Completing a Task with Interruptions

Jarrett Nantais
Myron Hlynka *
Percy Brill
CANQUEUE CONFERENCE 2020

Department of Mathematics and Statistics
University of Windsor
Windsor, ON, Canada.

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Outline

- Definition of Laplace Transform
- Probabilistic Interpretation
- Statement of problem: Tasks with Interruptions
- General Result
- Special Cases

Definition of Laplace Transform

DEFINITION: The Laplace transform $L(s)$ of a pdf $f(x)$ with positive support is given by

$$L_X(s) = \int_0^\infty e^{-sx} f(x) dx$$

where $s > 0$.

Catastrophe Process

THEOREM: Let X be a r.v. with positive support and with pdf $f(x)$. Let Y be a r.v. independent of X, such that $Y \sim$ exponential with rate s. Then

$$L_X(s) = P(X < Y).$$

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- The exponential random variable Y is called the catastrophe.
- The Laplace transform of a p.d.f of a random variable X is the probability that X occurs before the catastrophe.
- More precisely, the Laplace transform of a probability density function $f(x)$ of a random variable X can be interpreted as the probability that X precedes a catastrophe where the time to the catastrophe is an exponentially distributed random variable Y with rate s, independent of X.

Proof

Theorem

THEOREM: Let X be a r.v. with positive support and with pdf $f(x)$. Let Y be a r.v. independent of X, such that $Y \sim$ exponential with rate s. Then

$$L_X(s) = P(X < Y).$$

Proof.

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_x^\infty f(x) s e^{-sy} dy dx \\ &= \int_0^\infty f(x)(-e^{-sy})|_{y=x}^{y=\infty} dx \\ &= \int_0^\infty f(x) e^{-sx} dx \\ &= L_X(s). \end{aligned}$$

LT of Exponential r.v.

Corollary

Let X be an exponential r.v. with rate λ. $f(x) = \lambda e^{-\lambda x}$ for $x > 0$.
Then LT of X is $L_X(s) = P(X < Y) = \dfrac{\lambda}{\lambda + s}$

Statement of Problem

- We consider a task which has a completion time T (if not interrupted), with probability density function $f(t)$, $t > 0$.
- Before it is complete, the task may be interrupted by a Poisson process with rate λ.
- If that happens, then the task must begin again, with the same completion time random variable T, but with a potentially different realization.
- Interruptions can reoccur, until eventually the task is finished, with a total time of W.
- We find the Laplace Transform of W in several special cases.

Examples

- A pedestrian wants to cross a oneway road with traffic in a Poisson stream. Time to cross with no traffic is k. Pedestrian waits for a gap of size k time units in the traffic and then crosses. What is total time to cross the road? (see Ross, Taylor and Karlin, Cox and Miller)
- A person is watching a youtube lecture on queueing. But popups appear randomly during the lecture. Since the person has ADHD, the person must begin again after every interruption. How long to finish watching the complete lecture?
- A professor is busy preparing an exam for the next day. But students keep randomly interrupting the work. After an interruption, the professor must change the questions since the student has asked some of the exact questions already selected. How long to complete making the exam?

Diagram

— — — — — — —

$\leftarrow$ *Poisson traffic*

— — — — — — —

$\uparrow$

cross road

Theorem

Let $f(t)$ be a probability density function for some continuous random variable T, which is the time to complete the task uninterrupted, with positive support on $[a, b]$, for $0 \leq a \leq b$. Further, assume an interruption Poisson process of rate λ. Let W be the total time to complete the task, including interruptions. Then, the Laplace Transform for the total completion time W is given by

$$L_W(s) = \frac{\int_a^b e^{-(\lambda+s)t} f(t)dt}{1 - \frac{\lambda}{\lambda+s} \int_a^b (1 - e^{-(\lambda+s)t}) f(t)dt}$$

where $0 \leq a \leq b \leq \infty$.

PROOF:
Assume a catastrophe random variable Y, which is an exponential random variable with rate s. Then the Poisson process interruption coupled with the catastrophe will occur at a total rate of $\lambda + s$, assuming these events occur independently. Thus:

$$
\begin{aligned}
L_W(s) &= P(W < Y) \\
&= \int_a^b f(t) P(\textit{Poisson and catastrophe are later than } t) dt \\
&+ \int_a^b f(t) P(\textit{Poisson or catastrophe are early}) \\
&* P(\textit{Poisson beats catastrophe}) L_W(s) dt
\end{aligned}
$$

If the task beats the Poisson process interruption and the catastrophe, then we only need the first integrand. The probability of this event is $e^{-(\lambda+s)t}$. However, we also have the possibility of either the Poisson interruption or the catastrophe coming before the task completes, which has a probability $1 - e^{-(\lambda+s)t}$, and the Poisson process interruption occurs before the catastrophe, which has a probability of $\frac{\lambda}{\lambda+s}$. Then, we must multiply this integrand by the Laplace Transform, $L_W(s)$, as we must restart the task over again when an interruption occurs.

Thus :

$$L_W(s) = \int_a^b e^{-(\lambda+s)t} f(t)dt + \int_a^b (1 - e^{-(\lambda+s)t}) \frac{\lambda}{\lambda+s} f(t) L_W(s) dt$$

so

$$L_W(s) = \frac{\int_a^b e^{-(\lambda+s)t} f(t)dt}{1 - \frac{\lambda}{\lambda+s}\int_a^b (1 - e^{-(\lambda+s)t}) f(t)dt}$$

$\square$

Special case: T is uniform(a,b)

Theorem

Let $T \sim Uniform(a, b), a < b$ be the uninterrupted task time. Then the Laplace Transform for the total task W time (including interuptions) with a Poisson process interruption at rate λ is given by

$$L_W(s) = \frac{\left(-\mathrm{e}^{-(\lambda+s)a} + \mathrm{e}^{-(\lambda+s)b}\right)(\lambda + s)}{\lambda\left(-\mathrm{e}^{-(\lambda+s)a} + \mathrm{e}^{-(\lambda+s)b}\right) + s(\lambda + s)(a - b)}$$

Special case: T is deterministic $P(T = k) = 1$

Corollary

Let uninterrupted task time be exactly k. Then the Laplace Transform for the total task time W (including interuptions) with a Poisson process interruption at rate λ is given by

$$L_W(s) = \frac{e^{-(\lambda+s)k}(\lambda + s)}{e^{-(\lambda+s)k}\lambda + s}$$

Special case: T is deterministic $P(T = k) = 1$

Corollary

Let uninterrupted task time be exactly k. Then the expected value of the total task time W (including interruptions) with a Poisson process interruption at rate λ is given by

$$E(W) = -L'_w(0) = \frac{e^{\lambda k} - 1}{\lambda}$$

Note: This matches well known results in the literature.

Special case: T is exponential rate μ

Theorem

Let $T \sim$ Exponential rate μ be the uninterrupted task time. Then the Laplace Transform for the total task time W (including interruptions) with a Poisson process interruption at rate λ is given by

$L_W(s) =$

Special case: T is exponential rate μ

Theorem

Let $T \sim$ Exponential rate μ be the uninterrupted task time. Then the Laplace Transform for the total task time W (including interruptions) with a Poisson process interruption at rate λ is given by
$$L_W(s) = \frac{\mu}{\mu + s}$$

Special case: T is exponential rate μ

Theorem

Let $T \sim$ Exponential rate μ be the uninterrupted task time. Then the Laplace Transform for the total task time W (including interruptions) with a Poisson process interruption at rate λ is given by

$$L_W(s) = \frac{\mu}{\mu + s}$$

This is shocking!!. There is no λ in the answer.
And the result is the LT of an exponential with rate μ. i.e. The interruptions do not affect the time to complete the task.
Weird!

Special case: T is Gamma(2,1)

Theorem

Let $T \sim Gamma(2,1)$ (sum of 2 exp at rate 1) be the time to complete a task without interruption. Let W be the total time to complete the task, which can be interrupted by a Poisson process at rate λ. Then the Laplace Transform for W is given by

$$L_W(s) = \frac{1}{s^2 + (\lambda + 2)s + 1}.$$

Consider the special case where $\lambda = 1/2$. Then:

$$L_w(s) = \frac{1}{s^2 + (5/2)s + 1} = \frac{1/2}{(s + 1/2)} \frac{2}{(s + 2)}$$

When the LT is written in this form, we can see that the LT is the product of 2 exponential LT. But LT of a sum of rv is the product of their LT. So W is the sum of 2 exponentials at different rates.
Also, we could rewrite the LT using partial fraction decomposition.

$$L_W(s) = \frac{\frac{2}{3}}{s+\frac{1}{2}} + \frac{\frac{-2}{3}}{s+2}$$

But the inverse of a LT expressed as a sum of LT is the sum of the inverse of each. Inverting the LT, we have:

$$\begin{aligned} f_W(w) &= \frac{4}{3}L^{-1}\left(\frac{1/2}{s+\frac{1}{2}}\right) - \frac{1}{3}L^{-1}\left(\frac{2}{s+2}\right) \\ &= \frac{4}{3}(1/2)e^{-\frac{1}{2}w} - \frac{1}{3}2e^{-2w} \end{aligned}$$

so the pdf is a linear combination of two exponential pdf's.

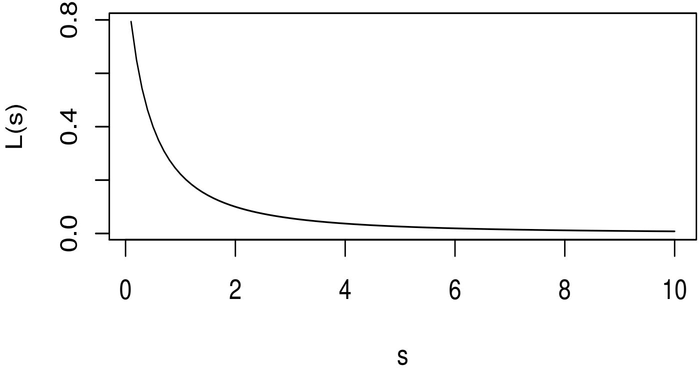

Figure: Laplace Transform

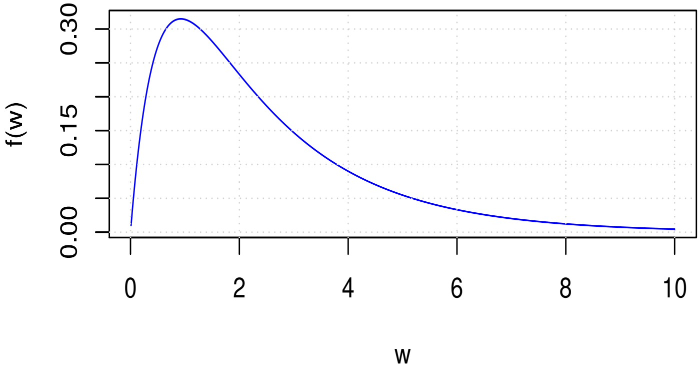

Figure: Inverse Laplace Transform using invlap in R

The End.

Thank you.