On the rough-smooth interface in the two-periodic Aztec diamond

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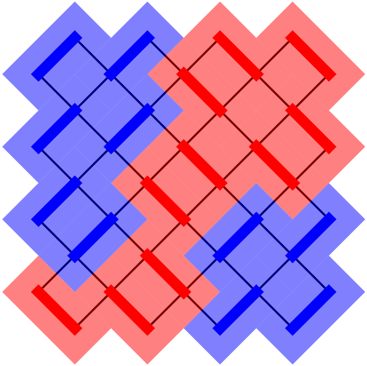
(Joint work with Sunil Chhita and Vincent Beffara)

BIRS, November 21, 2019

Aztec diamond

A **domino tiling** of an Aztec diamond shape corresponds to a **dimer configuration** on the Aztec graph.



Probability measure

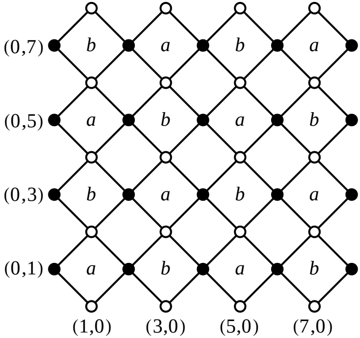
Let $\nu(e) > 0$ be the **weight** of the edge e in the graph $\mathcal{G}$. The probability of a certain **dimer cover** C, i.e. each vertex is covered exactly once, is

$$\frac{1}{Z} \prod_{e \in C} \nu(e).$$

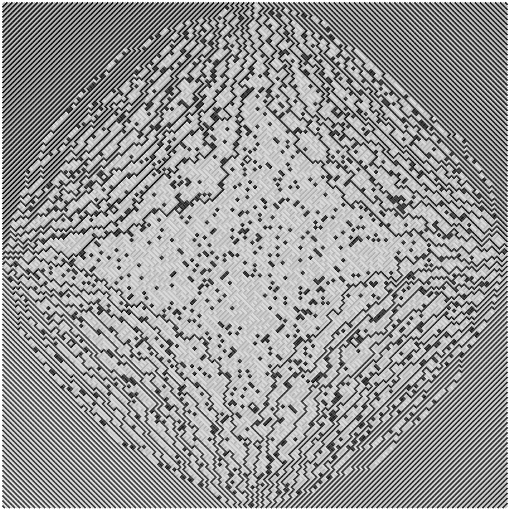
Z is the **partition function**.

Two Periodic Weighting

The **two-periodic weighting** of the Aztec diamond is defined in the following way. For a two-colouring of the faces, the edge weights around a particular coloured face alternate between a and b, we have *a-edges* and *b-edges*. E.g. for a size 4 Aztec diamond



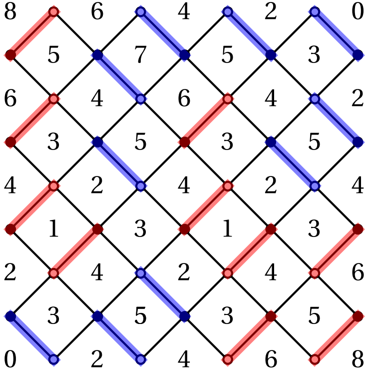
Random tiling of a two-periodic Aztec diamond



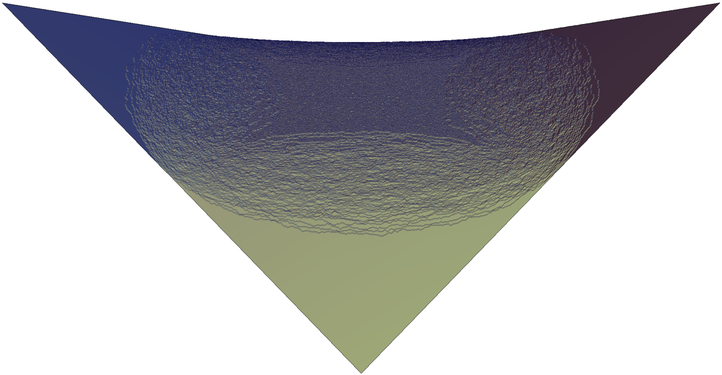
Aztec diamond height function

To each tiling of an Aztec diamond we can associate a **height function**. The heights sit on the faces of the Aztec graph. The height differences between two faces are given by

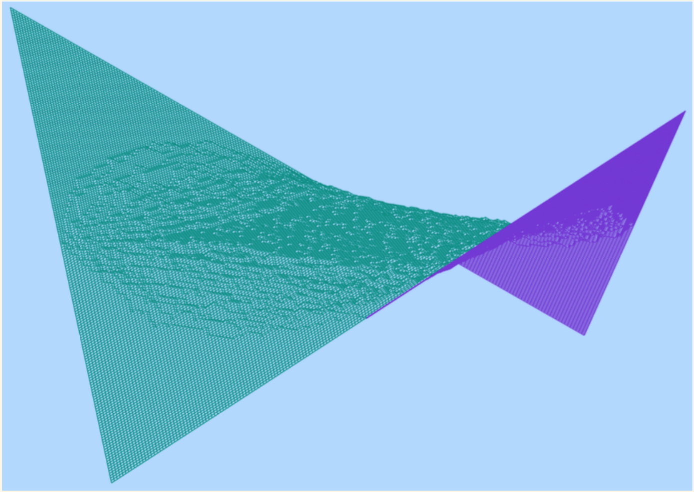
- $+3$ (-3) if we cross a dimer with a white vertex to the right (left)
- $+1$ (-1) if we do not cross a dimer and have a white vertex to the left (right)



Two-periodic Aztec diamond height function



Two-periodic Aztec diamond height function



Picture by V. Beffara

Kasteleyn Matrix

We choose a **Kasteleyn sign**, $s(e)$, $|s(e)| = 1$, for each edge with certain properties, and then define the **Kasteleyn matrix** $\mathbb{K}$ with elements

$$\mathbb{K}(b_i, w_j) = s(b_i, w_j)\nu(b_i, w_j).$$

This is a signed weighted adjacency matrix for the graph.
For the Aztec diamond graph we can take

$$\mathbb{K}(b, w) = \begin{cases} \nu(bw) & \text{if } e = (b, w) \text{ is horizontal} \\ \mathrm{i}\nu(bw) & \text{if } e = (b, w) \text{ is vertical} \\ 0 & \text{otherwise (i.e. no edge between } b \text{ and } w) \end{cases}$$

Kasteleyn's theorem

Let $\mathbb{K}$ be a Kasteleyn matrix

Theorem (Kasteleyn)

$$\det(\mathbb{K}) = SZ,$$

where Z is the partition function, and $|S| = 1$.

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It follows from Kasteleyn's theorem that

Theorem (Montroll-Potts-Ward, Kenyon)

If $e_i = (b_i, w_i)$, then the probability that $e_1, \ldots, e_m$ belong to a dimer cover is

$$\mathbb{P}(e_1, \ldots, e_m) = \det \left(\mathbb{K}(b_i, w_i)\mathbb{K}^{-1}(w_i, b_j)\right)_{1 \leq i,j \leq m}$$

This means that the dimers form a **determinantal point process** with correlation kernel $K(e_i, e_j) = \mathbb{K}(b_i, w_i)\mathbb{K}^{-1}(w_i, b_j)$, $e_i = (b_i, w_i)$.

A simulation of the two-periodic Aztec diamond

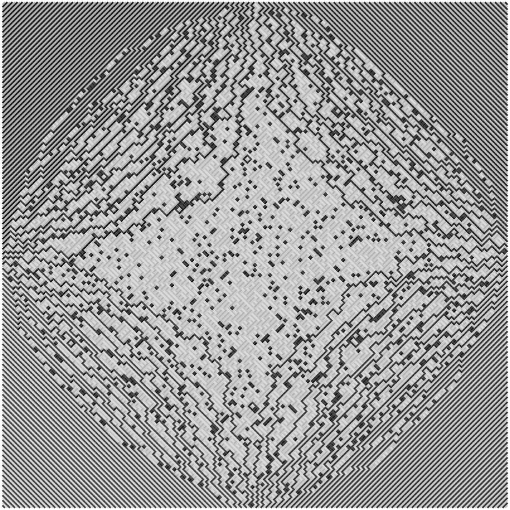
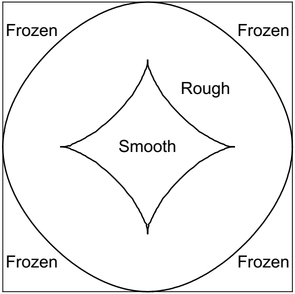


Figure: $n = 200$, $a = 0.5$, $b = 1$ with 8 grayscale colors

Phases



The curve in the picture is a degree 8 curve with two real components. We get three regions which are called **frozen, rough and smooth**.

Phases

Kenyon, Okounkov and Sheffield have characterized the different **limiting translation invariant Gibbs measures** that are possible for bipartite dimer models on the plane.

There are three classes of Gibbs measures, **frozen**, **rough** and **smooth**, given by an appropriate infinite, translation-invariant full-plane inverse Kasteleyn matrix $\mathbb{K}^{-1}$.

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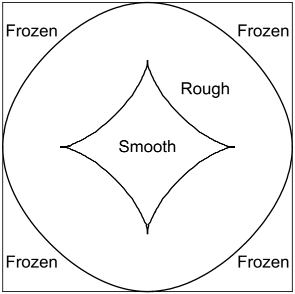
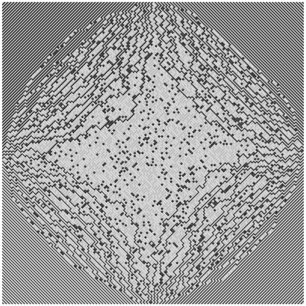
Correlations between dominos decay polynomially with distance in the rough region, and exponentially in the smooth region.

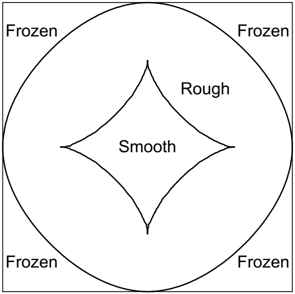
Rough-smooth boundary

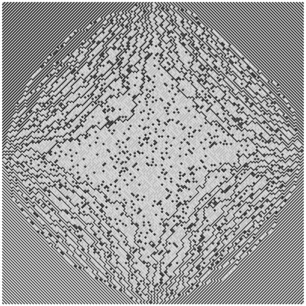
We now have two types of boundaries, the rough-frozen boundary and the rough-smooth boundary.





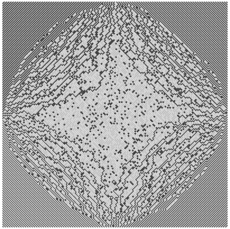
Rough-smooth boundary





What can we say about the interface fluctuations at the rough-smooth boundary? What is actually the interface?

Rough-smooth boundary

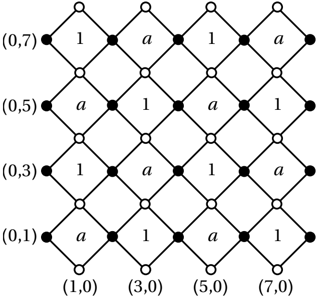


What can we say about the interface fluctuations at the rough-smooth boundary? What is actually the interface?

The rough-frozen interface is well-defined, the first place when the regular pattern is broken

At the rough-smooth boundary the situation is less clear. How should we define the interface combinatorially at the discrete level?

Formula for the inverse Kasteleyn matrix in the two-periodic case



The coordinate system that we use is indicated in the figure.

The inverse Kasteleyn Matrix

Theorem (Chhita-J. based on Chhita-Young)

Consider an Aztec diamond of size $n = 4m$ with the two-periodic weighting and let $\mathbb{K}_m$ be its Kasteleyn matrix. Then,

$$\mathbb{K}_m^{-1}((x_1, x_2), (y_1, y_2)) = \mathbb{K}_{sm}^{-1}((x_1, x_2), (y_1, y_2)) - \sum_{i=1}^{4} B_i((x_1, x_2), (y_1, y_2)),$$

where $\mathbb{K}_{sm}^{-1}$ is the full-plane inverse Kasteleyn matrix for the smooth phase, which has an explicit double integral formula, and $B_1, \ldots, B_4$ are contributions also given by explicit double integral formulas.

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Recently a more systematic approach has been developed to get the inverse Kasteleyn matrix or, more specifically, a closely related correlation kernel for an associated particle process, see *The two periodic Aztec diamond and matrix valued orthogonal polynomials*, by Maurice Duits, Arno B.J. Kuijlaars and *Correlation functions for determinantal processes defined by infinite block Toeplitz minors*, by T. Berggren, M. Duits.

Airy kernel point process

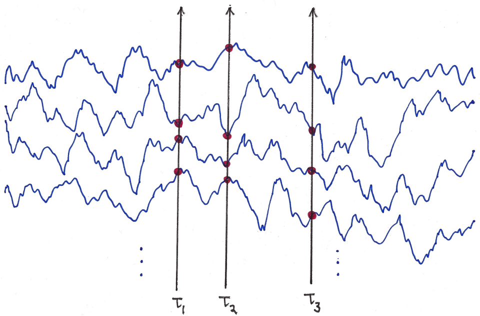

Figure: The **Airy line ensemble**. The top path is the **Airy process**.

Airy kernel point process

The **extended Airy point process** is a determinantal point process on parallel lines $\{\tau_q\} \times \mathbb{R}$, $1 \leq q \leq L_1$ in $\mathbb{R}^2$. We can think of it as a random measure μ_{Ai} defined via a Laplace transform. Let A_p, $1 \leq p \leq L_2$, be disjoint intervals in $\mathbb{R}$, $w_{p,q} \in \mathbb{C}$,

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\sum_{p=1}^{L_2}\sum_{q=1}^{L_1} w_{p,q}\mu_{\mathrm{Ai}}\left(\{\tau_q\} \times A_p\right)\right)\right] \\ &= \det\left(I + (e^{\Psi} - 1)K_{\mathrm{extAi}}\right)_{L^2(\{\tau_1,\dots,\tau_q\}\times\mathbb{R})},\end{aligned}$$

where

$$\Psi(x) = \sum_{q=1}^{L_1}\sum_{p=1}^{L_2} w_{p,q}\mathbb{I}_{\{\tau_q\}\times A_p}(x).$$

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where

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Recall that the **extended Airy kernel** is given by

$$K_{\mathrm{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2) = -\phi_{\tau_1,\tau_2}(\xi_1, \xi_2) + \tilde{K}_{\mathrm{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2),$$

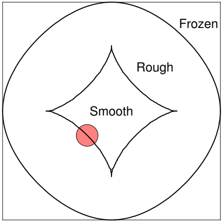
where

$$\tilde{K}_{\mathrm{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2) = \int_0^\infty e^{-\lambda(\tau_1-\tau_2)}\mathrm{Ai}\,(\xi_1 + \lambda)\mathrm{Ai}\,(\xi_2 + \lambda)\,d\lambda.$$

Asymptotics for the inverse Kasteleyn matrix at the rough-smooth boundary

Let $x = (x_1, x_2)$ be a white vertex and $y = (y_1, y_2)$ a black vertex. Scaling around a point at the rough-smooth boundary:

$$(x_1, x_2) = (4[\rho m] + 2[c_1 \xi_1 m^{1/3}](1,1) - 2[c_2 \tau_1 m^{2/3}](-1,1),$$
$$(y_1, y_2) = (4[\rho m] + 2[c_1 \xi_2 m^{1/3}](1,1) - 2[c_2 \tau_2 m^{2/3}](-1,1).$$



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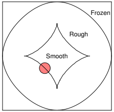
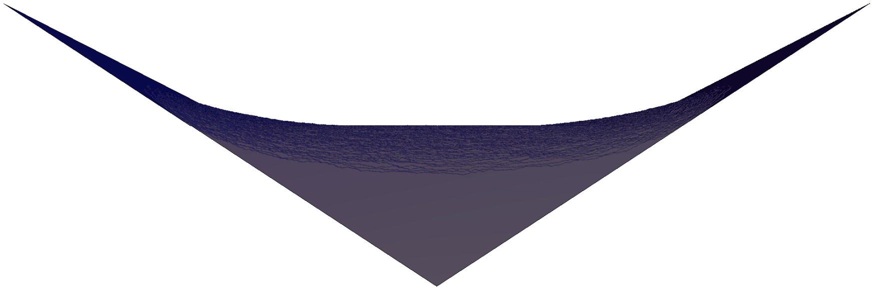


Asymptotics

$$\mathbb{K}_m^{-1}(x,y) = \mathbb{K}_{\text{sm}}^{-1}(x,y) - m^{-1/3}(\text{pre-factor})\tilde{K}_{\text{extAi}}(\tau_1, \xi_1 + \tau_1^2; \tau_2, \xi_2 + \tau_2^2)(1 + o(1))$$

as $m \to \infty$.

Random measure from the height function



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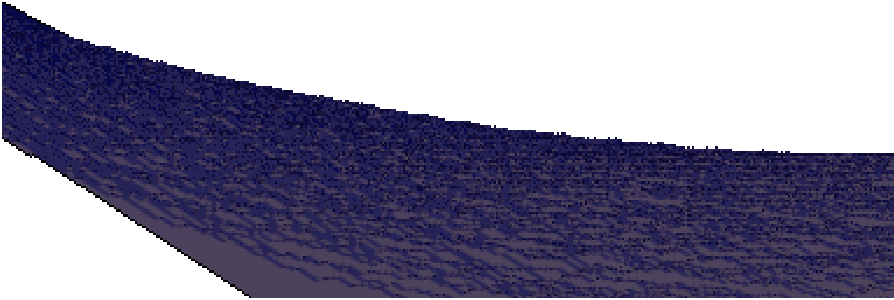
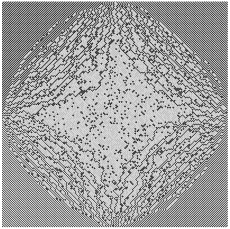


Random measure



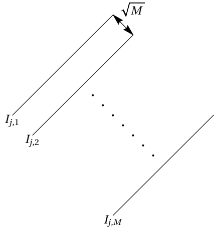
Height differences between two points in a vicinity of the rough-smooth boundary are due to two effects:

- Small and basically independent height fluctuations due to the "surrounding smooth phase".
- Long distance correlated effects due to the large scale structures that we see in the figure.
- By taking suitable averages of height differences we could hope to eliminate the small scale smooth phase effects. This is the idea behind the definition of a certain random signed measure.

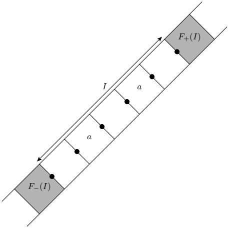
Random measure

Consider a two-periodic Aztec diamond of size $n = 4m$.

We want to imbed the intervals A_p as discrete intervals of length $\sim m^{1/3}$ in the Aztec diamond at the rough-smooth boundary. Consider only one interval, $A = [a^l, a^r]$. We want to imbed $M = [(\log m)^4]$ copies of it as discrete intervals starting and ending at a-faces a certain distance apart.



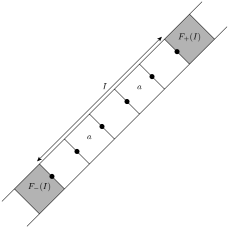
Random measure



The **height change** along a discrete interval

$$\Delta h(I) = h(F_+(I)) - h(F_-(I)).$$

Random measure



The **height change** along a discrete interval

$$\Delta h(I) = h(F_+(I)) - h(F_-(I)).$$

Random signed measure

$$\mu_m(\{\beta\} \times A) = \frac{1}{4M} \sum_{k=1}^{M} \Delta h(I_k).$$

Random measure

Random signed measure

$$\mu_m(\{\beta\} \times A) = \frac{1}{4M} \sum_{k=1}^{M} \Delta h(I_k).$$

Theorem (Beffara, Chhita, J., 18)

The random signed measure μ_m converges to μ_{Ai}.

For one interval the result is

$$\lim_{m \to \infty} \mathbb{E}\left[e^{w\mu_m(A)}\right] = \mathbb{E}\left[e^{w\mu_{\mathrm{Ai}}(A)}\right],$$

for $w \in \mathbb{C}$, $|w| < R$.

Squishing

An *a*-**dimer** is a dimer that covers an *a*-edge. They are **oriented** from white to black.

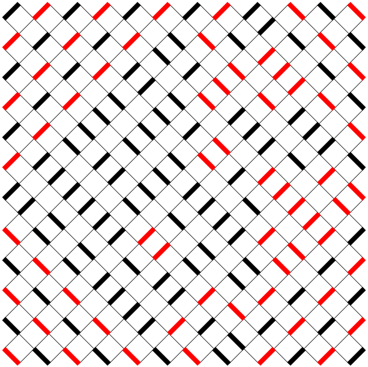
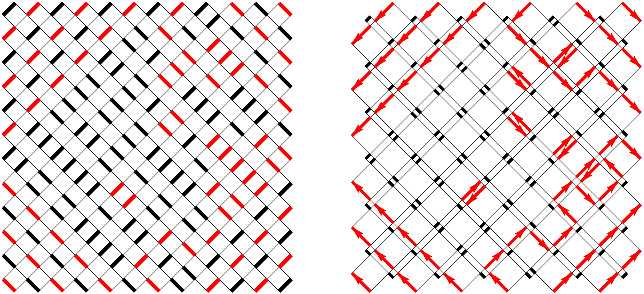


Figure: The red dimers are *a*-dimers, and the black *b*-dimers.

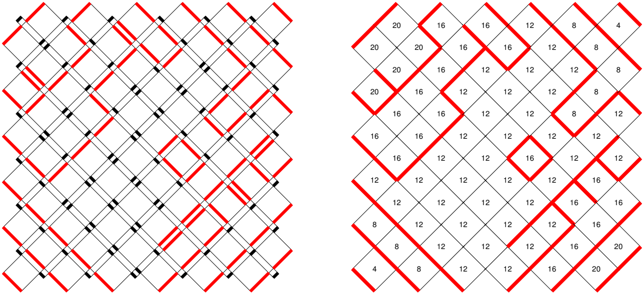
Squishing

We let the *b*-faces become smaller, go to zero in size.



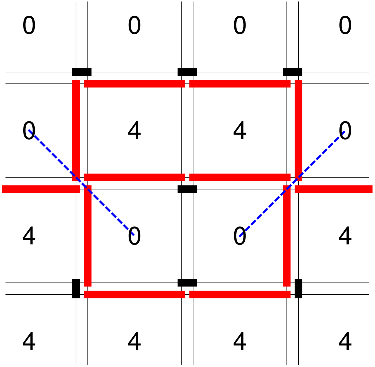
Squishing

We get **double edges**, **loops** and **paths**.



Paths and Loops

To get a unique split between paths and loops and get well-defined loops we need a convention. We use **mirrors**.



Paths

The paths go between the boundaries.

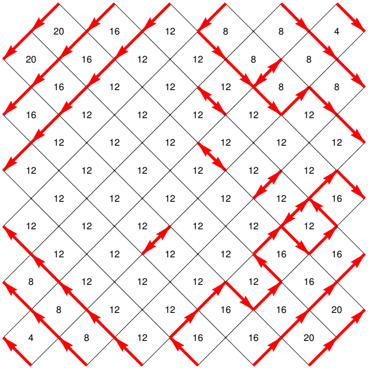


Figure: After squishing.

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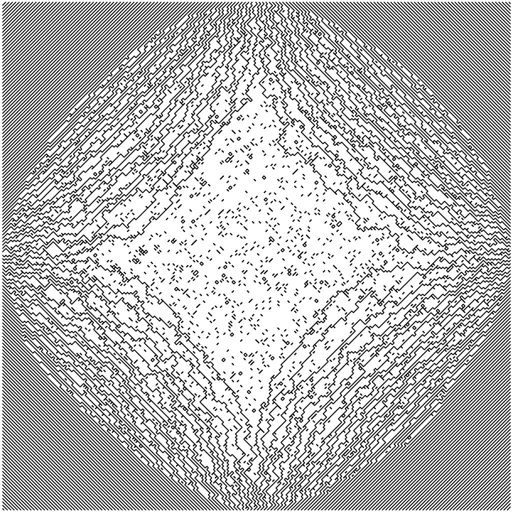
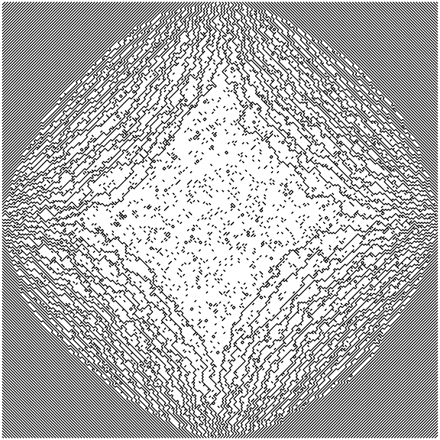


Figure: After squishing, $n = 300, a = 0.5$.

What we would like to prove

With high probability, if we go along the main diagonal there is a last path in the third quadrant close to the asymptotic rough-smooth boundary and this path converges to the Airy process.

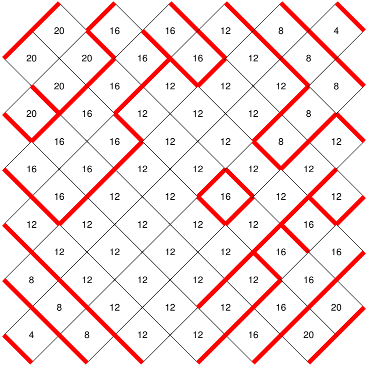


What we can prove

Let $h(f)$ be the height at the a-face f in the Aztec diamond. Then we can split it into two parts:

$$h(f) = h_\ell(f) + h_c(f)$$

where $h_\ell(f)$ is the **loop height** and $h_c(f)$ is the **corridor height**.



What we can prove

Assume that $a < 1/3$. Imbed the interval A as a discrete interval of length $\sim m^{1/3}$ in the Aztec diamond at the rough-smooth boundary. Define the **new random signed measure**

$$\kappa_m(\{\beta\} \times A) = \frac{1}{4}(h_c(F_+) - h_c(F_-)),$$

where F_+ and F_- are the end-faces of the discrete imbedded interval. Then $\kappa_m(\{\beta\} \times A)$ converges weakly to $\mu_{\text{Ai}}(\{\beta\} \times A)$ as $m \to \infty$, where μ_{Ai} is the Airy kernel point process.

We expect that with high probability κ_m is actually a positive measure. We should think of κ_m as counting the number of paths between the two faces.

Ingredients in the proof

- Show that the averaging in the random signed measure μ_m can be done along a single line instead of on parallel lines.
- We control the size of the loops; in a box of size L they are no bigger than $C \log L$. This uses a Peierls' type argument and requires $a < 1/3$.
- In a not too large region at the rough-smooth boundary the two-periodic Aztec measure can be replaced by the full-plane smooth measure.
- There are no bi-infinite paths in the full-plane smooth phase.

Thank you for listening!