Dimers on Riemann Surfaces

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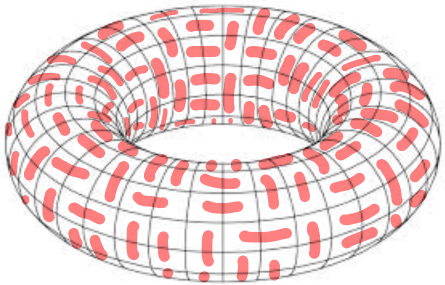

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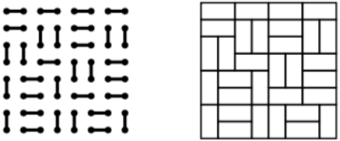
The dimer model

Definition

G = bipartite finite graph, planar
Dimer configuration = perfect matching on G:
each vertex incident to one edge
Dimer model: uniformly chosen configuration

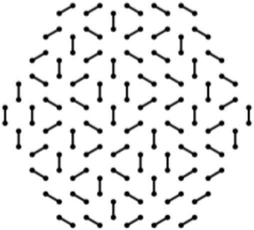


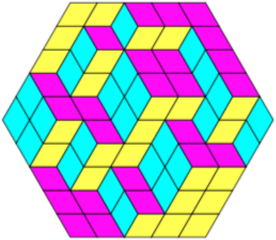
On square lattice, equivalent to domino tiling.

Dimer model as random surface

Example: honeycomb lattice

Dimer = lozenge tiling
Equivalently: stack of 3d cubes.





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Height function

Introduced by Thurston. Hence view as random surface.

Background

Classical model of statistical mechanics:

Kasteleyn, Temperley–Fisher 1960s
Kenyon, Propp, Lieb, Okounkov, Sheffield, Dubédat, de Tilière, Boutillier, Borodin, Petrov, Toninelli, Ferrari, Gorin, ... 1990s+

"Exactly Solvable": determinantal structure

$$\text{e.g.,}\quad Z_{m,n} = \prod_{j=1}^{m}\prod_{k=1}^{n}\left|2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right|^{1/2}$$

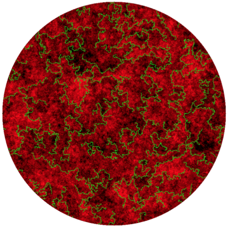
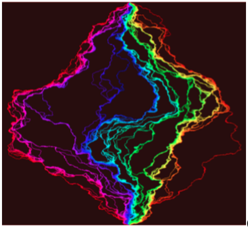
Mapping to other models:

Tilings, 6-vertex, Ising, **Uniform Spanning Trees** (UST)

Dimers and Imaginary Geometry

With B. Laslier and G. Ray, programme to obtain scaling limit <u>without</u> determinantal structure, in various geometries.

In simply connected regions, $\to$ Imaginary Geometry (Dubédat, Miller & Sheffield).

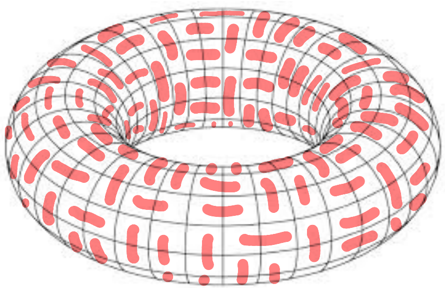
©G. Ray   ©J. Miller

GFF identified as field h whose flow lines are a continuum UST. This is a continuum form of Temperley's bijection.

This talk:

Dimers on Riemann surfaces.

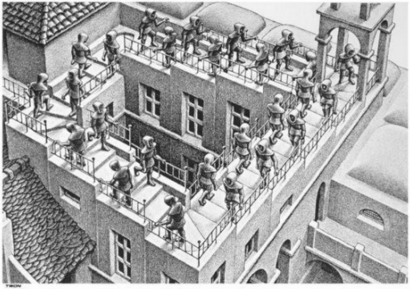
Goal: show existence of a universal limiting "height function" and conformal invariance.



(artistic rendering!)

Height as 1-form

In fact, "height function" is a closed 1-form (ie, ∇h well defined)



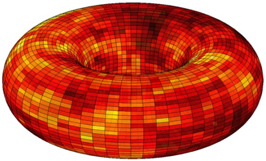
Hodge decomposition:

h consists of a function together with **instanton component** (a harmonic function on universal cover).

Some previous results:

Theorem (Boutillier and de Tilière, AoP 2009)

Convergence of instanton component for honeycomb lattice on torus + limit law: discrete Gaussian



Theorem (Dubédat, JAMS 2015)

Convergence of instanton <u>and</u> scalar component on torus for double isoradial graphs + limit law: compactified GFF

Theorem (Cimasoni, JEMS 2009)

On general surface, partition function = alternating sum of 2^{2g} determinants of Kasteleyn matrices. Coefficients given by Arf invariants.

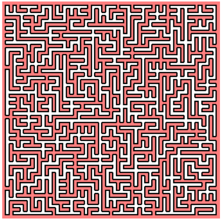
Temperley's bijection; Kenyon–Sheffield

Start with a UST on graph Γ.
Construct associated dimer config. on a modified graph G

Dimer configurations on $G \leftrightarrow$ UST on Γ
Height function $\leftrightarrow$ Winding of branches in tree

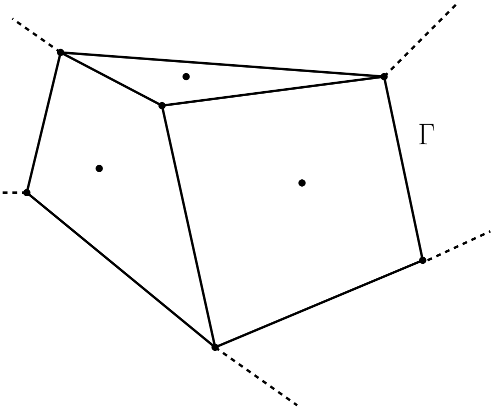
New goal:

Study winding of branches in UST.



Temperley's bijection: how does it work (1)?

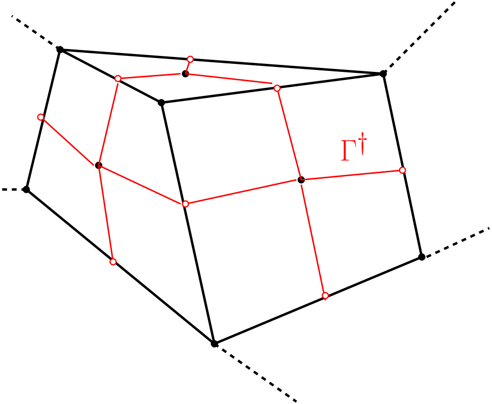
Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



Temperley's bijection: how does it work (2)?

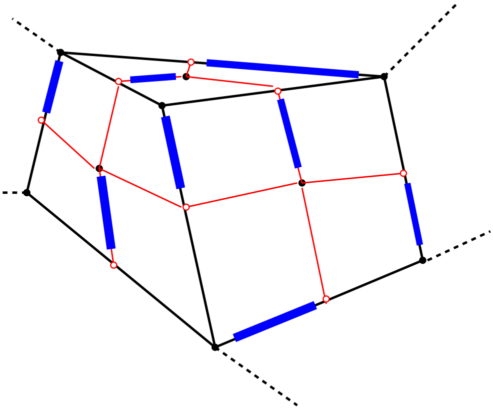
Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$:

(primal, dual _and_ medial lattice).



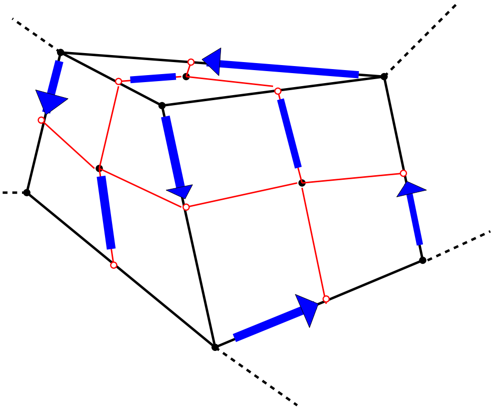
Temperley's bijection: how does it work (3)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



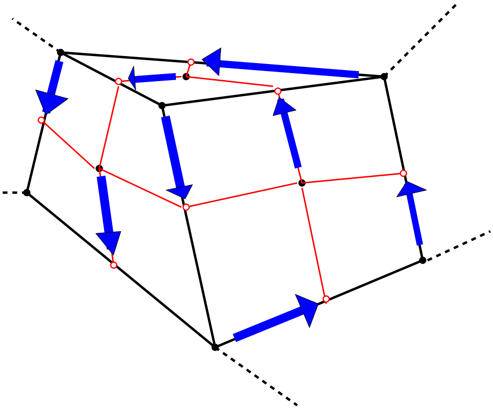
Temperley's bijection: how does it work (3.1)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



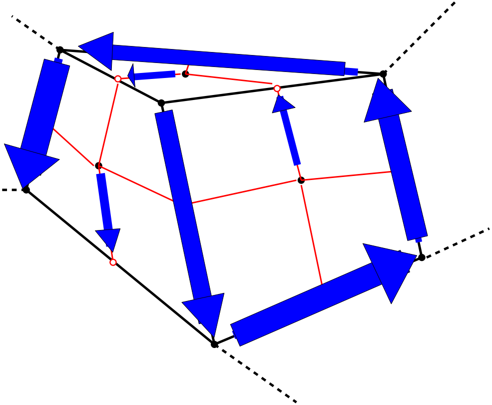
Temperley's bijection: how does it work (3.2)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



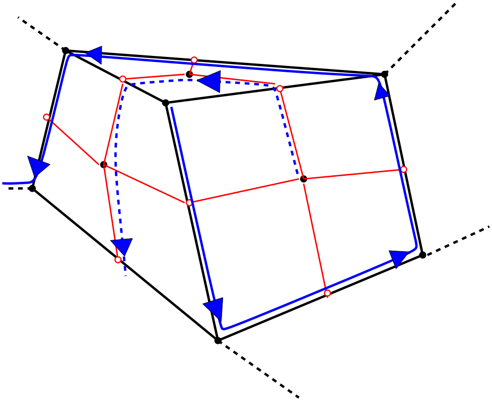
Temperley's bijection: how does it work (4)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



Temperley's bijection: how does it work (5)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



[Trees = oriented edges: each vertex has unique outgoing edge, except on boundary (wired).]

Starting point

Theorem 1 (B.–Laslier–Ray '19, ~~in preparation~~)

We extend Temperley's bijection to Riemann surfaces.
Instead of UST, "Temperleyan forests".

Extension of Temperley's bijection

If Γ a graph embedded on S, $\Gamma^\dagger$ its dual,
G = Superposition $\Gamma \cup \Gamma^\dagger$, + intermediate vertices.

Euler's formula for Γ:

$$V - E + F = \chi = 2 - 2g - b;$$

g = genus, b = boundary components.

However, for G to be dimerable, $V + F = E$ hence need:

$$\chi = 0, \text{ or } 2g + b = 2.$$

In simply connected case we need to remove one vertex (Kenyon–Propp–Wilson); and remove two on $\mathbb{S}^2$.

Extension of Temperley's bijection

Punctures

Lemma: Removing $2g + b - 2$ disjoint edges & medial vertices, G is dimerable.

Admit this for now.
Temperley's bijection is local, so can be applied here too.

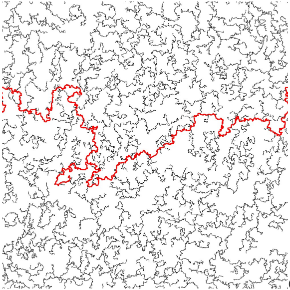
Instead of UST get a new object: "Temperleyan forests".

Extension of Temperley's bijection

Most natural generalisation of UST: Cycle Rooted Spanning Forest

Definition: CRSF

Oriented subgraph T of G:
– $\forall v \notin \partial G$, unique outgoing edge (except boundary: wired).
– Every cycle is non-contractible.



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Topological reasons: any component has at most one cycle;

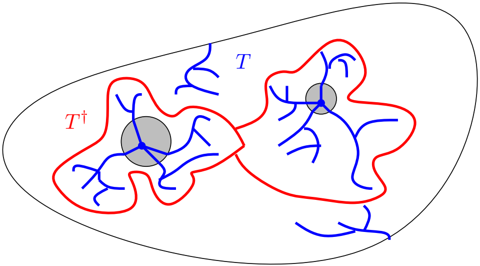
Temperleyan forest

Let T be oriented wired CRSF on Γ, $T^\dagger$ = dual.
Underline{Problem:} Can you orient dual $T^\dagger$? Not always possible!

Definition

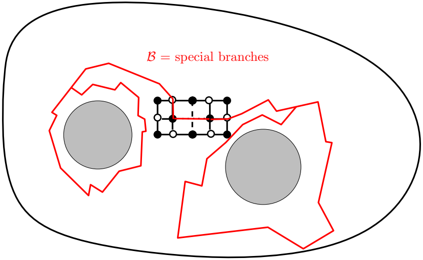
Call T **Temperleyan** if each connected component of $T^\dagger$ contains at most one cycle.

Ex: non-Temperleyan:



Characterisation of Temperleyan forests

Let $\mathcal{B}$ = special branches (emanating from punctures on either side)



Proposition

T is Temperleyan iff every component in the complement of $\mathcal{B}$ in M has the topology of an annulus or a torus.

Proof: "Pair of pants" decomposition.

Dimerability

Corollary

On torus/annulus, CRSF always Temperleyan (note $\chi = 0$).

(In fact, this is how the proposition is proved...)

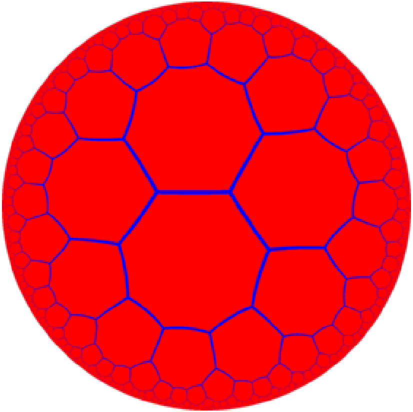
Corollary

Removing the $|\chi|$ punctures, the superposition graph is indeed dimerable.

Proof: apply pair of pants decomposition and Temperley's generalised bijection.

Height function and winding

Relation should persist if embed in flat surface: $\rightarrow$ universal cover.
But what to do if Temperleyan forest not connected?
Fuchsian theory: $S = \mathbb{D}/\Gamma$ (Fuchsian group).



Height function and winding

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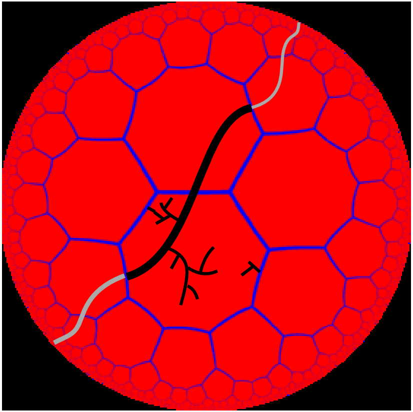


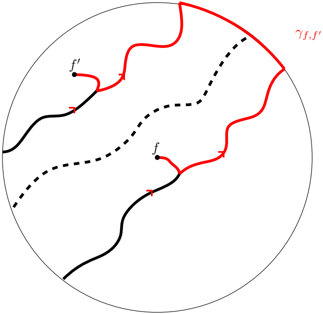
Height function and winding

With appropriate choice of reference flow we prove:

Theorem 2

$$h(f) - h(f') = W_{\text{int}}(\gamma_{f,f'}) + \pi \sum_{\sigma} (\varepsilon_\sigma + \delta_\sigma)$$

$\varepsilon_\sigma, \delta_\sigma \in \{-1, +1\}$ global topological terms (=algebraic count of primal and dual spines separating f and f').



⚠Note:
$\gamma_{f,f'}$ goes "through ∞"

Condition for convergence of height function

From [B.-Laslier-Ray, 2016], intrinsic winding is well-behaved.
Consequence:

Theorem 3

Let $\chi \leq 0$ be arbitrary. Suppose Temperleyan forest has a scaling limit (in Schramm topology).
Then both components of dimer height function converge.

Low Euler characteristic

When Euler's $\chi = 0$ (i.e., annulus or torus), by Proposition:
Temperleyan forest reduces to **Cycle Rooted Spanning Forest**:

$$\frac{d\mathbb{P}_{\text{Temp}}}{d\mathbb{P}_{\text{CRSF}}} = \frac{2^{\#\text{ dual cycles}}}{\mathbb{E}_{\text{CRSF}}(2^{\#\text{dual cycles}})} = \frac{2^{\#\text{ cycles}}}{\mathbb{E}_{\text{CRSF}}(2^{\#\text{ cycles}})}.$$

because 2 choices for orientation of each cycle.

Moreover, Wilson's algorithm for CRSF:
perform LERW but stop if make nontrivial loop.

Scaling limit of CRSF

Theorem 3 (B.–Laslier–Ray)

Let $\chi \leq 0$ arbitrary. Assume $(\star)$.

Then CRSF converges in Schramm space to universal, conformally invariant scaling limit.

Moreover, $\mathbb{E}_{\mathsf{CRSF}}(q^{\#\ \text{cycles}})$ uniformly bounded for any $q > 0$.

Solves some conjectures by Kassel-Kenyon.

$(\star)$ generic assumptions on graph:
SRW $\to$ BM on surface,
"rectangles" are crossed with positive probability ($\approx$ RSW).

Main conclusion (for now!)

Corollary

Dimer height function converges when $\chi = 0$.
Both components are **universal, conformally invariant**.

In torus case, proves conjecture by Dubédat, completing partial results by Dubédat-Ghessari.

In progress

Can handle Riemann surfaces of low complexity (Euler's $\chi = 0$)

Work on case $\chi < 0$ in progress ...
$\to$ existence of a conformally invariant scaling limit.

Conjecture: the limiting "height function" converges to the compactified GFF in the punctured surface

THANK YOU!