space:

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- **\(\Omega_0\)** is the sample space:
- **\(\mathcal{F}_0\)** is the \(\sigma\)-algebra of events:
  \[\emptyset, 1, 2, \ldots, 6\]
  \((2^6 = 64 \text{ sets})\)
- **\(\mathcal{P}_0\)** is the ambiguity set:
Ambiguous Probability Spaces

We model uncertainty via an *ambiguous* probability space:

\[(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\]

We denote by \(\mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) the real-valued random variables that are essentially bounded w.r.t. all \(\mathbb{P} \in \mathcal{P}_0\):

\[
\mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) = \bigcap_{\mathbb{P} \in \mathcal{P}_0} \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathbb{P})
\]
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\]

---

we think of

\(X\) as revenues
Ambiguous Probability Spaces

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We denote by $F_X^\mathbb{P}$ the *distribution function* of $X$ under $\mathbb{P} \in \mathcal{P}_0$:

$$F_X^\mathbb{P}(x) = \mathbb{P}(X \leq x) \quad \forall x \in \mathbb{R}$$

Let $\mathcal{D}$ be the set of all distribution functions
Ambiguous Probability Spaces

We model uncertainty via an *ambiguous* probability space:

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Ambiguous Probability Spaces

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We denote by \(F_X^\mathbb{P}\) the distribution function of \(X\) under \(\mathbb{P} \in \mathcal{P}_0:\)

\[F_X^\mathbb{P}(x) = \mathbb{P}(X \leq x) \quad \forall x \in \mathbb{R}\]

An ambiguous probability space \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is *non-atomic* if:

\[\exists U_0 \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\] that follows a uniform distribution on \([0, 1]\) under every probability measure \(\mathbb{P} \in \mathcal{P}_0.\)
A risk measure assigns each random variable a risk index:

\[ \rho_0 : \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \rightarrow \mathbb{R} \]
A **risk measure** assigns each random variable a risk index:

\[
\rho_0 : \mathcal{L}_\infty (\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \to \mathbb{R}
\]

\[
\begin{pmatrix}
  \begin{array}{cc}
    1 & \rightarrow -10 \\
    2 & \rightarrow 10 \\
    3 & \rightarrow 10 \\
    4 & \rightarrow -10 \\
    5 & \rightarrow 10 \\
    6 & \rightarrow 10 \\
  \end{array}
\end{pmatrix}
= 1
\]
A risk measure assigns each random variable a risk index:

\[ \rho_0 : \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \rightarrow \mathbb{R} \]

\[
\begin{array}{cccc}
\rho_0 & \rightarrow -10 & \rightarrow 10 \\
\rightarrow 10 & \rightarrow -10 \\
\rightarrow -10 & \rightarrow 10 \\
\end{array}
\]

= 1

\[
\begin{array}{cccc}
\rho_0 & \rightarrow -50 & \rightarrow 10 \\
\rightarrow 10 & \rightarrow -50 \\
\rightarrow -50 & \rightarrow 10 \\
\end{array}
\]

= 2
A risk measure assigns each random variable a risk index:

\[ \rho_0 : L_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \to \mathbb{R} \]

A risk measure \( \rho_0 \) is law invariant if it satisfies:

\[ \{ F^P_X : P \in \mathcal{P}_0 \} = \{ F^P_Y : P \in \mathcal{P}_0 \} \Rightarrow \rho_0(X) = \rho_0(Y) \]

\[ \forall X, Y \in L_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \]
A risk measure assigns each random variable a risk index:

\[ \rho_0 : L_{\infty}(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \to \mathbb{R} \]

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Risk Measures

A risk measure assigns each random variable a risk index:

\[ \rho_0 : \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \to \mathbb{R} \]

A risk measure \( \rho_0 \) is law invariant if it satisfies:

\[ \{ F_P^X : \mathbb{P} \in \mathcal{P}_0 \} = \{ F_P^Y : \mathbb{P} \in \mathcal{P}_0 \} \Rightarrow \rho_0(X) = \rho_0(Y) \]

\[ \forall X, Y \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \]
Proposition: Assume that \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is non-atomic and that \(\rho_0\) is law invariant.

- For all \(F \in \mathcal{D}\) there is \(X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) with \(F_X^\mathcal{P} = F\) for all \(\mathcal{P} \in \mathcal{P}_0\).
- There exists a unique \(\varrho_0 : \mathcal{D} \rightarrow \mathbb{R}\) satisfying

\[
\rho_0(X) = \varrho_0(F_X) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) : F_X^\mathcal{P} = F_X \quad \forall \mathcal{P} \in \mathcal{P}_0.
\]
Proposition: Assume that \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is non-atomic and that \(\rho_0\) is law invariant.

- For all \(F \in \mathcal{D}\) there is \(X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) with \(F_X^P = F\) for all \(P \in \mathcal{P}_0\).
- There exists a unique \(\varrho_0 : \mathcal{D} \to \mathbb{R}\) satisfying
  \[
  \rho_0(X) = \rho_0(F_X) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) : F_X^P = F_X \quad \forall P \in \mathcal{P}_0.
  \]

\[F_X^P = F_X \quad \forall P\]

\[F_X^P \neq F_X^{P'}\]
Ambiguity Averse Risk Measures

**Definition:** A risk measure $\rho_0$ is called ambiguity averse if it satisfies for all $X, Y \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$:

- **Ambiguity aversion:** If $\{F_X^\mathbb{P} : \mathbb{P} \in \mathcal{P}_0\} \subseteq \{F_Y^\mathbb{P} : \mathbb{P} \in \mathcal{P}_0\}$, then $\rho_0(X) \leq \rho_0(Y)$.

- **Ambiguity monotonicity:** If $\varrho_0(F_X^\mathbb{P}) \leq \varrho_0(F_Y^\mathbb{P})$ for all $\mathbb{P} \in \mathcal{P}_0$, then $\rho_0(X) \leq \rho_0(Y)$. 
Ambiguity Averse Risk Measures

**Definition:** A risk measure \( \rho_0 \) is called **ambiguity averse** if it satisfies for all \( X, Y \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \):

- **Ambiguity aversion:** If \( \{F^p_X : P \in \mathcal{P}_0\} \subseteq \{F^p_Y : P \in \mathcal{P}_0\} \), then \( \rho_0(X) \leq \rho_0(Y) \).

- **Ambiguity monotonicity:** If \( \xi_0(F^p_X) \leq \xi_0(F^p_Y) \) for all \( P \in \mathcal{P}_0 \), then \( \rho_0(X) \leq \rho_0(Y) \).
Ambiguity Averse Risk Measures

**Definition:** A risk measure $\rho_0$ is called *ambiguity averse* if it satisfies for all $X, Y \in L_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$:

1. **Ambiguity aversion:** If $\{F^P_X : P \in \mathcal{P}_0\} \subseteq \{F^P_Y : P \in \mathcal{P}_0\}$, then $\rho_0(X) \leq \rho_0(Y)$.

2. **Ambiguity monotonicity:** If $\varrho_0(F^P_X) \leq \varrho_0(F^P_Y)$ for all $P \in \mathcal{P}_0$, then $\rho_0(X) \leq \rho_0(Y)$.

\[
\begin{align*}
\varrho_0(F^P_X) &\leq \varrho_0(F^P_Y) \\
\Rightarrow \rho_0(X) &\leq \rho_0(Y)
\end{align*}
\]
Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that $\rho_0$ is ambiguity averse and translation invariant.
Then the risk measure satisfies

$$\rho_0(X) = \sup_{P \in \mathcal{P}_0} \varrho_0(F_X^P) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0).$$
**Representation of Risk Measures**

**Proposition:** Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that $\rho_0$ is ambiguity averse and translation invariant.

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$$\rho_0(X) = \sup_{P \in \mathcal{P}_0} \varrho_0(F^P_X) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0).$$
**Proposition:** Assume that \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is **non-atomic** and that \(\rho_0\) is **ambiguity averse** and **translation invariant**.

Then the risk measure satisfies

\[
\rho_0(X) = \sup_{P \in \mathcal{P}_0} \rho_0(F_X^P) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0).
\]
Agenda

1. **Motivation**

2. **Randomization under Distributional Ambiguity**
   - Background
   - Problem Setup
   - The Power of Randomization

3. **Discussion**
We consider the **abstract optimization problem**

\[
\text{minimize} \quad \rho_0(X) \\
\text{subject to} \quad X \in X_0
\]

where \( X_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \) denotes the **feasible region**.
Pure Strategy Problem

We consider the abstract optimization problem

\[
\begin{align*}
\text{minimize } & \quad \rho_0(X) \\
\text{subject to } & \quad X \in \mathcal{X}_0
\end{align*}
\]  

where \( \mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0,\mathcal{F}_0,\mathcal{P}_0) \) denotes the feasible region.

**Example:** Facility location

\[ \mathcal{X}_0 = \left\{ \right. \}

\[
\begin{array}{cccc}
\text{UK} & \text{Spain} & \text{Italy} & \text{France} & \text{Germany}
\end{array}
\]  

\[ \left. \right\} \]
Pure Strategy Problem

We consider the abstract optimization problem

\[
\minimize_{X \in \mathcal{X}_0} \rho_0(X)
\]

(PSP)

where \( \mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \) denotes the feasible region.

**Example: Portfolio optimization**

\[
\mathcal{X}_0 = \{ r^\top w : w \geq 0, e^\top w = 1 \} \quad \text{with} \quad r = \begin{pmatrix} r_{\text{IBM}} \\ r_{\text{Walmart}} \\ r_{\text{P&G}} \end{pmatrix}
\]
From Deterministic to Random Decisions

We consider the abstract optimization problem

$$\minimize_{X \in \mathcal{X}_0} \rho_0(X)$$

(PSP)

where $\mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ denotes the feasible region.
From Deterministic to Random Decisions

We consider the abstract optimization problem

$$\minimize_{X \in \mathcal{X}_0} \rho_0(X)$$

(PSP)

where $\mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ denotes the feasible region.

How can we randomize over decisions?

What is the risk of randomized decisions?
We assume we have a randomization device that generates uniform samples from $[0, 1]$:

**pure strategy problem**

$(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$

$X_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$

**randomized strategy problem**

$(\Omega, \mathcal{F}, \mathcal{P})$

with

- $\Omega = \Omega_0 \times [0, 1]$
- $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{B}_{[0,1]}$
- $\mathcal{P} = \{\mathbb{P} \times \mathcal{U} : \mathbb{P} \in \mathcal{P}\}$

$\mathcal{X} = \{X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) : X(\cdot, u) \in X_0 \ \forall u \in [0, 1]\}$
**Proposition:** Assume that \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is non-atomic and that \(\rho_0\) is ambiguity averse and translation invariant.

The unique extension of \(\rho_0\) to an ambiguity averse risk measure \(\rho\) on \(\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P})\) is given by

\[
\rho(X) = \sup_{P \in \mathcal{P}} \rho_0(F^P_X) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).
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**Proposition:** Assume that \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is non-atomic and that \(\rho_0\) is ambiguity averse and translation invariant.

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\[
\rho(X) = \sup_{\mathcal{P} \in \mathcal{P}} \rho_0(F_X^\mathcal{P}) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).
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Proposition: Assume that \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is non-atomic and that \(\rho_0\) is ambiguity averse and translation invariant.

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\[
\rho(X) = \sup_{P \in \mathcal{P}} \rho_0(F_X^P) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).
\]
Randomized Strategy Problem

We define the randomized strategy problem

\[
\minimize_{X \in \mathcal{X}} \rho(X) \quad \text{(RSP)}
\]

where the extended risk measure \( \rho \) is defined via

\[
\rho(X) = \sup_{P \in \mathcal{P}} \varrho_0(F^P_X) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).
\]

and \( \mathcal{X} \) denotes the enlarged feasible region:

\[
\mathcal{X} = \left\{ X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) : X(\cdot, u) \in \mathcal{X}_0 \quad \forall u \in [0, 1] \right\}
\]
Randomized Strategy Problem

We define the **randomized strategy problem**

\[
\min_{X \in \mathcal{X}} \rho(X) \tag{RSP}
\]

where the **extended risk measure** \(\rho\) is defined via

\[
\rho(X) = \sup_{P \in \mathcal{P}} \rho_0(F_X^P) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).
\]

and \(\mathcal{X}\) denotes the **enlarged feasible region**:

\[
\mathcal{X} = \{ X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) : X(\cdot, u) \in \mathcal{X}_0 \quad \forall u \in [0, 1] \}
\]

The feasible region contains all **pure strategies**:

\[
X_0 \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \quad \Rightarrow \quad X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) \text{ with } \\
X(\omega, u) = X_0(\omega) \quad \forall u \in [0, 1]
\]
Agenda

1. Randomization under Distributional Ambiguity
   - Background
   - Problem Setup
   - The Power of Randomization

2. Discussion
**Definition:** The ambiguity averse risk measure $\rho_0$ has the Lebesgue property if

$$\lim_{k \to \infty} \varrho_0(F_k) = \varrho_0(F) \quad \text{whenever} \quad F_k \to F.$$
The Power of Randomization

**Definition:** The ambiguity averse risk measure \( \rho_0 \) has the Lebesgue property if

\[
\lim_{k \to \infty} \varphi_0(F_k) = \varphi_0(F) \quad \text{whenever} \quad F_k \to F.
\]

**Theorem:** Assume that

- \((\Omega_0, \mathcal{F}_0, \mathcal{P}_0)\) is **non-atomic** and has a maximally ambiguous random variable,
- \(\rho_0\) is ambiguity averse and satisfies the Lebesgue property.

Then there is \(\mathcal{X}_0\) such that \((\text{PSP}) > (\text{RSP})\).
The Rainbow Urn Game

Consider an urn with balls of $K$ different colors where:

- the number of balls is unknown
- the proportions of colors are unknown
The Rainbow Urn Game

Consider an urn with balls of $K$ different colors where:
- the number of balls is unknown
- the proportions of colors are unknown

A player is offered the following game:

- player names color
- ball is drawn
- player receives:
  - $-1$ if ball is of stated color
  - $+1$ if ball is not of stated color
The Rainbow Urn Game

Assume the player uses an ambiguity averse risk measure $\rho_0$: 
Assume the player uses an ambiguity averse risk measure $\rho_0$:

Any *pure strategy* is as good as losing $1$ with certainty

All balls are of stated color
Assume the player uses an ambiguity averse risk measure $\rho_0$:

**Strategy**

- Any *pure strategy* is as good as losing $1$ with certainty.
- The *randomized strategy* that names each color with probability $1/K$ suppresses the ambiguity.

**Worst-case outcome**

- All balls are of stated color.
- The probability is:
  - $-1$ with probability $\frac{1}{K}$
  - $+1$ with probability $\frac{K-1}{K}$
The Rainbow Urn Game

Assume the player uses an ambiguity averse risk measure $\rho_0$:

**Strategy**
- Any *pure strategy* is as good as losing $1$ with certainty
- The *randomized strategy* that names each color with probability $1/K$ suppresses the ambiguity

**Worst-case outcome**
- All balls are of stated color
- $-\$1$ with probability $\frac{1}{K}$
- $+\$1$ with probability $\frac{K - 1}{K}$

If $\rho_0$ has the Lebesgue property, then this is as attractive as receiving $+\$1$ for sure as $K \to \infty$!

Randomization can serve as a cure for ambiguity.
Agenda

1. Motivation
2. Randomization under Stochastic Uncertainty
3. Discussion
Summary

Randomization
Receptive

Randomization
Proof

Stochastic
Uncertainty

Distributional
Ambiguity
Summary

Randomization

Receptive

Randomization Proof

Mixture quasi-convex RM’s

convex RM’s with convex $\mathcal{X}_0$
Randomization
Receptive

Mean moment &
mean deviation RM’s

Mean semi-moment &
mean semi-deviation RM’s

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Stochastic
Uncertainty

Distributional
Ambiguity
Summary

Randomization
  Receptive

Stochastic Uncertainty
  Mean moment & mean deviation RM’s
  Mean semi-moment & mean semi-deviation RM’s

Distributional Ambiguity
  every amb. av. RM with Lebesgue property

Randomization Proof
  Mixture quasi-convave RM’s
  convex RM’s with convex $\chi_0$
The Issue of Time Consistency
Remember the randomized strategy problem:

\[
\min_{X \in \mathcal{X}} \rho(X)
\]

(RSP)
The Issue of Time Consistency

Remember the randomized strategy problem:

\[
\minimize_{X \in \mathcal{X}} \rho(X)
\]

Once we observe the outcome of the randomization, we have an incentive to deviate in favour of the optimal pure choice!
The Issue of Time Consistency

Remember the randomized strategy problem:

\[
\text{minimize } \rho(X) \quad \forall X \in \mathcal{X}
\]

(RSP)

Once we observe the outcome of the randomization, we have an incentive to deviate in favour of the optimal pure choice!
Randomized decisions in economics:


Randomized decisions in algorithm design:


Randomized decisions in Markov decision processes:


Randomized decisions in SP and DRO:
