Liouville type theorems and local behaviour of solutions to degenerate or singular problems

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Degenerate and singular operators

Let $z = (x, y) \in \Omega \subset \mathbb{R}^{n+1}$, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, $n \geq 1$, $a \in \mathbb{R}$. We are concerned with qualitative properties of solutions to a class of problems involving the operator in divergence form given by

$$\mathcal{L}_a u := \mathrm{div}(|y|^a A(x, y) \nabla u) \ ,$$

where the matrix A is symmetric, continuous, even in y, and satisfy the uniform ellipticity condition:

$$\lambda_1 |\xi|^2 \leq A(x, y)\xi \cdot \xi \leq \lambda_2 |\xi|^2 \ ,$$

for all $\xi \in \mathbb{R}^{n+1}$, for every $(x, y) \in \Omega$ and some ellipticity constants $0 < \lambda_1 \leq \lambda_2$.
We denote by $\Sigma := \{y = 0\} \subset \mathbb{R}^{n+1}$ the characteristic manifold that we assume to be invariant with respect to A.

Muckenhoupt weights

Our class of elliptic operators may be degenerate or singular, in the sense that the coefficients of the differential operator may vanish or be infinite over Σ, and this happens respectively when $a > 0$ and $a < 0$.
Such behaviour affects the regularity of solutions: indeed $u(x, y) = |y|^{-a} y$ is $\mathcal{L}_a$ harmonic, when $A \equiv \mathbb{I}$ and lacks smoothness whenever a is not an integer.
We recall that a function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is said an A_p weight if the following holds

A_p weights

$$\sup_{B \subset \mathbb{R}^{n+1}} \left(\frac{1}{|B|} \int_B w \right) \left(\frac{1}{|B|} \int_B w^{-1/(p-1)} \right)^{p-1} < \infty$$

where B is any ball. The weights $\rho(y) = |y|^a$ belong to the Muckenhoupt class A_p when $a \in (-1, p-1)$. In particular, they belong to the Muckenhoupt class A_2 when $a \in (-1, 1)$, and have studied in a seminal paper by Fabes, Kenig and Serapioni.

Motivations

- Such elliptic operators as those above arise in the study of fractional powers of elliptic operators as well as in applications to Physics, Ecology and biological sciences.
- Our study wants to point out that boundary conditions at Σ of Neumann and Dirichlet types play an important role in the regularity and qualitative behaviour of solutions when degeneratation or singularity affect the diffusion.
- We are particularly interested in qualitative properties which hold uniformly with respect to parameter of regularization of the problem.
- Our starting point for the study of degenerate and singular equations in divergence form is a series of papers written in the 80's by E. Fabes, C. Kenig, D. Jerison and R. Serapioni, which cover the range $a \in (-1, 1)$, proving Hölder continuity of all solutions.

A_2 weights

The fundamental properties of A_2 weights include:

Weighted Sobolev embeddings:

$$\left(\frac{1}{w(B)}\int_B |\varphi|^{2k} w(x)dx\right)^{1/2k} \le C\mathrm{diam}(B)\left(\frac{1}{w(B)}\int_B |\nabla\varphi|^2 w(x)dx\right)^{1/2}$$

for all $\varphi \in C_0^\infty(B)$.

Poincaré inequality:

$$\frac{1}{w(B)}\int_B |\varphi-\varphi_B|^2 w(x)dx \le C\mathrm{diam}(B)^2\frac{1}{w(B)}\int_B |\nabla\varphi|^2 w(x)dx$$

for all $\varphi \in C^1(\overline{B})$, where $\varphi_B = (1/w(B))\int_B \varphi w(x)dx$.

These properties entail the validity of Maximum principle, unique continuation priciple, Harnack inequality and eventually the Hölder continuity of all energy solutions.

The superdegenerate case

We are going to deal with even solutions and we will solve completely the problem in the energy space for all values $a \in (-1, \infty)$ and this range is wider than the A_2 one, i.e. $(-1, 1)$. Moreover, in the super degenerate range $a \geq 1$, the evenness assumption can not be removed for Hölder regularity (also for continuity). In fact, we have the following counterexample:

Example

When $a \geq 1$, the jump function

$$\overline{u}(z) = \begin{cases} 1 & \text{in } B_1^+ \\ -1 & \text{in } B_1^-, \end{cases}$$

is an energy (not even) $\mathcal{L}_a$-harmonic function. Even more, replacing the constant 1 (say) in B_1^+ by 0, one produces also an energy $\mathcal{L}_a$-harmonic function for which the unique continuation principle does not hold.

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Regularized operators for approximation

In order to better understand the regularity of solutions to degenerate and singular problems involving the operator L_a, but also the local behaviour near their nodal set and the geometric structure of the nodal set itself, we introduce a family of regularized operators. For $a \in \mathbb{R}$ fixed, let us consider the family in $\varepsilon \geq 0$ of functions $\rho_\varepsilon^a(y) : \Omega \to \mathbb{R}_+$ defined by

$$\rho_\varepsilon^a(y) := \begin{cases} (\varepsilon^2 + y^2)^{a/2} \min\{\varepsilon^{-a}, 1\} & \text{if } a \geq 0, \\ (\varepsilon^2 + y^2)^{a/2} \max\{\varepsilon^{-a}, 1\} & \text{if } a \leq 0, \end{cases}$$

and that of the associated operators

$$\mathcal{L}_{\rho_\varepsilon^a} u = \mathrm{div}\left(\rho_\varepsilon^a(y) \nabla u\right).$$

Inhomogenous equations

We are going to follow a perturbative method, actually allowing us to deal with more general equations with right hands in possibly divergence form, and to deal with an entire class of (possibly) regularised problems in the form ($\varepsilon \geq 0$):

$$-\mathrm{div}((\varepsilon^2 + y^2)^{a/2} A(x,y)\nabla u) = \begin{cases} \text{either} & (\varepsilon^2 + y^2)^{a/2} f(x,y)\ , \\ \text{or} & \mathrm{div}\left((\varepsilon^2 + y^2)^{a/2} F(x,y)\right)\ , \end{cases}$$

and derive both $C^{0,\alpha}$ and $C^{1,\alpha}$ estimates which are uniform with respect to the parameter $\varepsilon \geq 0$ (we shall refer to this fact as a *ε-stable property*).

The family $\{\rho_\varepsilon^a\}_\varepsilon$ satisfies the following conditions:

1) $\rho_\varepsilon^a(y) \to |y|^a$ as $\varepsilon \to 0^+$ almost everywhere in Ω,
2) $\rho_\varepsilon^a(y) = \rho_\varepsilon^a(-y)$,
3) for any $\varepsilon > 0$, the operator $-L_{\rho_\varepsilon^a}$ is uniformly elliptic,
4) fix $a \in \mathbb{R}$, then for any $\varepsilon > 0$

$$H^1(\Omega, \rho_\varepsilon^a(y)\mathrm{d}z) \subseteq H^1(\Omega; |y|^{\max\{a;0\}}\mathrm{d}z),$$

with a constant of immersion $c = c(a) > 0$ which does not depend on ε.

Condition 4) is due to the fact that on Ω, if $a \geq 0$, for $0 < \varepsilon_1 < \varepsilon_2 < +\infty$

$$|y|^a \leq \rho_{\varepsilon_1}^a(y) \leq \rho_{\varepsilon_2}^a(y) < (1 + \mathrm{diam}\Omega)^{a/2},$$

and if $a \leq 0$, for $0 < \varepsilon_1 < \varepsilon_2 < +\infty$

$$(1 + \mathrm{diam}(\Omega))^{a/2} < \rho_{\varepsilon_2}^a(y) \leq \rho_{\varepsilon_1}^a(y) \leq |y|^a.$$

Dual problems

The direct computation shows the following duality relation between the weights ρ_ε^a and ρ_ε^{-a} and the solutions of the associated elliptic equations:

Lemma

Let $a \in \mathbb{R}$, $\varepsilon > 0$ and let w be solution to

$$-\mathrm{div}\left(\rho_\varepsilon^a \nabla w\right) = \rho_\varepsilon^a f \qquad \text{in } B_1.$$

Then $v = \rho_\varepsilon^a \partial_y w$ solves

$$-\mathrm{div}\left(\rho_\varepsilon^{-a} \nabla v\right) = \partial_y f \qquad \text{in } B_1.$$

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Sobolev embeddings

Sobolev inequalities for weighted Sobolev spaces have been deeply studied in many contexts and by many authors as they play a key role in the regularity theory for elliptic PDEs.
We are concerned with a class of weighted Sobolev inequalities - not necessarily with the best constant but with the best exponent - for the class of approximating weights ρ_ε^a with constants which are uniform as $\varepsilon \to 0$. For this aim we can use the known results of Hajłasz about Sobolev spaces involving general measures, where, in our context the measure is naturally defined as $\mathrm{d}\mu = \rho_\varepsilon^a(y)\mathrm{d}z$. The basic requirement is a local growth on the measure of balls, which reflects in a local uniform in ε integrability condition of the weights. When $a \in (-1, +\infty)$, then a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ has $\mu(\Omega) < +\infty$ for any $\varepsilon \geq 0$. According with Hajłasz, a domain Ω is said to be d-regular with respect to μ if there exists $b > 0$ such that for any $z \in \Omega$, for any $r < \mathrm{diam}(\Omega)$,

$$\mu(B_r(z)) \geq br^d.$$

In our context, let us consider, up to rescaling the domain, that $\mathrm{diam}(\Omega) \leq 1$. If $a \in (-1, +\infty)$, then any bounded Ω is d-regular with respect to μ and the effective dimension

$$d = n + 1 + \max\{a, 0\}.$$

We remark that the constant $b > 0$ can be taken independent from $\varepsilon \geq 0$. Moreover, since we are interested in Sobolev inqualities with $p = 2$, we obtain that for functions $C^1_c(\Omega)$ there exists a constant which does not depend on $\varepsilon \geq 0$ such that

$$\left(\int_\Omega \rho_\varepsilon^a |u|^{2^*(a)} \right)^{2/2^*(a)} \leq c(d, b, p) \int_\Omega \rho_\varepsilon^a |\nabla u|^2,$$

where the optimal embedding exponent is

$$2^*(a) = \frac{2(n + 1 + \max\{a, 0\})}{n + \max\{a, 0\} - 1},$$

Energy solutions

Now, for any $a \in (-1, +\infty)$, we are in a position to give a notion of energy solution to the elliptic equation

$$-\mathcal{L}_a u = |y|^a f \qquad \text{in } B_1. \tag{E1}$$

Let $f \in L^p(B_1, |y|^a \mathrm{d}z)$ with $p \geq (2^*(a))'$ the conjugate exponent of $2^*(a)$; that is,

$$(2^*(a))' = \frac{2(n+1+\max\{a,0\})}{n+\max\{a,0\}+3}.$$

Definition

We say that $u \in H^1(B_1; |y|^a \mathrm{d}z)$ is an energy solution to (E1) if

$$\int_{B_1} |y|^a \nabla u \cdot \nabla \phi = \int_{B_1} |y|^a f \phi, \qquad \forall \phi \in H^1(B_1; |y|^a \mathrm{d}z).$$

Energy solutions

Moreover, for any $a \in (-1, +\infty)$ we can give a natural notion of energy solutions to

$$-\mathcal{L}_a u = \mathrm{div}(|y|^a F) \qquad \text{in } B_1. \tag{E2}$$

Definition

Let $F = (f_1, ..., f_{n+1})$ be defined on B_1 such that $F \in L^p(B_1, |y|^a \mathrm{d}z)$ with $p \geq 2$, then $u \in H^1(B_1; |y|^a \mathrm{d}z)$ is an energy solution to (E2) if

$$\int_{B_1} |y|^a \nabla u \cdot \nabla \phi = \int_{B_1} |y|^a F \cdot \nabla \phi, \qquad \forall \phi \in H^1(B_1; |y|^a \mathrm{d}z).$$

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Moser and Schauder estimates 1

Theorem

Let $a \in (-1, +\infty)$ and let $u \in H^{1,a}(B_1)$ be an even in y energy solution to

$$-\mathcal{L}_a u = |y|^a f \qquad \text{in } B_1,$$

with $f \in L^p(B_1, |y|^a \mathrm{d}z)$. Then

i) *If A is continuous, $\alpha \in (0,1) \cap (0, 2 - \frac{n+1+a^+}{p}]$, $p > \frac{n+1+a^+}{2}$, $\beta > 1$ and $r \in (0,1)$ one has: there exists a constant $c > 0$ such that*

$$\|u\|_{C^{0,\alpha}(B_r)} \leq c \left(\|u\|_{L^\beta(B_1, |y|^a \mathrm{d}z)} + \|f\|_{L^p(B_1, |y|^a \mathrm{d}z)} \right).$$

ii) *If A is α-Hölder continuous, $\alpha \in (0, 1 - \frac{n+1+a^+}{p}]$, $p > n+1+a^+$, $\beta > 1$ and $r \in (0,1)$ one has: there exists a constant $c > 0$ such that*

$$\|u\|_{C^{1,\alpha}(B_r)} \leq c \left(\|u\|_{L^\beta(B_1, |y|^a \mathrm{d}z)} + \|f\|_{L^p(B_1, |y|^a \mathrm{d}z)} \right).$$

Moreover, both these properties are ε-stable.

Moser and Schauder estimates for r.h.s. in divergence form

Theorem

Let $a \in (-1, +\infty)$ and let $u \in H^{1,a}(B_1)$ be an even in y energy solution to

$$-\mathcal{L}_a u = \mathrm{div}\left(|y|^a F\right) \qquad \text{in } B_1,$$

with $F = (f_1, ..., f_{n+1})$. Then

i) *If A is continuous, $\alpha \in (0, 1 - \frac{n+1+a^+}{p}]$, $F \in L^p(B_1, |y|^a \mathrm{d}z)$ with $p > n+1+a^+$, $\beta > 1$ and $r \in (0,1)$ one has: there exists a constant $c > 0$ such that*

$$\|u\|_{C^{0,\alpha}(B_r)} \le c\left(\|u\|_{L^\beta(B_1,|y|^a\mathrm{d}z)} + \|F\|_{L^p(B_1,|y|^a\mathrm{d}z)}\right).$$

ii) *If A is α-Hölder continuous and $F \in C^{0,\alpha}(B_1)$ with $\alpha \in (0,1)$, $f_{n+1}(x,0) = f^y(x,0) = 0$, $\beta > 1$ and $r \in (0,1)$ one has that there exists a constant $c > 0$ such that*

$$\|u\|_{C^{1,\alpha}(B_r)} \le c\left(\|u\|_{L^\beta(B_1,|y|^a\mathrm{d}z)} + \|F\|_{C^{0,\alpha}(B_1)}\right).$$

Moreover, both these estimates are ε-stable.

Higher order Schauder estimates

Theorem

Let $a \in (-1, +\infty)$, $k \in \mathbb{N} \cup \{0\}$ and $f \in C^{k,\alpha}(B_1)$ for $\alpha \in (0,1)$ and even in y. Let also $u \in H^{1,a}(B_1)$ be an even energy solution to (see the next section for the precise definition)

$$-\mathrm{div}(|y|^a \nabla u) = |y|^a f \qquad \text{in } B_1.$$

Then, $u \in C^{k+2,\alpha}_{\mathrm{loc}}(B_1)$. If moreover $f \in C^\infty(B_1)$, then, $u \in C^\infty(B_1)$.

This result is somewhat surprising, in view of the lack of regularity of the coefficients of the operators and can be attributed to the joint regularising effect of the equation and the Neumann boundary condition in the half ball, associated with evenness. We stress that odd solutions may indeed lack regularity, as shown by the example $u(x,y) = |y|^{-a}y$ which solves $-\mathrm{div}(|y|^a \nabla u) = 0$ whenever $a \in (-1,1)$.

Schauder estimates

Theorem

Let $a \in (-1,+\infty)$, $k \in \mathbb{N} \cup \{0\}$ *and* $f \in C^{k,\overline{\alpha}}(B_1)$ *for* $\overline{\alpha} \in (0,1]$ *and even in* y. *Let also* $u \in H^{1,a}(B_1)$ *be an even energy solution to*

$$-\mathrm{div}\left(|y|^a \nabla u\right) = |y|^a f(x,y) \qquad \text{in } B_1 \ .$$

Then, $u \in C^{k+2,\alpha}_{\mathrm{loc}}(B_1)$ *for any* $\alpha \in (0,\overline{\alpha})$. *If moreover* $f \in C^\infty(B_1)$, *then,* $u \in C^\infty(B_1)$.

Scheme of the proof

1. As already said, we first regularize the problem by introducing a parameter ε such that the operator becomes uniformly elliptic when $\varepsilon > 0$;
2. by means of appropriate Liouville-type theorems, which may be of independent interest, we then obtain uniform estimates in $\varepsilon \geq 0$ in Hölder spaces $C^{0,\alpha}$ and $C^{1,\alpha}$ for even solutions in y. This is the main part of the paper and relies heavily on some spectral properties;
3. we prove that all solution to the singular/degenerate equation can be obtain as limits of solutions to a sequence of regularized problems;
4. to provide higher regularity in the case $A = \mathbb{I}$, we use the structure of the operator $\mathcal{L}_a$, the evenness of the solutions and algebraic manipulations.

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The mother Liouville theorem

Theorem

Let $a \in (-\infty, 1)$*,* $\varepsilon \geq 0$ *and* w *be a (locally) energy solution to*

$$\begin{cases} -\operatorname{div}(\rho_\varepsilon^a(y)\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1}$$

and let us suppose that for some $\gamma \in [0, 1)$*,* $C > 0$ *there holds*

$$|w(z)| \leq C\rho_\varepsilon^{-a}(y)(1 + |z|^\gamma) \tag{2}$$

for every $z = (x, y)$*. Then* w *is identically zero.*

The baby Liouville theorem 1

Corollary

Let $a \in (-1, +\infty)$, $\varepsilon \geq 0$ *and* w *be a solution to*

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a(y)\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \rho_\varepsilon^a \partial_y w = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases}$$

and let us suppose that for some $\gamma \in [0,1)$, $C > 0$ *it holds*

$$|w(z)| \leq C(1 + |z|^\gamma)$$

for every z. *Then* w *is constant.*

The baby Liouville theorem 2

Corollary

Let $a \in (-1, +\infty)$, $\varepsilon \geq 0$ *and* w *be a solution to*

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a(y)\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \rho_\varepsilon^a \partial_y w = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases}$$

and let us suppose that for some $\gamma \in [0, 1)$, $C > 0$ *it holds*

$$|\nabla w(z)| \leq C(1 + |z|^\gamma)$$

for every z. *Then* w *is a linear function.*

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Non homogenous Neumann boundary conditions

$\mathcal{L}_a$-harmonic functions have been widely studied in connection with fractional Laplacians, in view of the well known realization of $(-\Delta)^s$ as a Dirichlet-to-Neumann operator, when $s \in (0,1)$. Following this approach, H^s-functions over $\mathbb{R}^n$ are uniquely extended in $\mathbb{R}^{n+1}_+$ by convolution with the Poisson kernel of $\mathcal{L}_a$, where $a = 1-2s \in (-1,1)$. Thus, in order to study the equation $(-\Delta)^s \cdot = f$, one is lead to deal with global finite energy solutions to

$$\begin{cases} -\mathcal{L}_a u = 0 & y > 0 \\ -\lim_{y\to 0} y^a \partial_y u(x,y) = f(x) & y = 0\,. \end{cases}$$

In this light, we are naturally lead to extend our analysis to inhomogeneous Neumann boundary value problems associated with $\mathcal{L}_a$. Again, we shall mainly (though not exclusively) seek ε-stable estimates.

$C^{0,\alpha}$ estimates

Theorem

Let $a \in (-1,1)$ and let $u \in H^{1,a}(B_1^+)$ be an energy solution to

$$\begin{cases} -\mathcal{L}_a u = 0 & \text{in } B_1^+ \\ -\lim_{y\to 0} y^a \partial_y u(x,y) = f(x) & \text{on } \partial^0 B_1^+ \end{cases}, \qquad \text{(INBC)}$$

with $f \in L^p(\partial^0 B_1^+)$. Then, if A is continuous, $p > \frac{n}{1-a}$, $\alpha \in (0, 1-a-\frac{n}{p}] \cap (0,1)$, $\beta > 1$ and $r \in (0,1)$, there exists a constant $c > 0$ such that

$$\|u\|_{C^{0,\alpha}(B_r)} \le c\left(\|u\|_{L^\beta(B_1,|y|^a \mathrm{d}z)} + \|f\|_{L^p(\partial^0 B_1^+)}\right).$$

Moreover, if $p > \frac{n}{1-a^+}$ and $\alpha \in (0, 1-a^+-\frac{n}{p}] \cap (0,1)$ this estimate is ε-stable.

$C^{1,\alpha}$ estimates

In order to prove the $C^{1,\alpha}$ estimates, we have to restrict ourselves to the cases $a<0$.

Theorem

Let $a \in (-1,0)$ and let $u \in H^{1,a}(B_1^+)$ be an energy solution to (INBC). Assume A is α-Hölder continuous. Then,

i) *if $f \in L^p(\partial^0 B_1^+)$ with $p > \frac{n}{-a}$, $\alpha \in (0, -a-\frac{n}{p}]$, $\beta > 1$ and $r \in (0,1)$ one has: there exists a constant $c>0$ such that*

$$\|u\|_{C^{1,\alpha}(B_r^+)} \leq c\left(\|u\|_{L^\beta(B_1^+, y^a \mathrm{d}z)} + \|f\|_{L^p(\partial^0 B_1^+)}\right).$$

ii) *If $f \in C^{0,\alpha}(\partial^0 B_1^+)$ with $\alpha \in (0,-a]$, $r \in (0,1)$ and $\beta > 1$ one has: there exists a constant $c>0$ such that*

$$\|u\|_{C^{1,\alpha}(B_r^+)} \leq c\left(\|u\|_{L^\beta(B_1^+, \rho_\varepsilon^a(y) \mathrm{d}z)} + \|f\|_{C^{0,\alpha}(\partial^0 B_1^+)}\right).$$

Moreover, this property is ε-stable.

These results should be compared with the known $C^{0,\alpha}$ and $C^{1,\alpha}$ local estimates for solutions to inhomogeneous fractional Laplace equations by Silvestre, Caffarelli-Stinga, and other authors. Their method is essentially based on singular integrals involving Riesz potentials. It is worthwhile noticing, however that we take a different perspective; first of all we seek for regularity in all the $n+1$ variables, while the quoted papers deal with the regularity in the x-variable only. The presence of the special solution y^{1-a} gives a necessary bound $\alpha \leq -a$ in the $C^{1,\alpha}$ estimate. Moreover, our results apply to the whole family of regularizing weights, with constants which are uniform in ε. Eventually we remark that our results provide local C^∞-regularity for extensions of s-harmonic functions (homogeneous Neumann boundary condition) in any variable (also the extension variable y) up to the characteristic manifold Σ.