Eigenfunction concentration and its connection to geometry

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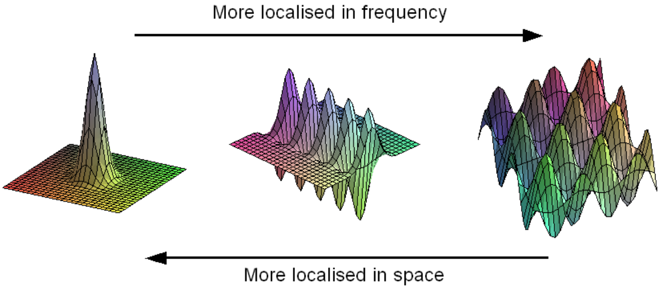
Eigenfunction asymptotics

On a compact, Riemannian manifold (M, g) consider u

$$-\Delta_g u = \lambda^2 u$$

How does u behave as $\lambda \to \infty$?



Can u display concentrations?

Why to do we care?

Laplacian eigenfunctions are useful building blocks. One important way they come up is as the stationary states of a quantum system.

$$\psi(t,x) = e^{iEt}u(x)$$

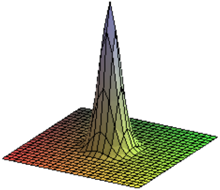
satisfies Schrödinger's equation with $E = \lambda^2$.

- E is interpreted as the energy of the system
- Concentration of u implies concentration of ψ
- Concentration of ψ is interpreted as a high probability that the system is found in the concentration region.

Measuring Concentration

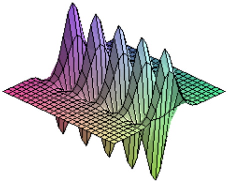
There are many ways to measure eigenfunction concentration. We will focus on L^p estimates

Point



- High L^∞ norm
- Sharp change in L^p norm when $p < \infty$

Tube



- Lower L^∞ norm
- Change in L^p norm more gentle

L^p Estimates for Eigenfunctions

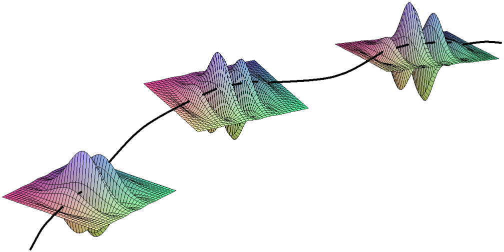
Let X be some subset (not necessarily of full dimension) of M. Seek estimates of the form

$$\|u\|_{L^p(X)} \lesssim f(n, p, \lambda) \|u\|_{L^2(M)}$$

- For what f is the inequality valid?
- Are there sharp examples?
- Does f depend on the geometry of X?
- What about concentration of $q(x, hD)u$ where $q(x, hD)$ is the quantisation of a dynamical quantity.

Heuristic - Concentration/Dynamics

Heuristically think of eigenfunction as being made of of wave packets tracking the classical flow.



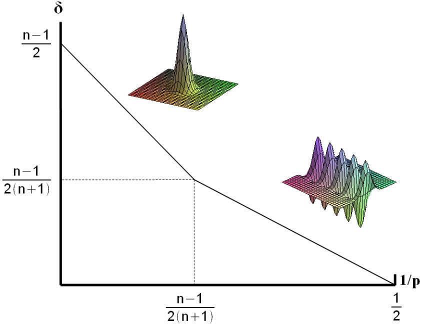
- Packets are localised both physically and in momentum
- Concentration in a region is related to time packets spend there
- Heuristic breaks down in time due to dispersion

L^p concentration on manifolds

Sogge 1988

$$\|\chi_\lambda u\|_{L^p(M)} \lesssim \lambda^{\delta(n,p)} \|u\|_{L^2}$$

χ_λ a spectral cluster operator.



- Two different regimes for sharp results.
- On the sphere sharp for actual eigenfunctions.
- Can be extended to semiclassical results for quasimodes (Koch-Tataru-Zworski 2007).

Improvements with negative curvature

- Bérard (1977)

$$\|u\|_{L^\infty} \lesssim \frac{\lambda^{\frac{n-1}{2}}}{\log^{1/2}(\lambda)} \|u\|_{L^2}$$

- Hassell-Tacy (2015)

$$\|u\|_{L^p} \lesssim \frac{\lambda^{\delta(n,p)}}{\log^{1/2}(\lambda)} \|u\|_{L^2} \quad p > p_c = \frac{2n}{n-1}$$

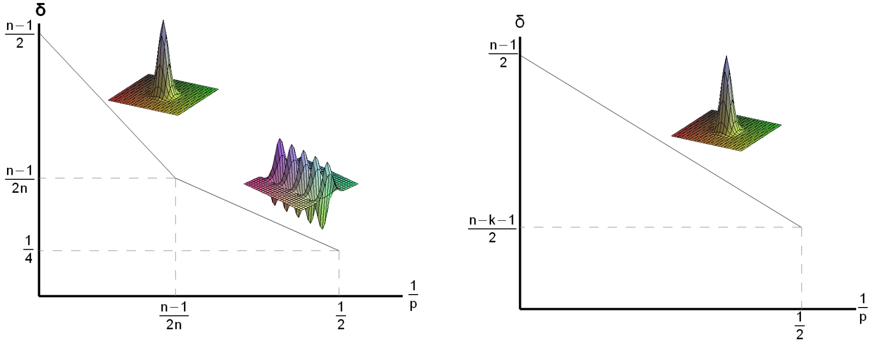
- Blair-Sogge (2017)

$$\|u\|_{L^{p_c}} \lesssim \frac{\lambda^{\delta(n,p_c)}}{(\log(\lambda))^{\epsilon_0}} \|u\|_{L^2}$$

$$\|u\|_{L^p(X)} \lesssim \lambda^{\delta(n,p,k)} \|u\|_{L^2(M)}$$

Hypersurfaces

Low dimensional submanifolds



Established by Burq-Gerard-Tvetkov 2005 for eigenfunctions of Δ and by Tacy 2010 for semiclassical quaismodes.

Concentration of dynamical quantities

Classical flow defined by

$$\begin{cases} \dot{x}(t) = \partial_\xi p(x,\xi) \\ \dot{\xi}(t) = -\partial_x p(x,\xi) \end{cases}$$

The function $p(x,\xi)$ is the classical energy function. Other observables $q(x,\xi)$ evolve under

$$\dot{q}(x,\xi) = \{p(x,\xi), q(x,\xi)\}$$

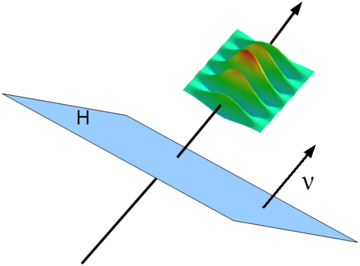
Quantum analogue, semiclassical pseudo

$$q(x,hD)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} q(x,\xi)u(y)d\xi dy$$

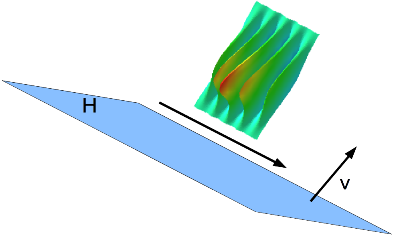
Concentration of normal velocity

Let H be a hypersurface in M, normal ν.

- Normal velocity $\nu(x,\xi) = \partial_{\xi_\nu} p(x,\xi)$
- Quantisation of normal velocity $\nu(x,hD)u$



Normal velocity is large but packets spend only a short time near the surface.



Packets spend a long time near the surface (high concentration) but the normal velocity is small so $\nu(x,hD)u$ decays.

Theorem (T 17)

Suppose u is an approximate solution to $p(x, hD)u = 0$ then

$$\|\nu(x, hD)u\|_{L^2(H)} \lesssim \|u\|_{L^2(M)}.$$

and

$$\left\|\nu^{1/2}(x, hD)u\right\|_{L^2(H)} \lesssim \|u\|_{L^2(M)}$$

where $\nu^{1/2}(X, hD)$ is the quatisation of a suitable regularisation of $\nu^{1/2}(x, \xi)$.

- Can allow error up to $O_{L^2}(h)$.
- Estimate only require $p(x, \xi)$ is smooth, other semiclassical estimates require Laplace like condition.

Flat Models

Gain insight into behaviour by considering approximate eigenfunctions on $\mathbb{R}^n$. Rescale the problem, setting $h = \lambda^{-1}$ look for u so that

$$(-h^2\Delta - 1)u = O_{L^2}(h)$$

Exploit constant coefficients to use the (scaled) Fourier transform

$$\mathcal{F}_h f = \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\langle x,\xi\rangle} f(x)dx$$

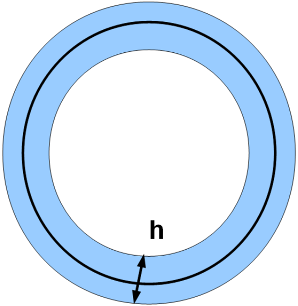
$$hD_{x_i} \to \xi_i$$

$$\|\mathcal{F}_h f\|_{L^2} = \|u\|_{L^2}$$

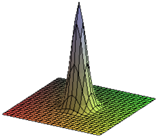
Solving on the Fourier side

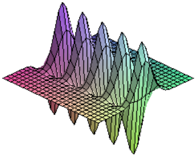
We require

$$(|\xi|^2 - 1)\mathcal{F}_h u = O(h)$$



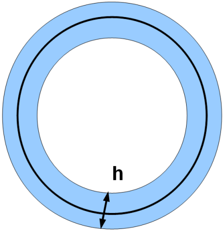
- Must place the support of $\mathcal{F}_h u$ close to $|\xi| = 1$
- By spreading $\mathcal{F}_h u$ as much as possible can make u large at a point
- On the other hand to spread u out we concentrate $\mathcal{F}_h u$.

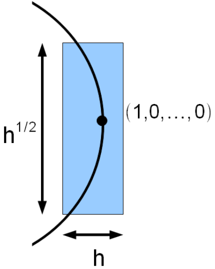




Spread $\mathcal{F}_h u$ evenly throughout annular region

Concentrate $\mathcal{F}_h u$ around one point





What about intermediate spread?

Family of examples

Let

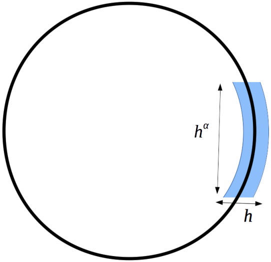
$$\chi_\alpha^h(r,\omega) = \begin{cases} 1 & \text{if } |r-1| < h, |\omega - \omega_0| < h^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then set

$$f_\alpha^h(r,\omega) = h^{-1/2-\alpha(n-1)/2}\chi(r,\omega).$$

Note that f_α^h is L^2 normalised.

$$T_\alpha^h(x) = \mathcal{F}_h^{-1}[f_\alpha^h](x)$$



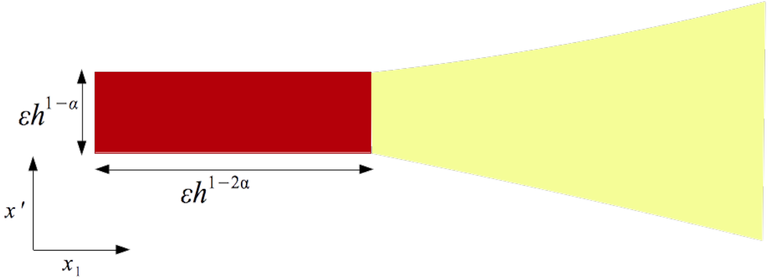
$$T^h_\alpha(x) = \frac{h^{-1/2-\alpha(n-1)/2-n/2}e^{\frac{i}{h}x_1}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{\frac{i}{h}(x_1(\xi_1-1)+\langle x',\xi'\rangle)}\chi_\alpha(\xi)\,d\xi.$$

If $|x_1| < \epsilon h^{1-2\alpha}$ and $|x'| < \epsilon h^{1-\alpha}$ the factor

$$e^{\frac{i}{h}(x_1(\xi_1-1)+\langle x',\xi'\rangle)}$$

does not oscillate so in this region

$$|T^h_\alpha(x)| > ch^{-(n-1)/2+\alpha(n-1)/2}$$



Connections to spherical harmonics

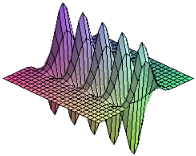
For each α we can produce an exact eigenfunction on the sphere S^{n-1} which has the same size properties as T_α^h. Build them out of highest weight harmonics.

$$\phi(x) = j^{\frac{n-1}{4}}(x_1 + ix_2)^j$$

is a solution to the spherical Laplacian eigenfunction equation with $j(j+n-1) = \lambda^2 = h^{-2}$.

Further if $x = (x_1, x_2, \bar{x})$ then

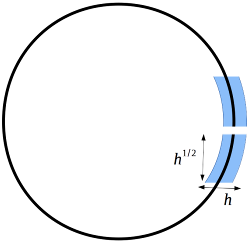
$$|\phi(x)|^2 = j^{\frac{n-1}{2}}(1 - |\bar{x}|^2)^j = j^{\frac{n-1}{2}} e^{j\log(1-|\bar{x}|^2)}$$



Resembles the tube

Rotations of highest weight harmonics

Can think of T^h_α as a sum of $T^h_{1/2}$ with the principal direction rotated.



So produce a function u_α where

$$u_\alpha = \sum_j \phi(R_j(x))$$

where R_j is a rotation. Since u has same concentration properties as T^h_α we can use the flat model examples to test for saturation and know how to produce an exact eigenfunction example.

Uses of the family T_α^h

- Checking sharpness for linear estimates, what kind of cross sections do we expect?
- Analysing bilinear estimates for sharpness. Here we estimate

  $$\|uv\|_{L^p} \leq G(\lambda, \mu) \|u\|_{L^2} \|v\|_{L^2}$$

  where u and v are eigenfunctions with eigenvalues λ^2 and μ^2. Can find all sharp examples by considering combinations of the T_α^h.
- Understanding the effect of geometry on L^p estimates. Negative curvature improves the estimates in a logarithmic fashion. These examples allow us to see exactly what sort of concentrations need to be considered.

Inverse problems?

Consider the hypersurface estimates. If we know $\|u\|_{L^2(H)}$ what can we say about u? If H is a hyperplane in $\mathbb{R}^n$

$$\|u\|_{L^2(H)} = R_H(|u|^2)$$

where R_H is the Radon transform evaluated at H. Therefore we could in fact reproduce $|u(x)|^2$ via

$$|u(x)|^2 = c_n(-\Delta)^{\frac{n-1}{2}} R^\star \circ R[|u|^2]$$

What if we only know estimates for $\|u\|_{L^2(H}$ can we hope to say anything about $|u(x)|^2$ or $\|u\|_{L^p(M)}$.

Why do we want to do this anyway?

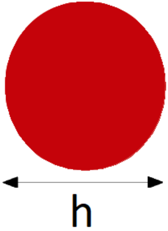
Eigenfunctions (and quasimodes) oscillate very rapidly. Taking L^2 norms allows us to take advantage of that. Consider $e^{\frac{i}{h}\langle x,\xi\rangle}$ and $e^{\frac{i}{h}\langle x,\eta\rangle}$ where $\xi, \eta \in S^{n-1}$ and $|\xi - \eta| > \epsilon$.

$$\int_H e^{\frac{i}{h}\langle x,\xi-\eta\rangle} dx$$

- If
$$|(\xi - \eta) - \nu \cdot (\xi - \eta)\nu| > c$$
we can integrate by parts in the hypersurface variables to show the contribution is $O(h^\infty)$.
- Even if
$$|(\xi - \eta) - \nu \cdot (\xi - \eta)\nu| > ch^\alpha \quad \alpha < 1$$
can still get $O(h^\infty)$ decay.
- Means we can restrict our attention to contributions to $\int_H |u(x)|^2$ that are bilinear combinations with $\xi - \eta$ being exactly in direction ν.

What do the flat models tell us?

The $\alpha = 0$ case.
Produces the highest L^∞ norm, a peak $h^{-\frac{n-1}{2}}$ concentrated on an $O(h)$ set.



Therefore

$$c_1 \leq \left\| T_0^h \right\|_{L^2(H)} \leq c_2$$

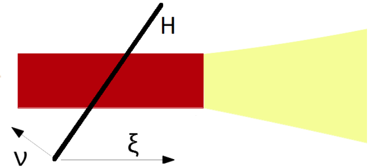
So this concentration is 'invisible' to hypersurfaces.

The $\alpha > 0$ cases

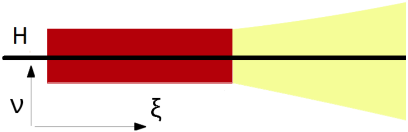
Depends how H is aligned.

- Let ν be the unit norm of H.
- Let ξ be the long direction of T_α^h.

$\langle \nu, \xi \rangle \neq 0$



$\langle \nu, \xi \rangle = 0$

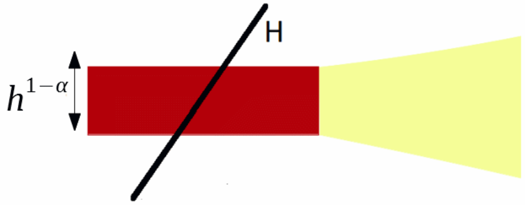


Greater concentration on H when $\langle \nu, \xi \rangle = 0$

The case $\langle \nu, \xi \rangle \neq 0$

$$|T_\alpha^h| \sim h^{-\frac{n-1}{2}+\frac{\alpha(n-1)}{2}}$$

and is supported on a region of measure approximately $h^{(1-\alpha)(n-1)}$



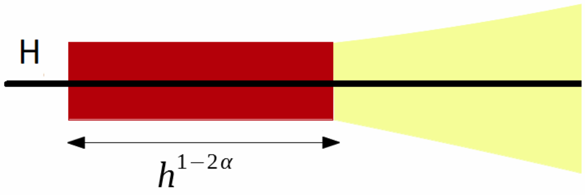
$$c_1 \leq \left\| T_\alpha^h \right\|_{L^2(H)} \leq c_2$$

So similar to the $\alpha = 0$ case these hypersurfaces don't 'see' the concentration.

The case $\langle \nu, \xi \rangle = 0$

$$|T_\alpha^h| \sim h^{-\frac{n-1}{2}+\frac{\alpha(n-1)}{2}}$$

and is supported on a region of measure approximately $h^{(1-\alpha)(n-1)+1-2\alpha}$



$$c_1 h^{-\frac{\alpha}{2}} \leq \left\| T_\alpha^h \right\|_{L^2(H)} \leq c_2 h^{-\frac{\alpha}{2}}$$

So these hypersurfaces do ‘see’ the concentration.

What does this tell us about L^p estimates?

- Estimates for high p, that is $p \geq \frac{2(n+1)}{n-1}$ saturated by the $\alpha = 0$ cases. So we can't recover information about these from $\|u\|_{L^2(H)}$.
- Reversing the information from the examples suggests that if there is a hypersurface with

$$c_1 h^{-\frac{\alpha}{2}} \leq \|u\|_{L^2(H)} \leq h^{-\frac{\alpha}{2}}$$

then

$$c_1 h^{-\mu(n,p,\alpha)} \leq \|u\|_{L^p} \leq c_2 h^{-\mu(n,p,\alpha)}$$

$$\mu(n,p,\alpha) = (n-1)\left(\frac{1}{2} - \frac{1}{p}\right) + \alpha\left(\frac{n-1}{2} - \frac{n}{p}\right)$$

- Difficult to prove without a stability condition.

Stability

Need to have control on near hypersurfaces as well.

- This allows us to create a thickened region around the hypersurface
- Fix a point x_0 and associate the set of hypersurfaces through x with S^{n-1}
- Then condition is that there is some x_0 so that

$$\{H \mid x_0 \in H, \|u\|_{L^2} \sim h^{-\frac{\alpha}{2}}\} \subset S^{n-1}$$

contains a ball of radius h^{α}

Connection with Kakeya tubes

- Sogge and Blair-Sogge show that growth in L^p for $p < p_c$ depends on growth in Kakeya tubes.
- In two dimensions these Kakeya tubes are just thickened hypersurfaces and are associated with the $\alpha = 1/2$ case of T^h_α.
- Similar sorts of ideas, also based on bilinear estimates and exploiting the relationship between dimension and p value.

Where next?

- Could we work with a weaker stability condition. For instance one that only gave a lower bound on the measure of

$$\{H \mid x_0 \in H, \|u\|_{L^2} \sim h^{-\frac{\alpha}{2}}\}$$

rather than requiring it to contain a ball.

- Can we get the other direction. That is can we say that the $\|u\|_{L^p}$ ONLY grows if the $\|u\|_{L^2(H)}$ grows for some collection of H.

- If we can obtain such a result for a range of p can we apply it to situations where we expect better L^p norms.