Tensor network representations from the geometry of entangled states

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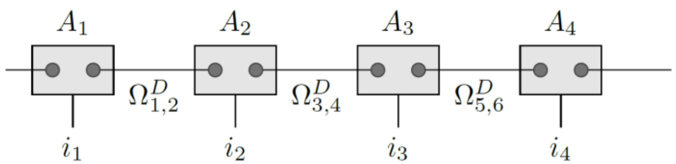
arXiv:1809.08185

Alexander von Humboldt
Stiftung / Foundation

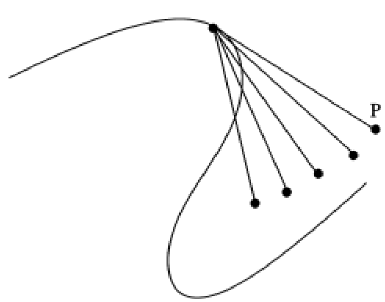
BIRS workshop: Quantum Walks and Information Tasks

Outline

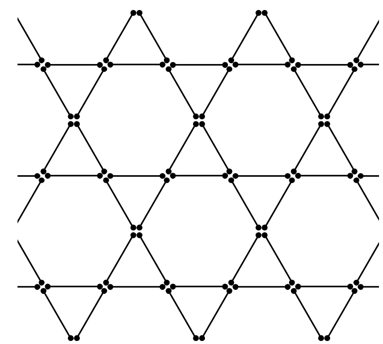
1. Matrix product states



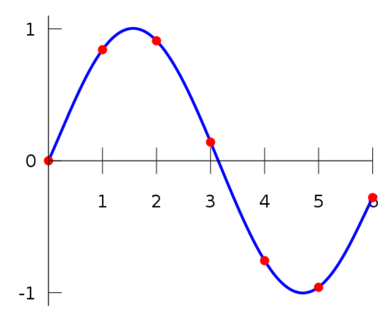
2. Geometry of entanglement



3. Tensor Networks



4. Reducing the bond dimension



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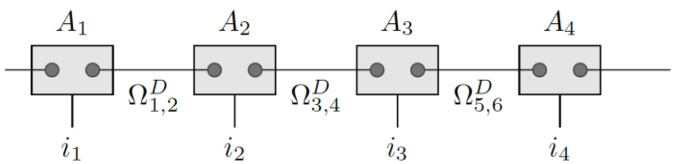

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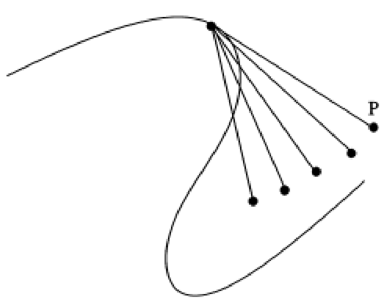

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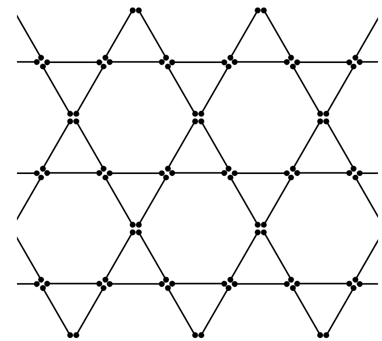

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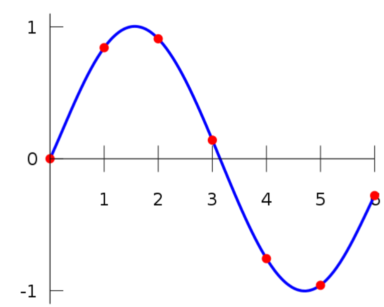
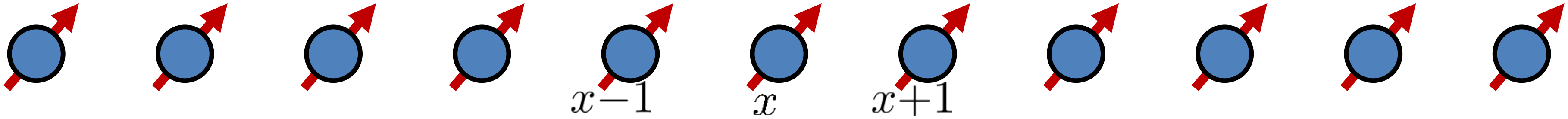
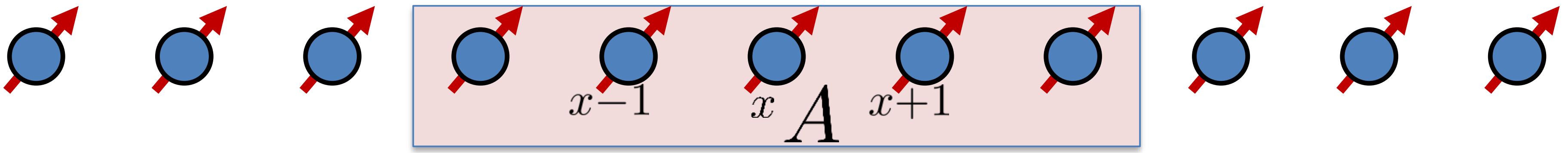


Quantum many-body systems



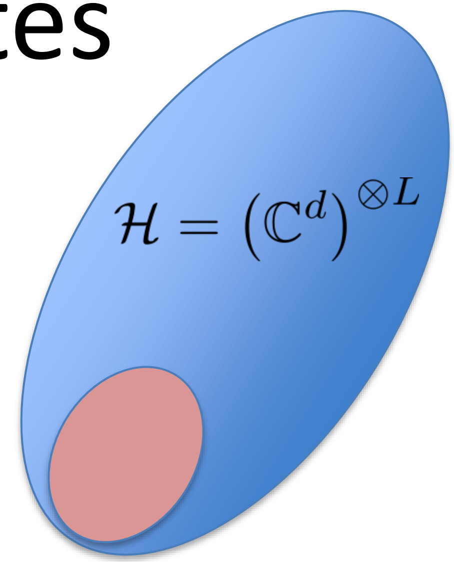
- state of L spins/qudits: $\phi \in \mathcal{H} = \left(\mathbb{C}^d\right)^{\otimes L}$
  - $\dim(\mathcal{H}) = d^L$
  - describes physically possible
  - what about physically reasonable?

Quantum many-body systems

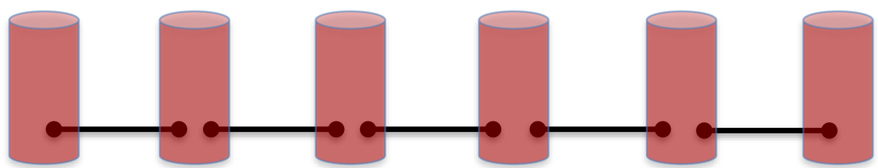


- state of L spins/qudits: $\phi \in \mathcal{H} = \left(\mathbb{C}^d\right)^{\otimes L}$
  - $\dim(\mathcal{H}) = d^L$
  - describes physically possible
  - what about physically reasonable?
- Entanglement entropy: $S(A) = \mathrm{tr}\left(\rho_A \log(\rho_A)\right)$
  - random state: $S(A) \sim \mathrm{Vol}(A)$
  - ground states of local Hamiltonians $S(A) \sim \mathrm{Area}(A)$
  - area vs. volume law

Matrix product states



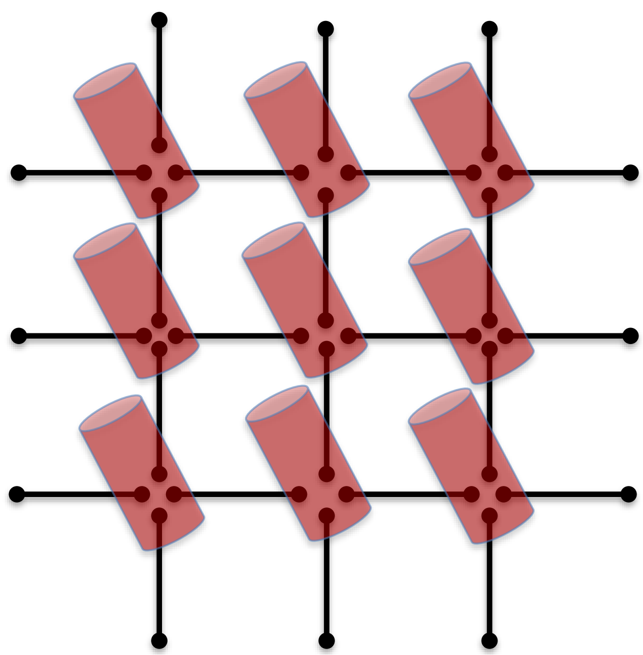
- physical corner of Hilbert space
  - find efficient parametrization
  - buildin area law
- Matrix product states (MPS)

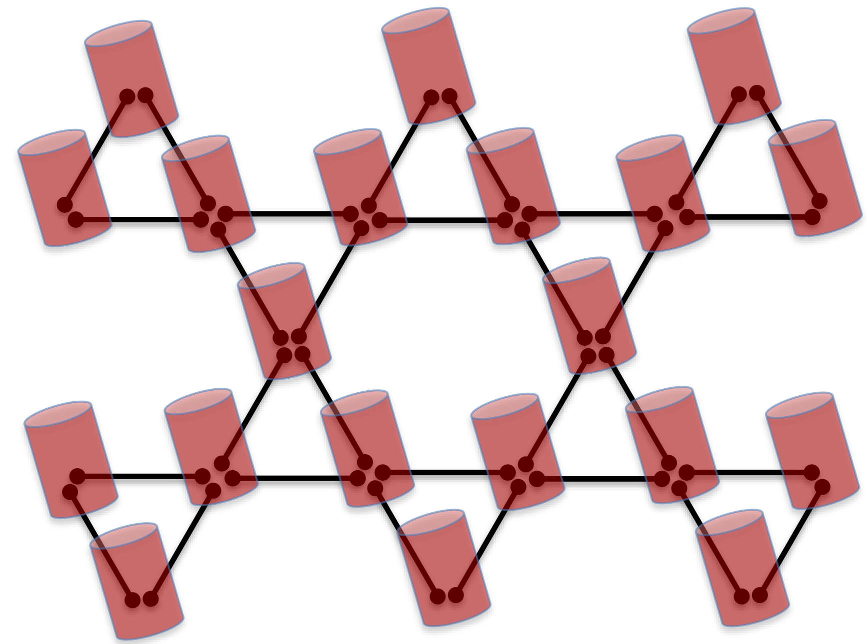
 $= \phi \in \mathcal{H} = \left(\mathbb{C}^d\right)^{\otimes L}$

  - network of max. entangled states $\Omega^D = \sum_{l=1}^{D} |l, l\rangle =$ 
  - apply local maps:  $: \mathbb{C}^D \otimes \mathbb{C}^D \mapsto \mathbb{C}^d$
  - # of parameters $\sim dD^2L$
  - efficient approximation of groundstates

Higher dimensions

- projected entangled pair states (PEPS)





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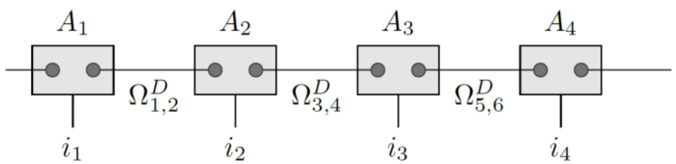

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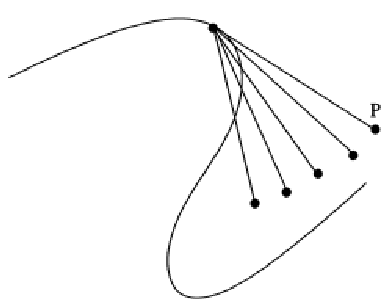
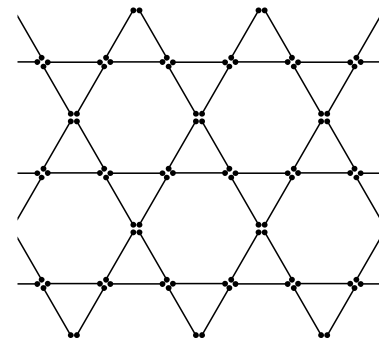


3. Tensor Networks



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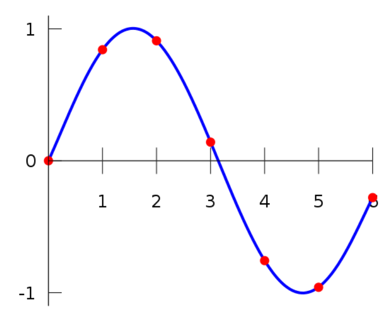
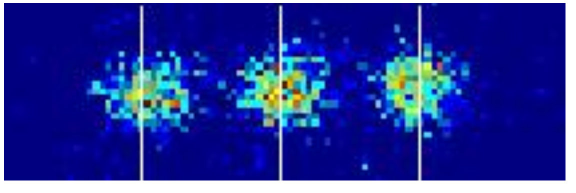
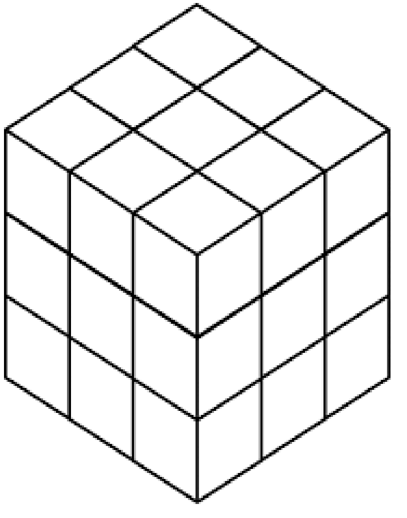


Quantum state=tensor

$$t \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$$

$$t = \sum_{i,j,k=1}^{d} t_{ijk} e_i \otimes e_j \otimes e_k$$

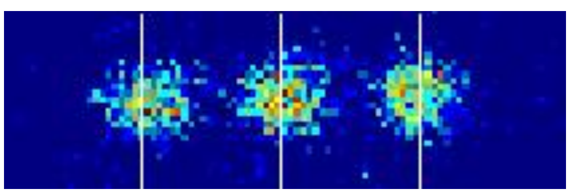


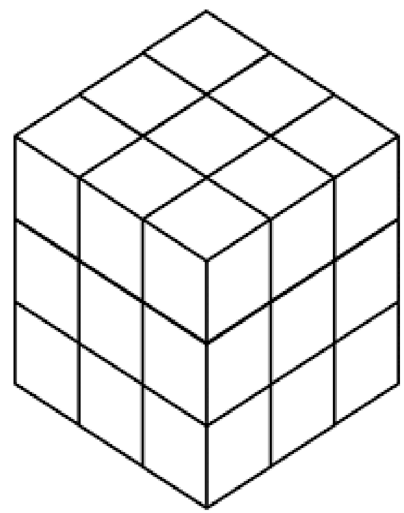


Local operations=restrictions

$$t \geq t' \text{ if } (a \otimes b \otimes c)\ t = t'$$

$$\text{for some matrices } a, b, c$$





Linear combination of slices

Restriction

$t \geq t'$ if $(a \otimes b \otimes c)\ t = t'$

for some matrices a, b, c

$t \cong t'$ if $t \geq t'$ and $t' \geq t$

iff $(a \otimes b \otimes c)\ t = t'$

for invertible a, b, c

$G = GL(d) \times GL(d) \times GL(d)$

iff $G.t = G.t'$

Deciding restriction ⟺ Classifying orbits and their relations

3 qubits

Greenberger-Horne-Zeilinger GHZ-state

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

W-state

$$\approx e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

Einstein-Podolsky-Rosen (EPR)-state

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0$$

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1$$

$$e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$$

$$e_0 \otimes e_0 \otimes e_0$$

unentangled state

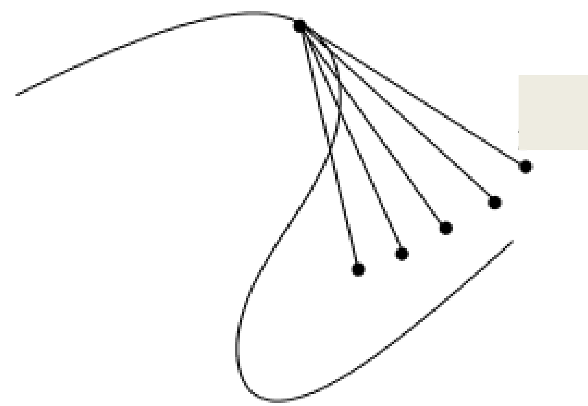
GHZ state

Degeneration

W state

$$(e_0 + \epsilon e_1)^{\otimes 3} - e_0^{\otimes 3}$$
$$= \epsilon(e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0) + O(\epsilon^2)$$

$$t \trianglerighteq t' \text{ if } t_\epsilon \underset{\epsilon \mapsto 0}{\rightarrow} t', t \geq t_\epsilon$$



Deciding degeneration ⟺ Classifying orbit closures and their relations

Deciding degeneration

- Orbit closures are G-invariant algebraic varieties

$$t \not\trianglerighteq t' \text{ iff there exists}$$

$$G-\text{covariant polynomial } f:$$

$$f(t)=0, \text{ but } f(t')\neq 0$$

- Example: $e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$

$$\downarrow \quad \not\uparrow$$

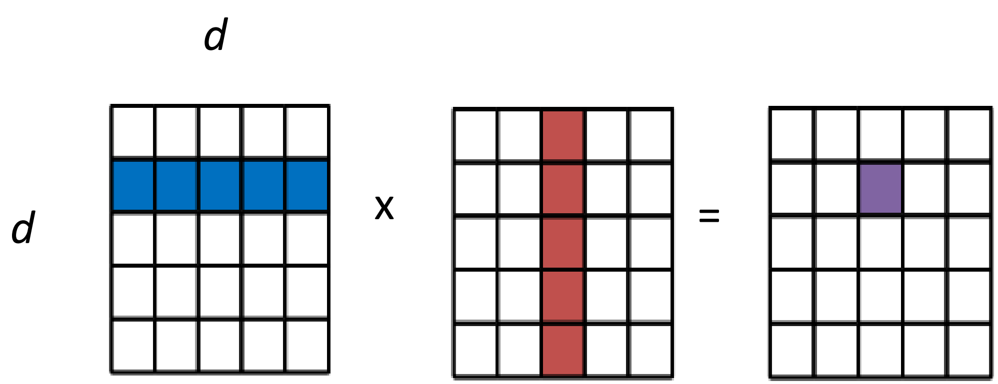
f=Cayley hyperdeterminant

$$\approx\ e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

Algebraic Complexity

$$M(d) = \text{ algebra of } d \times d \text{ complex matrices}$$

$$\begin{aligned} Mamu(d) : M(d) \times M(d) &\to M(d) \qquad \text{bilinear} \\ (A, B) &\mapsto A \cdot B \end{aligned}$$



d^3 multiplications

Bilinear maps=tensors

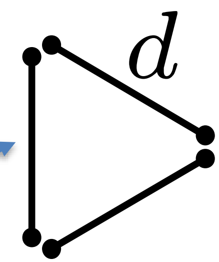
$$Mamu(d) : M(d) \times M(d) \times M(d)^* \to \mathbf{C}$$

$$(A, B, C) \mapsto tr A \cdot B \cdot C$$

$$Mamu(d) = \sum_{i,j,k=1}^{d} e_{ij} \otimes e_{jk} \otimes e_{ki}$$

$$e_{ij} = e_i \otimes e_j$$

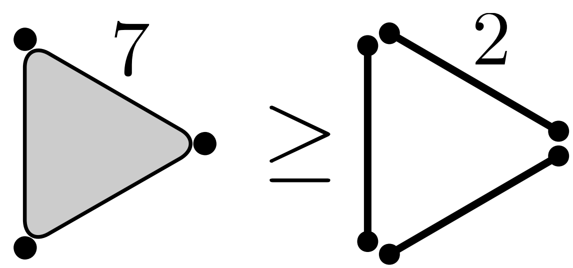
$$= \sum_{i,j,k=1}^{d} (e_i \otimes e_j) \otimes (e_j \otimes e_k) \otimes (e_k \otimes e_i)$$



EPR states

Complexity=Tensor rank

Strassen: # elementary multiplications = tensor rank



In this context, many techniques have been developed to understand restriction and degeneration

$$e_{00}\otimes e_{00}\otimes e_{00}+e_{11}\otimes e_{11}\otimes e_{11}$$
$$e_{01}\otimes e_{10}\otimes e_{00}+e_{10}\otimes e_{01}\otimes e_{11}$$
$$e_{01}\otimes e_{11}\otimes e_{10}+e_{10}\otimes e_{00}\otimes e_{01}$$
$$e_{00}\otimes e_{01}\otimes e_{10}+e_{11}\otimes e_{10}\otimes e_{01}$$

$$e_{\pm}:=e_0\pm e_1$$

$$\begin{aligned}=&e_{-1}\otimes e_{1+}\otimes e_{00}+e_{1+}\otimes e_{00}\otimes e_{-1}+e_{00}\otimes e_{-1}\otimes e_{1+}\\ &-e_{-0}\otimes e_{0+}\otimes e_{11}-e_{0+}\otimes e_{11}\otimes e_{-0}-e_{11}\otimes e_{-0}\otimes e_{0+}\\ &+(e_{00}+e_{11})\otimes(e_{00}+e_{11})\otimes(e_{00}+e_{11})\end{aligned}$$

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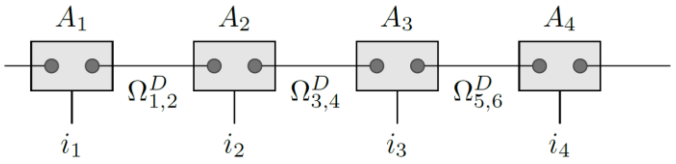

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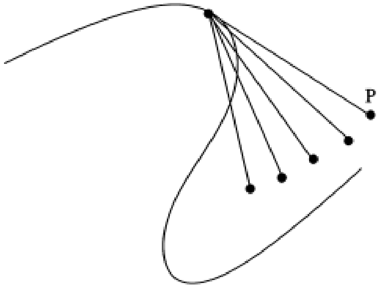

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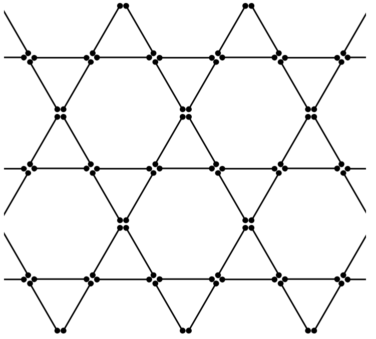

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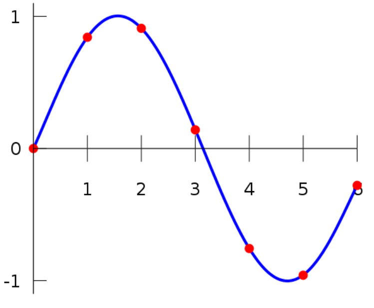
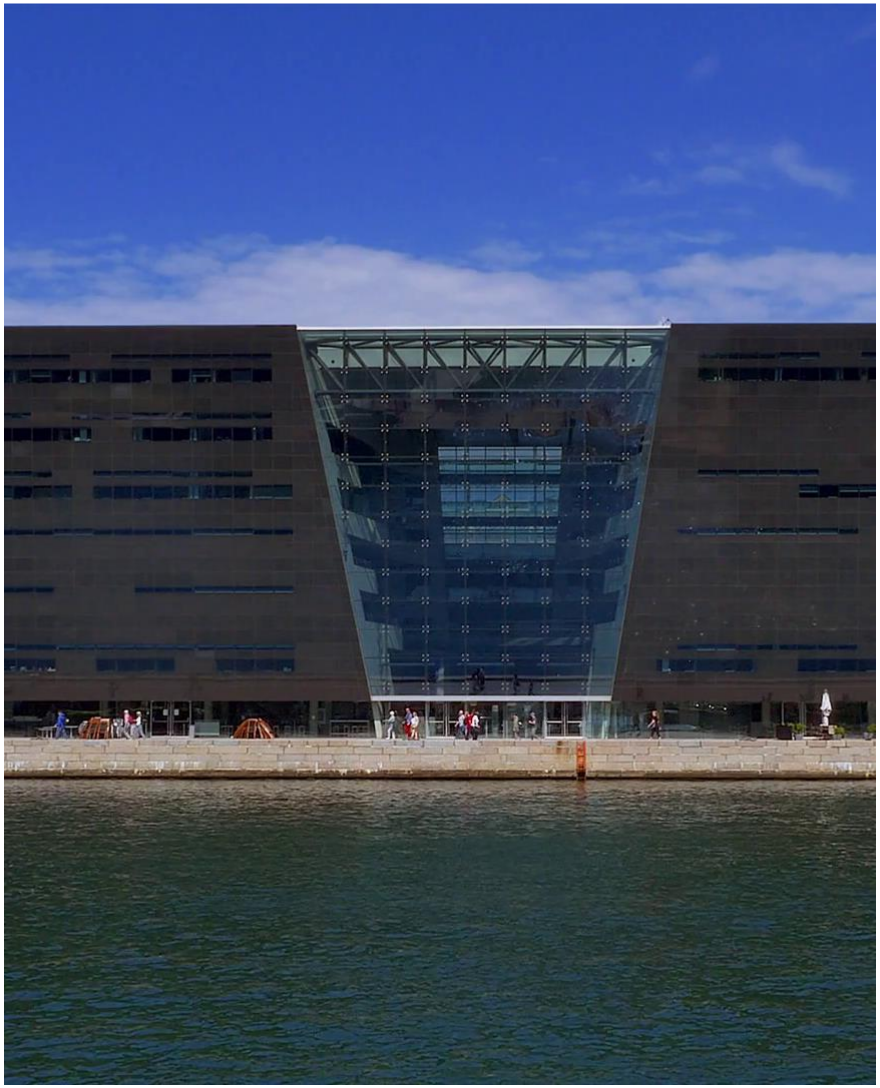


Graph or Hypergraph

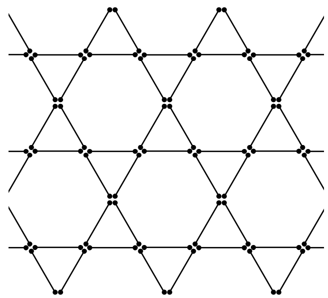


Tensor networks

line or circle=MPS
lattice=PEPS

lattices: e.g. Chen et al.’11, Xie et al.’14
Schuch et al.’12, Molnar et al.’18

- Choose graph

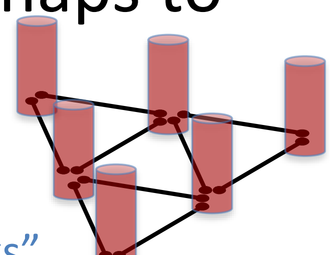


- Associate entangled state to edges

bond dimension

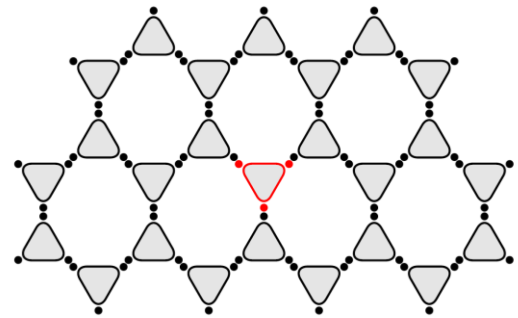
$$\Omega^D = \sum_{l=1}^{D} |l, l\rangle$$

- Apply linear maps to vertices



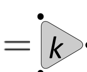
tensors in
“tensor networks”

- Choose hypergraph

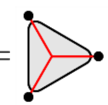


- Associate entangled state to hyperedges

$$\sum_{i=1}^{k} |i\rangle\,|i\rangle\,|i\rangle =$$



$$\sum_{i,j,k=0}^{2} \varepsilon_{i,j,k}\,|i,j,k\rangle + |2,2,2\rangle =$$



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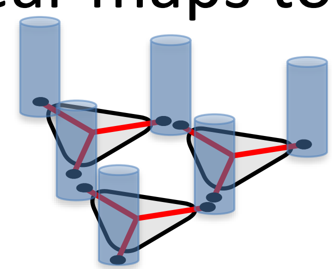
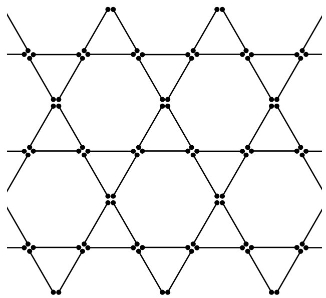
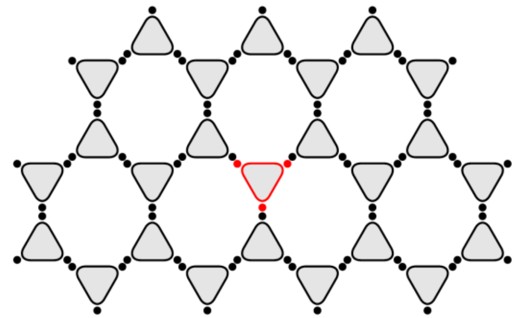
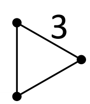


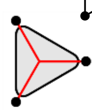
Tensor networks

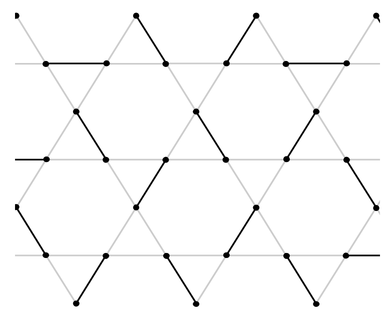
- Underlying "entanglement structure"





- Does the job
  - E.g. represents
    Resonating Valence Bond state
  - Verstraete et al.'06 

  - Schuch et al.'12 



- You don't like it
  - Too large bond dimension
  - Weird entangled state

Contraction is too slow
Code does not work at all
(you are computational physicist)

Representation is too ugly
(you are mathematical physicist)

Let’s find a better tensor network!

- Start from scratch
  - I.e. from physical state
  - Pro: super-optimized
  - Con: are you kidding?
- Switch entanglement structure
  - Independent from projectors
  - Pro: Works for any state with same structure
  - Pro: Tight for injective ones
  - Con: Tailored optimization could be better

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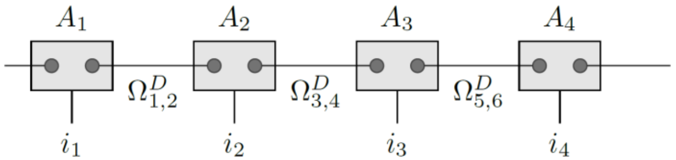

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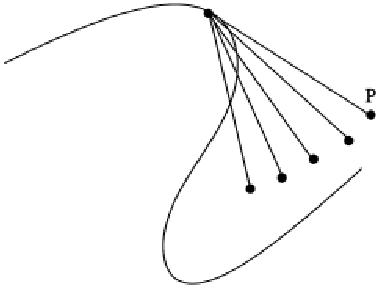

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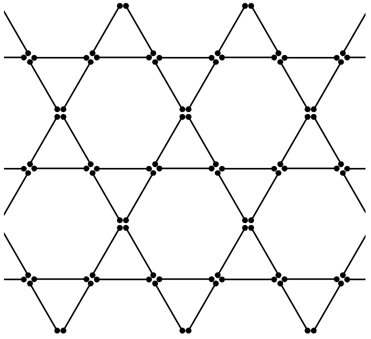

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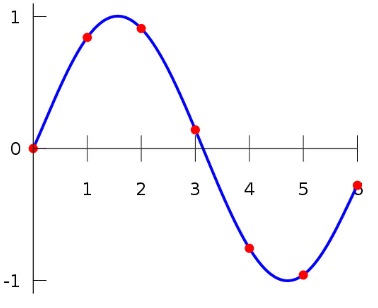
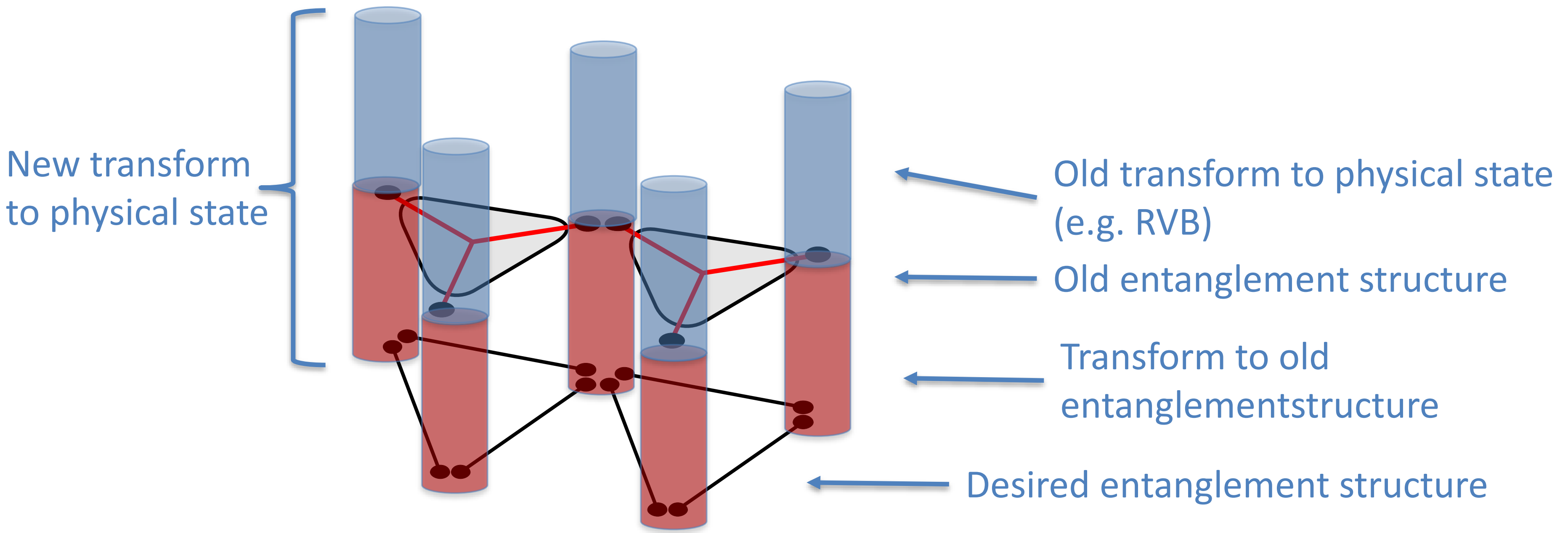
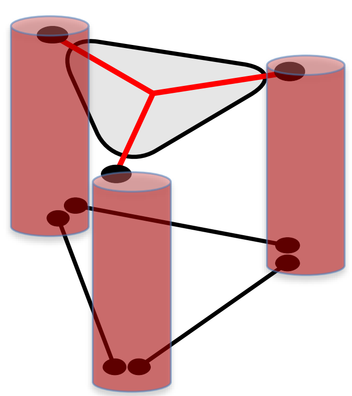


Switch entanglement structure

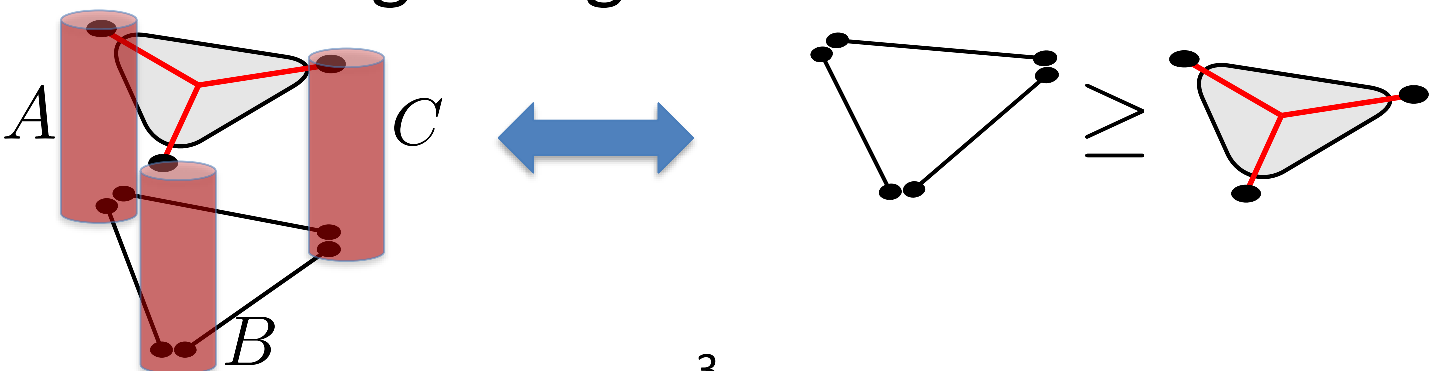


- Let us just focus on transforming the entanglement structure
- Plaquette by plaquette

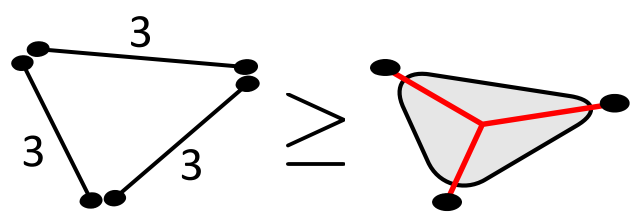


Entanglement

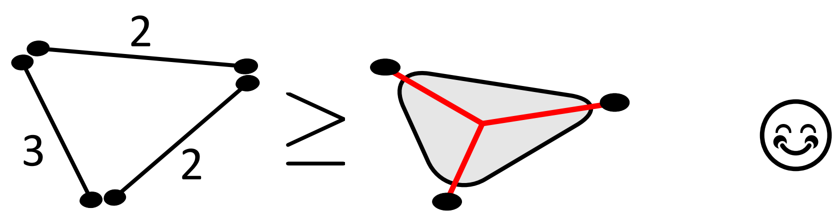
- Back to the beginning



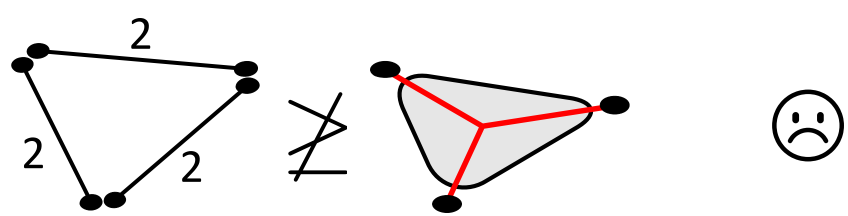
- State of the art
  - Molnar et al.’18



- Improvement

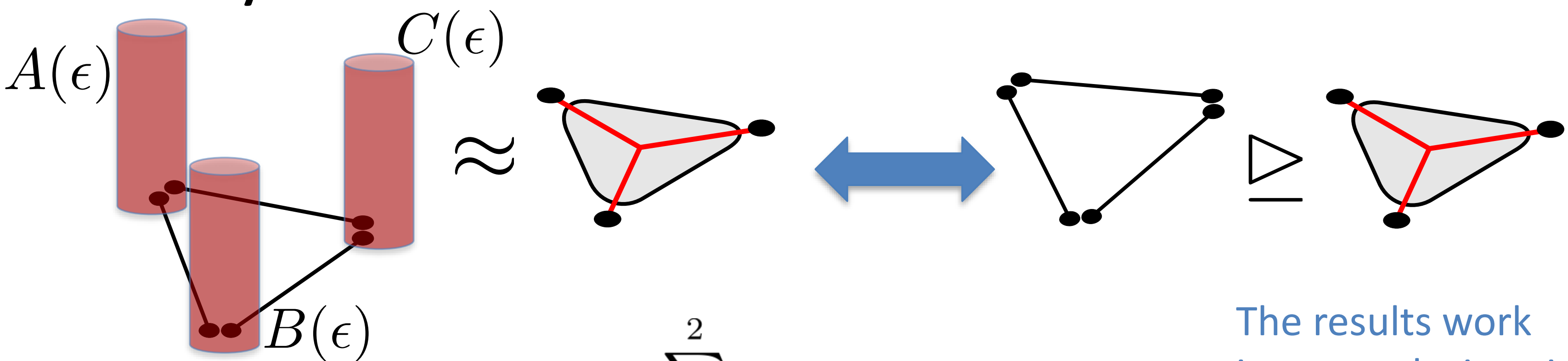


- But not all the way



Entanglement

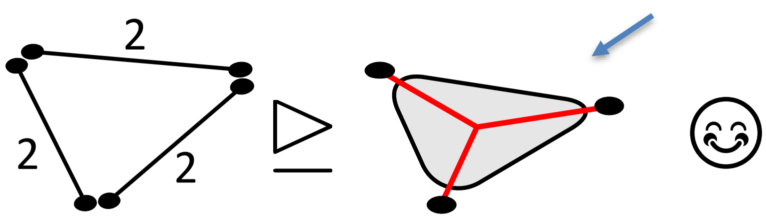
- Why not relax?



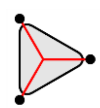
The results work
in any such situation

- It works!

$$\sum_{i,j,k=0}^{2} \varepsilon_{i,j,k}\,|i,j,k\rangle + |2,2,2\rangle$$

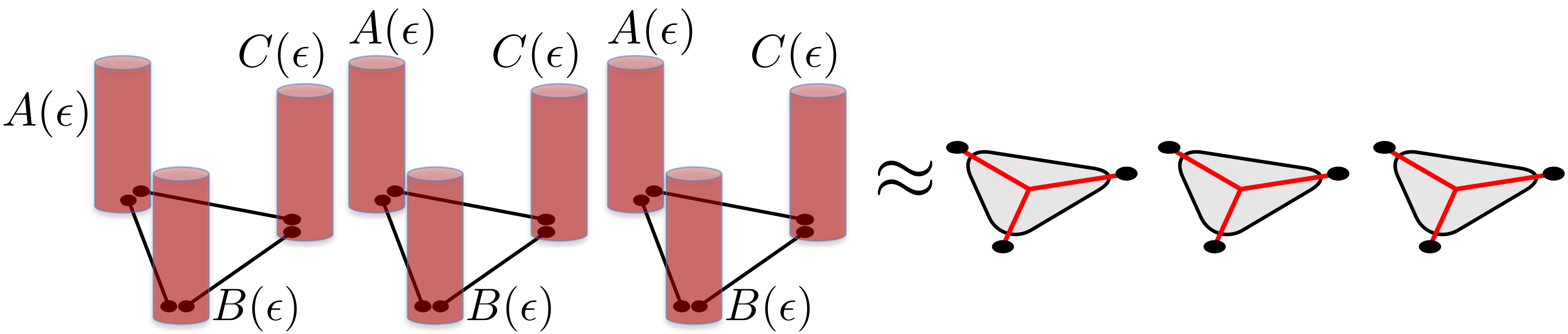


- We are looking for approximate MPS
- Pauli's give exact MPS for antisymm
- Squeeze in an epsilon

ε^2  $+\varepsilon^4\,|2\rangle \otimes \left(\frac{1}{4}\,|00\rangle - |11\rangle\right)$

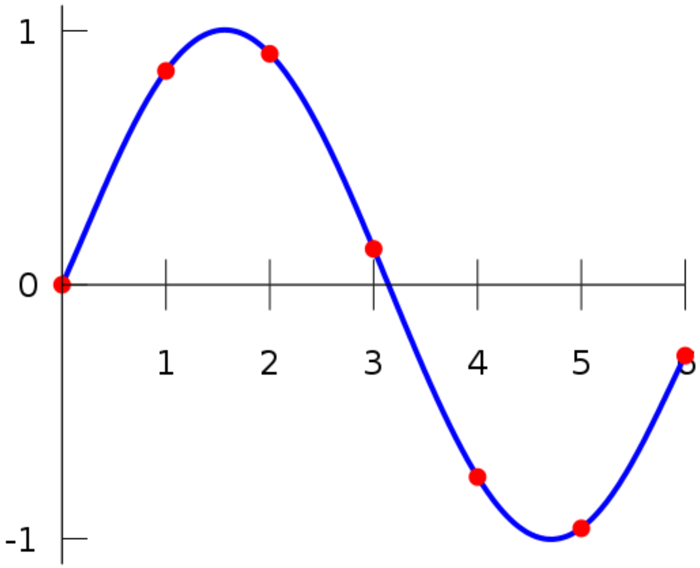
$$M_0^{[j]}(\varepsilon) = \frac{1}{2}\begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}, \quad M_1^{[j]}(\varepsilon) = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}, \quad M_2^{[j]}(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\delta_{j,3}\varepsilon^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Roll it out



- Leads to approximate tensor network representation
- Who cares?
- Let’s convert it to an exact one!
  – Only need to pay a small price

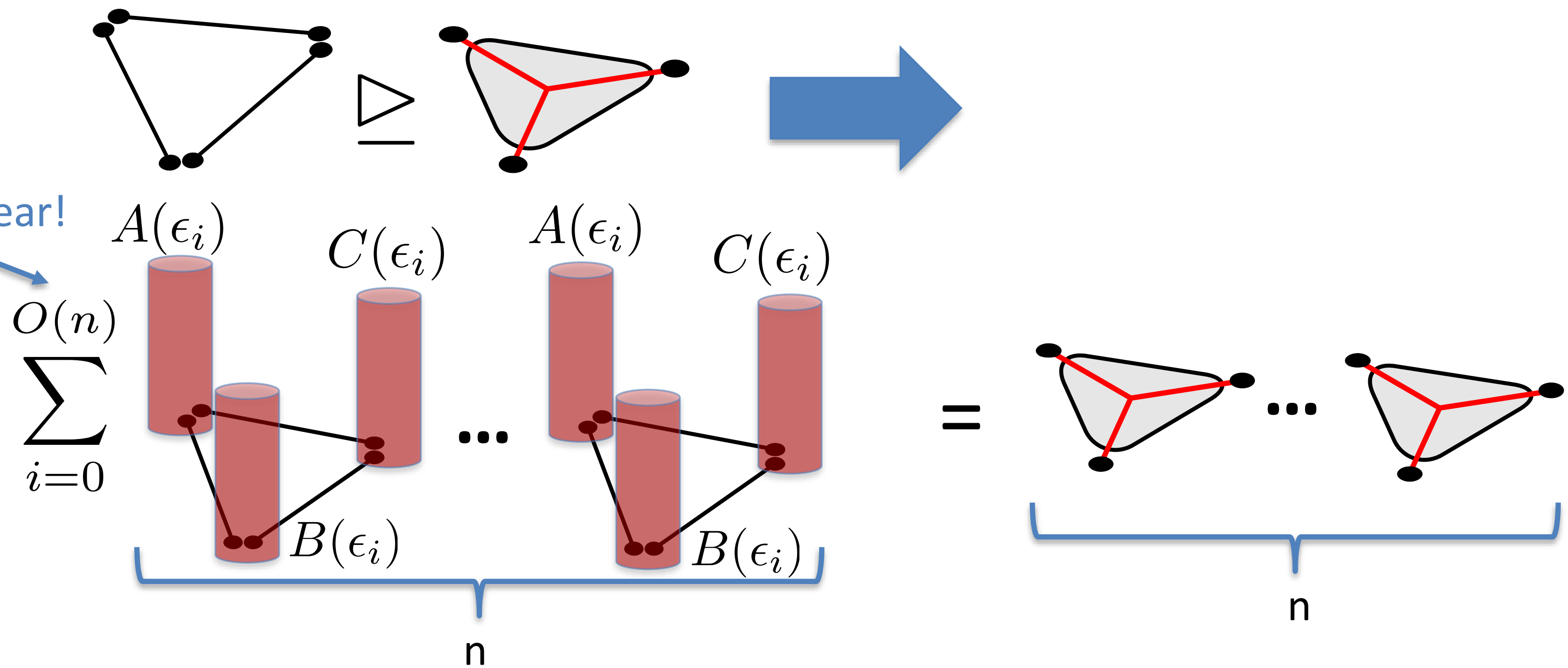
Interpolation



- Given a polynomial p(x) of degree n
- Obtain p(0) by Lagrange interpolation
  - Evaluate at n+1 points
  - Determines the entire polynomial
  - Value at 0 can be easily obtained (see Wiki)
- Our transformation matrices are polynomial in epsilon!

Lemma: Degeneration → Restriction

Only linear!

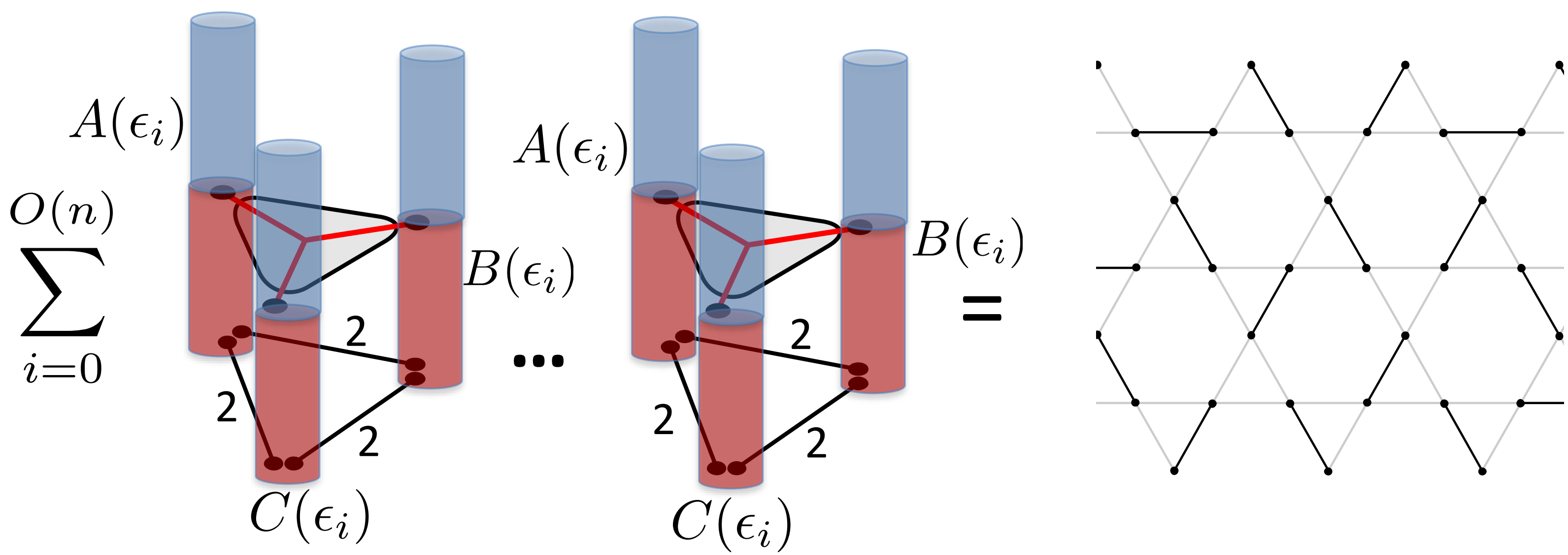


- Proof based on interpolation
  - Bini, Lotti and Romani, SIAM J.Comp. 1980
  - Christandl, Jensen & Zuiddam, Lin.Alg. App 2018

It's about tensor rank

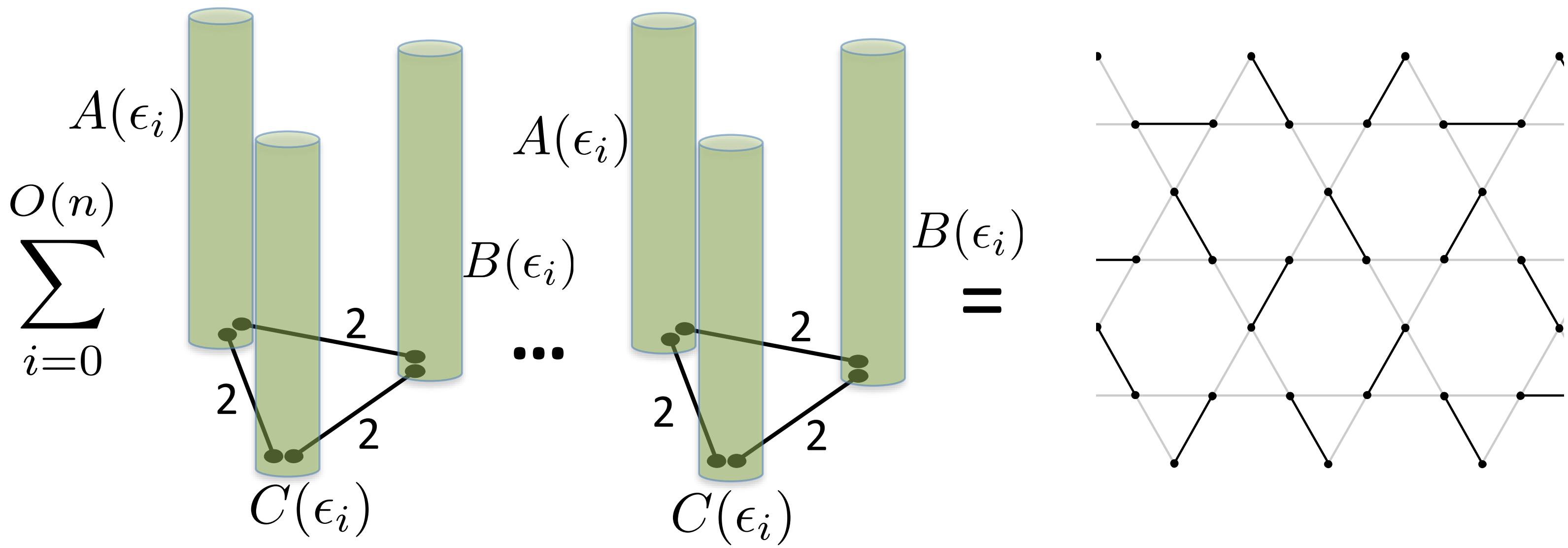
Application

- Resonating Valence Bond State



Application

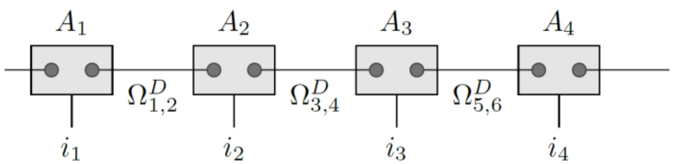
- Resonating Valence Bond State



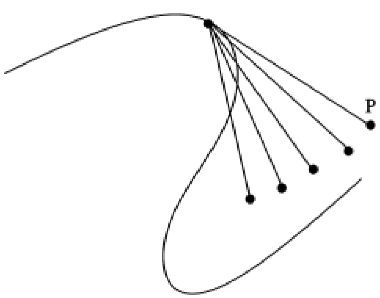
- Parallel algorithm for faster contraction
  - also for expectation values

Summary

1. Matrix product states

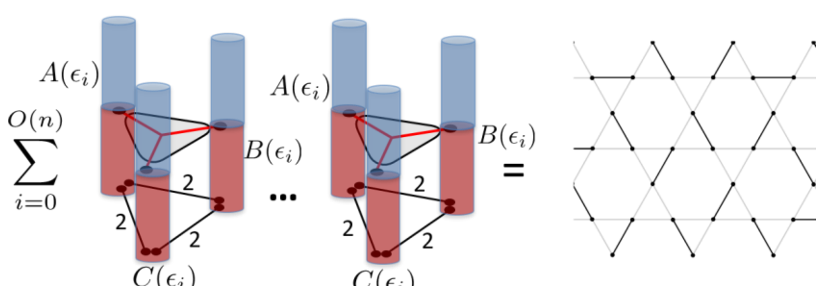


2. Geometry of entanglement

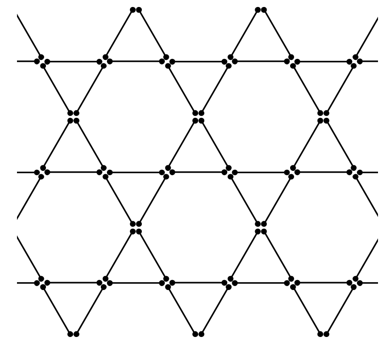


Open questions

- In practice?
- Other examples?
- Sums of tensor networks as new variational class?



3. Tensor Networks



4. Reducing the bond dimension