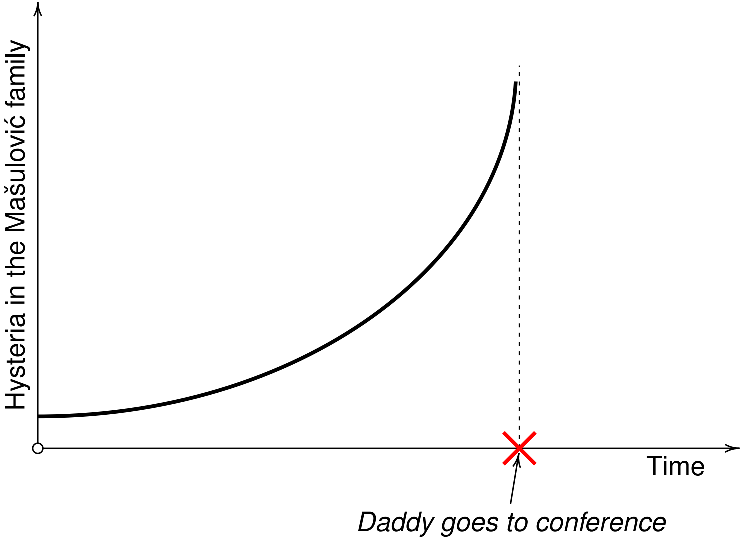
Structural Ramsey Theory from the Point of View of Category Theory

Dragan Mašulović

Department of Mathematics and Informatics
University of Novi Sad, Serbia

Unifying Themes in Ramsey Theory
BIRS, Banff, 23 Nov 2018


Hysteria in the Mašulović family
Time
Daddy goes to conference

Thank you!

In this talk

- "Quest for Ramsey classes" (see J. Nešetřil's talk)
- Overview of several strategies to obtain new Ramsey and dual Ramsey results *using the toolbox of Category Theory*.
- **Our final results are always combinatorial statements about finite structures and appropriate maps between them.**

Structural Ramsey Theory and Category Theory

K. LEEB: *The categories of combinatorics.* Combinatorial structures and their applications. Gordon and Breach, New York (1970).

R. L. GRAHAM, K. LEEB, B. L. ROTHSCHILD: *Ramsey's theorem for a class of categories.* Adv. Math. 8 (1972) 417–443.

J. NEŠETŘIL, V. RÖDL: *Dual Ramsey type theorems.* In: Z. Frolík (ed), Proc. Eighth Winter School on Abstract Analysis, Prague, 1980, 121–123.

H. J. PRÖMEL: *Induced partition properties of combinatorial cubes.* J. Combinat. Theory Ser. A, 39 (1985) 177–208.

Structural Ramsey Theory and Category Theory

H. J. PRÖMEL: *Induced partition properties of combinatorial cubes.* J. Combinat. Theory Ser. A, 39 (1985) 177–208.

$\mathbf{C}$ – a category; $A, B, C \in \mathrm{Ob}(\mathbf{C})$

Subobjects in a category:

- for $f, g \in \hom_{\mathbf{C}}(A, B)$:
  $f \sim g \quad \text{if} \quad f = g \cdot \alpha$ for some $\alpha \in \mathrm{Aut}(A)$;
- $\binom{B}{A} = \hom_{\mathbf{C}}(A, B)/\sim$.

Structural Ramsey Theory and Category Theory

H. J. PRÖMEL: *Induced partition properties of combinatorial cubes.* J. Combinat. Theory Ser. A, 39 (1985) 177–208.

C – a category; $A, B, C \in \mathrm{Ob}(\mathbf{C})$

Ramsey property for subobjects:

- $C \longrightarrow (B)^A_k$:
  for every coloring $\chi : \binom{C}{A} \to k$ there is a $w \in \hom_{\mathbf{C}}(B, C)$ such that $|\chi(w \cdot \binom{B}{A})| \leqslant 1$.
- **C** has the Ramsey property (for subobjects) if for every $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbf{C})$ there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $C \longrightarrow (B)^A_k$.

Structural Ramsey Theory and Category Theory

If the objects in **C** are rigid then all the $\sim$'s are trivial, so

$$\binom{B}{A} = \hom_{\mathbf{C}}(A, B).$$

Ramsey property for morphisms:

- $C \xrightarrow{mor} (B)^A_k$:
  for every coloring $\chi : \hom_{\mathbf{C}}(A, C) \to k$ there is a $w \in \hom_{\mathbf{C}}(B, C)$ such that $|\chi(w \cdot \hom_{\mathbf{C}}(A, B))| \leqslant 1$.
- **C** has the Ramsey property (for morphisms) if for every $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbf{C})$ there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $C \xrightarrow{mor} (B)^A_k$.

Structural Ramsey Theory and Category Theory

Benefits of categorification:

Technical: Category Theory has duality built into its foundations.

Psychological: Specifying a category brings morphisms explicitly to our attention.

Technological: Category Theory has many transfer principles.

Duality

$\mathbf{C}^{op} = \mathbf{C}$ with arrows and composition formally reversed

$\varphi^{op} = \varphi$ with notions replaced by the dual notions

The Duality Principle.
A statement φ is true in $\mathbf{C}$ if and only if φ^{op} is true in $\mathbf{C}^{op}$.

- $\mathbf{C}$ has the dual Ramsey property for subobj's (mor's) if $\mathbf{C}^{op}$ has the Ramsey property for subobj's (mor's).

Duality

Example.

Theorem. [Kechris, Pestov, Todorčević 2005; translation by M]
Let **C** be a category such that:

- morphisms are mono;
- there is a subcat of "finite obj's" and it is rich enough.

Let F be an ultrahomogeneous locally finite object in **C** whose automorphisms are finitely separated. TFAE:

1. Aut(F) endowed with "pointwise convergence topology" is extremely amenable;
2. Age(F) has the Ramsey property for morphisms.

Duality

Example.

Theorem. [for free]
Let **C** be a category such that:

- morphisms are epi;
- there is a subcat of "finite obj's" and it is rich enough.

Let F be a projectively ultrahomogeneous projectively locally finite object in **C** whose automorphisms are finitely projectively separated. TFAE:

1. Aut(F) endowed with "pointwise convergence topology" is extremely amenable;
2. Age(F) has the dual Ramsey property for morphisms.

Morphisms and the Ramsey property

Structural Ramsey Theory is not only about structures, but also about morphisms between them.

Sometimes, we have to add or fine-tune morphisms in order to get the Ramsey property.

Morphisms and the Ramsey property

Example. (adding morphisms)

Theorem. [M 2018+]
Let $\mathbf{V}$ be a nontrivial locally finite variety of lattices (as algebras) distinct from $\mathbf{L}$ and $\mathbf{D}$. Then no reasonable (JEP)-expansion of $\mathbf{V}^{fin}$ has the Ramsey property for morphisms.

Theorem. [M 2018+]
Let $\mathbf{V}$ be a nontrivial variety of lattices or semilattices (as algebras). Then $\overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ has both the Ramsey property and the ordering property.

NB. Semilattices as algebras $\to$ Sokić

Morphisms and the Ramsey property

Example. (fine-tuning morphisms)

Theorem. [M 2019+]
The following categories of finite structures whose morphisms are *strong rigid quotient maps* have the dual Ramsey property:

- linearly ordered graphs;
- posets with a linear extension;
- linearly ordered L-structures where L is a relational language.

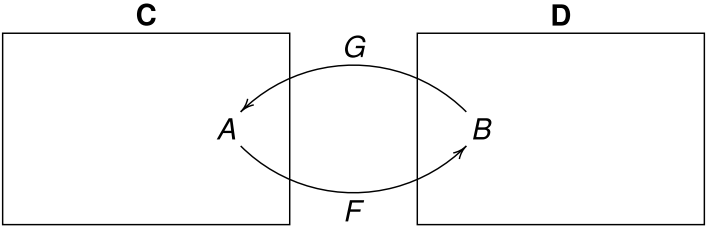
Open Problem.
Is it true that the category of finite linearly ordered graphs whose morphisms are *all rigid quotient maps* has the dual Ramsey property?

Transfer principles

Use categorical machinery to transfer the Ramsey property from one category onto the other.

- Isomorphism of categories.
- Categorical equivalence.
- Adjunctions.
- Pre-adjunctions.
- Products of categories.
- Passing to a "closed" subcategory.

Isomorphism of categories



Fact. If two categories are isomorphic and one of them has some kind of Ramsey property then so does the other.

Isomorphism of categories

Example. Canonical Ramsey Property.

- A category $\mathbf{C}$ has the canonical Ramsey property if for all $A, B \in \mathrm{Ob}(\mathbf{C})$ there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $C \xrightarrow{can} (B)^A$.
- $C \xrightarrow{can} (B)^A$:
  For every $\chi : \hom_{\mathbf{C}}(A, C) \to \omega$ there is a $w \in \hom_{\mathbf{C}}(B, C)$, a $Q \in \mathrm{Ob}(\mathbf{C})$ and a $q \in \hom_{\mathbf{C}}(Q, A)$ such that, for all $f, g \in \hom_{\mathbf{C}}(A, B)$:

$$\chi(w \cdot f) = \chi(w \cdot g) \text{ if and only if } f \cdot q = g \cdot q.$$

$$\begin{tikzcd} Q \arrow[r, "q"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, "w"] & C \end{tikzcd}$$

Isomorphism of categories

Example. Canonical Ramsey Property.

Proposition. [M 2018+]
The category of finite linearly ordered tournaments has the canonical Ramsey property.

Proof.

$$\mathbf{OTour} \cong \mathbf{OGra}$$

and

H. J. Prömel, B. Voigt: *Canonizing Ramsey theorems for finite graphs and hypergraphs.* Discrete Math. 54(1985), 49–59.

Isomorphism of categories

Example. The Problem of Kechris, Sokić and Todorčević.

Open Problem. [Homogeneous Dual Ramsey]
Prove that the category of finite chains and *homogeneous* rigid surjections has the dual Ramsey property.

- A rigid surjection $f : n \to m$ is *homogeneous* if $|f^{-1}(i)| = |f^{-1}(j)|$ for all $i, j < m$.

Isomorphism of categories

Example. The Problem of Kechris, Sokić and Todorčević.

Open Problem. [Homogeneous Dual Ramsey]
Prove that the category of finite chains and *homogeneous* rigid surjections has the dual Ramsey property.

$$\Updownarrow$$

Open Problem. Prove that the class

$$\{(\{0,1\}^n, d_n, \vec{0}, \prec_{lex}) : n \in \mathbb{N}\}$$

of linearly ordered metric spaces has the Ramsey property, where $\prec_{lex}$ is the lexicographic ordering of 01-strings and

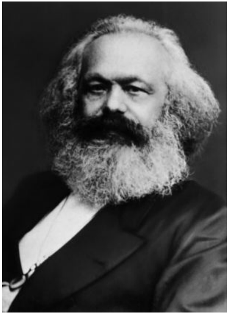
$$d_n(\vec{x}, \vec{y}) = \frac{\mathrm{Hamming}(\vec{x}, \vec{y})}{n}.$$

Isomorphism of categories

No extra work

$\Downarrow$

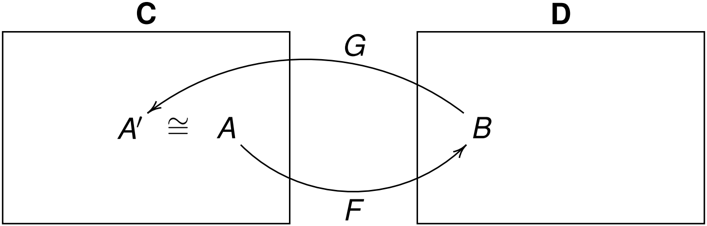
No added value



Karl Marx
1818–1883

Image courtesy of Wikipedia

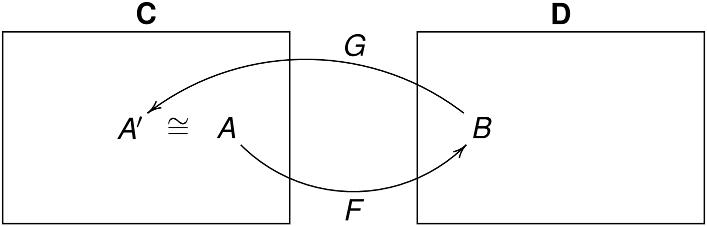
Categorical equivalence



Theorem. [M, Scow 2017]
If **C** and **D** are equivalent categories then one of them has the (dual) Ramsey property iff the other one does.

Example. [M, Scow 2017]
The category of finite naturally ordered powers of a primal algebra $+$ embeddings has the Ramsey property.

Categorical equivalence

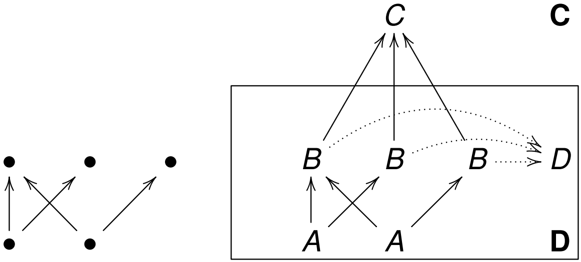


Theorem. [M, Scow 2017]
If **C** and **D** are dually equiv cat's then one of them has the Ramsey prop iff the other one has the dual Ramsey prop.

Example. [M, Mudrinski 2017]
The category of finite naturally ordered distrib lattices $+$ *positive surj lattice hom's* has the dual Ramsey property.

Passing to a “closed” subcategory

A subcategory “closed with respect to certain diagrams”:



$$\Delta \xrightarrow{F} \mathbf{D}$$

Passing to a “closed” subcategory

Theorem. [M 2017]
Let **D** be a subcategory of **C** “closed with respect to certain diagrams”.

- If **C** has the (dual) Ramsey property for morphisms, then so does **D**.
- If **D** is hereditary and **C** has the canonical Ramsey property, then so does **D**.
- If **D** is an age and **C** “has finite big Ramsey degrees”, then so does **D**.

Passing to a “closed” subcategory

Example. [M 2019+]
Canonical Ramsey propery for posets with linear extension.

Example. [M 2019+]
Every finite permutation has finite big Ramsey degree in $(\mathbb{Q}, <, \sqsubset)$ where $\sqsubset$ is a linear order of order type ω.

Example. [M 2019+]
Finite big Ramsey deg’s for a class of finite metric spaces.

K $\sqsubseteq_{\mathit{closed}}$ **EdgeColGra** [Sauer 2006]

Product of categories

Theorem. [Sokić 2012, translation by M 2017]
Assume that $\mathbf{C}_1$ and $\mathbf{C}_2$ are categories with the Ramsey property for morphisms where morphisms are monic and hom-sets are finite. Then $\mathbf{C}_1 \times \mathbf{C}_2$ has the Ramsey property for morphisms.

Product of categories

Theorem. [for free]
Assume that $\mathbf{C}_1$ and $\mathbf{C}_2$ are categories with the dual Ramsey property for morphisms where morphisms are epi and hom-sets are finite. Then $\mathbf{C}_1 \times \mathbf{C}_2$ has the dual Ramsey property for morphisms.

Product of categories

Strategy. If all the $\mathbf{C}_i$'s have the (dual) Ramsey property for morphisms and

$$\mathbf{D} \sqsubseteq_{closed} \mathbf{C}_1 \times \ldots \times \mathbf{C}_n$$

then $\mathbf{D}$ has the (dual) Ramsey property for morphisms.

Example. [M 2017]
Dual Ramsey propery for permutations (structures with two independent lin orders).

Example. [Draganić, M 2019+]
Ramsey propery for multiposets (structures with several partial orders conforming to a "template").

NB. This generalizes a recent result of Solecki and Zhao.

Pre-adjunctions

Definition. A pre-adjunction between **C** and **D** consists of

- a pair of maps $F : \mathrm{Ob}(\mathbf{D}) \rightleftarrows \mathrm{Ob}(\mathbf{C}) : G$, and
- a family of maps $\Phi_{Y,X} : \hom_{\mathbf{C}}(F(Y), X) \to \hom_{\mathbf{D}}(Y, G(X))$

such that:

$$\begin{tikzcd}
F(D) \arrow[r, "\forall u"] & \forall C \\
F(E) \arrow[u, "\exists v"] \arrow[ur, "u \cdot v"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\forall D \arrow[r, "\Phi_{D,C}(u)"] & G(C) \\
\forall E \arrow[u, "\forall f"] \arrow[ur, "\Phi_{E,C}(u \cdot v)"'] &
\end{tikzcd}$$

$$\Phi_{D,C}(u) \cdot f = \Phi_{E,C}(u \cdot v).$$

Pre-adjunctions

Theorem. [M 2018]
If **C** has the (dual) Ramsey property for morphisms and there is a pre-adjunction Ob(**D**) $\rightleftarrows$ Ob(**C**) then **D** has the (dual) Ramsey property for morphisms.

The first proof in this fashion: Ramsey property for **OGra**

H. J. PRÖMEL: *Ramsey Theory for Discrete Structures.*
Springer 2013.

Pre-adjunctions

Theorem. [M 2018]
If **C** has the (dual) Ramsey property for morphisms and there is a pre-adjunction $\mathrm{Ob}(\mathbf{D}) \rightleftarrows \mathrm{Ob}(\mathbf{C})$ then **D** has the (dual) Ramsey property for morphisms.

Example. [Nešetřil 2005, M 2018]
The category of linearly ordered metric spaces $+$ isometric embeddings has the Ramsey property.

$$\mathbf{GR} \quad \dashv \quad \mathbf{EPos} \quad \dashv \quad \mathbf{OMet}$$

Pre-adjunctions

Example. [M 2019+]
The following categories of finite structures whose morphisms are *strong rigid quotient maps* have the dual Ramsey property:

- linearly ordered graphs;
- linearly ordered hypergraphs;
- posets with a linear extension;
- linearly ordered L-structures where L is a relational language.

Example. [M 2018]
A purely categorical proof (modulo Graham-Rothschild Theorem) of the Nešetřil-Rödl Theorem *without forbidden substructures*.

Canonical pre-adjunctions

A technical modification of the notion of pre-adjunction.

Theorem. [M 2019+]
If **C** has the canonical Ramsey property and there is a canonical pre-adjunction Ob(**D**) $\rightleftarrows$ Ob(**C**) then **D** has the canonical Ramsey property.

Example. [M 2019+]
Canonical Ramsey property for

- metric spaces with "tight" distance sets; in particular rational and integral metric spaces;
- Canonical Nešetřil-Rödl Theorem *without forbidden substructures*.

Baire pre-adjunctions

C – enriched over **Top**, that is:

- hom-sets are topological spaces, and
- the composition is continuous.

$C \overset{\textit{Baire}}{\longrightarrow} (B)^A_k$ if for every Baire coloring ...

Theorem. [Prömel, Voigt 1985]
Let **C** be the category of chains and rigid surjections enriched over **Top** in the usual way. Then for every n and $k \geqslant 2$:

$$\omega \overset{\textit{Baire}}{\longrightarrow} (\omega)^n_k \ \text{ in } \ \mathbf{C}^{op}.$$

Baire pre-adjunctions

Definition. A Baire pre-adjunction between two categories enriched over **Top** is a pre-adjunction where all the maps

$$\Phi_{Y,X} : \hom_{\mathbf{C}}(F(Y), X) \to \hom_{\mathbf{D}}(Y, G(X))$$

are Baire maps.

Example. [M, unpublished]
Let **C** be the category of linearly ordered graphs and strong rigid surjections, enriched over **Top** in the usual way. Then for every finite linearly ordered graph G and $k \geqslant 2$:

$$\omega \cdot K_2 \xrightarrow{\textit{Baire}} (\omega \cdot K_2)^G_k \text{ in } \mathbf{C}^{op}.$$

Baire pre-adjunctions

Definition. A Baire pre-adjunction between two categories enriched over **Top** is a pre-adjunction where all the maps

$$\Phi_{Y,X} : \hom_{\mathbf{C}}(F(Y), X) \to \hom_{\mathbf{D}}(Y, G(X))$$

are Baire maps.

Example. [M, unpublished]
Let **C** be the category of posets with a linear extension and strong rigid surjections, enriched over **Top** in the usual way. Then for every finite poset P with a linear extension and $k \geqslant 2$:

$$\omega \cdot 2 \xrightarrow{\textit{Baire}} (\omega \cdot 2)^P_k \ \text{ in } \ \mathbf{C}^{op}.$$

Conclusion

This approach:

- *not self-sufficient*
  (we need a Ramsey result as an initial point);
- *not as powerful as the standard methods*
  (cannot handle classes defined by forbidden substruct's).

Conclusion

This approach:

- *not self-sufficient*
  (we need a Ramsey result as an initial point);
- *not as powerful as the standard methods*
  (cannot handle classes defined by forbidden substruct's).

This approach:

- can say something about Ramsey property;
- can say something about dual Ramsey property;
- can say something about canonical Ramsey property;
- can say something about Baire colorings;
- can say something about big Ramsey degrees.

  (And the small ones, but I was unable to squeeze this in.)