Volume estimates for some random convex sets

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$$\|\mathbf{t}\|_{\mathcal{C},K} = \frac{1}{\prod_{j=1}^s |C_j|} \int_{C_1} \cdots \int_{C_s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \, dx_s \cdots dx_1,$$

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where $\mathbf{t} = (t_1, \ldots, t_s)$. If $\mathcal{C} = (C, \ldots, C)$ then we write $\|\mathbf{t}\|_{C^s,K}$ instead of $\|\mathbf{t}\|_{\mathcal{C},K}$.

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Question (V. Milman)

To examine if, in the case $C = K$, one has that

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is equivalent to the standard Euclidean norm up to a term which is logarithmic in the dimension. In particular, if under some cotype condition on the norm induced by K to $\mathbb{R}^n$ one has equivalence between $\|\cdot\|_{K^s,K}$ and the Euclidean norm.

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- Since $\|\mathbf{t}\|_{K^s,K} = \|\mathbf{t}\|_{(TK)^s,TK}$ for any $T \in GL(n)$, we may choose any position of K.

- We may assume that $|C_1| = \cdots = |C_s| = |K| = 1$.

Lower bounds

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- Bourgain, Meyer, V. Milman and Pajor (1987) obtained the lower bound

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- Assuming, additionally, that C is isotropic they also obtained the lower bound

$$\int_C \cdots \int_C \int_\Omega \Big\| \sum_{j=1}^{s} g_j(\omega)x_j \Big\|_K d\omega\, dx_s \cdots dx_1 \geqslant c\sqrt{s}\, L_C \sqrt{n} M(K),$$

where L_C is the isotropic constant of C and $M(K) = \int_{S^{n-1}} \|\xi\|_K \, d\sigma(\xi)$.

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Gluskin-Milman

Let $A_1, \ldots, A_s$ be measurable sets in $\mathbb{R}^n$ and K be a star body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Assume that $|A_1| = \cdots = |A_s| = |K|$.

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$$\|\mathbf{t}\|_{\mathcal{A},K} := \frac{1}{\prod_{j=1}^{s} |A_j|} \int_{A_1} \cdots \int_{A_s} \Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K \, dx_s \cdots dx_1 \geqslant c \, \|\mathbf{t}\|_2.$$

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- The proof uses the Brascamp-Lieb-Luttinger rearrangement inequality.

Lower bounds: alternative proof

G.-Chasapis-Skarmogiannis

Let $\mathcal{C} = (C_1, \ldots, C_s)$ be an s-tuple of symmetric convex bodies and K be a symmetric convex body in $\mathbb{R}^n$ with $|C_j| = |K| = 1$. Then, for any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,

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Let $X_1, \ldots, X_s$ be independent random vectors, uniformly distributed on $C_1, \ldots, C_s$ respectively. Given $\mathbf{t} = (t_1 \ldots, t_s) \in \mathbb{R}^s$, we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1 X_1 + \cdots + t_s X_s$. Then,

$$\|\mathbf{t}\|_{\mathcal{C},K} = \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x).$$

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$$\|\mathbf{t}\|_{\mathcal{C},K} = \int_{\mathbb{R}^n} \|x\|_K d\nu_{\mathbf{t}}(x).$$

- Note that $\nu_{\mathbf{t}}$ is an even log-concave probability measure on $\mathbb{R}^n$ We write $g_{\mathbf{t}}$ for the density of $\nu_{\mathbf{t}}$.

Lower bounds: alternative proof

Lemma 1

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If g is log-concave then

$$\log(\|g\|_\infty^{-1}) \leqslant h(X) \leqslant n + \log(\|g\|_\infty^{-1}).$$

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- Let $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$ and $t_1, \ldots, t_s \geqslant 0$. Then, if $X_1, \ldots, X_s$ are independent random vectors with densities $g_1, \ldots, g_s$, by an equivalent form of the Shannon-Stam inequality, we have that $h(t_1X_1 + \cdots + t_sX_s) \geqslant \sum_{j=1}^s t_j^2 h(X_j)$.

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- Since the density $g_{\mathbf{t}}$ of $t_1X_1 + \cdots + t_sX_s$ is also log-concave, we may write

$$\sum_{j=1}^{s} t_j^2 \log(\|g_j\|_\infty^{-1}) \leqslant \sum_{j=1}^{s} t_j^2 h(X_j) \leqslant h(t_1X_1 + \cdots + t_sX_s) \leqslant n + \log(\|g_{\mathbf{t}}\|_\infty^{-1}),$$

which implies that $\|g_{\mathbf{t}}\|_\infty \leqslant e^n \prod_{j=1}^{s} \|g_j\|_\infty^{t_j^2}$.

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  which implies that $\|g_{\mathbf{t}}\|_\infty \leqslant e^n \prod_{j=1}^{s} \|g_j\|_\infty^{t_j^2}$.
- In our case, $g_j = \mathbf{1}_{C_j}$, therefore $\|g_j\|_\infty = 1$ and the lemma follows.

Lower bounds: alternative proof

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If $\|\mathbf{t}\|_2 = 1$ then $\|g_{\mathbf{t}}\|_\infty \leqslant e^n$.

Lemma 2

Let f be a bounded positive density on $\mathbb{R}^n$. For any symmetric convex body K of volume 1 in $\mathbb{R}^n$ we have

$$\frac{n}{n+1} \leqslant \|f\|_\infty^{1/n} \int_{\mathbb{R}^n} \|x\|_K f(x)\, dx.$$

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- For any $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$ we have $\|g_{\mathbf{t}}\|_\infty \leqslant e^n$, therefore

$$\frac{n}{n+1} \leqslant e \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x) = e\, \|\mathbf{t}\|_{\mathcal{C},K}.$$

Isotropic convex bodies and log-concave measures

- A convex body C in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_C > 0$ such that

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- We say that a log-concave probability measure μ with density f_μ on $\mathbb{R}^n$ is isotropic if it is centered, i.e. if

$$\int_{\mathbb{R}^n}\langle x,\xi\rangle d\mu(x) = \int_{\mathbb{R}^n}\langle x,\xi\rangle f_\mu(x)dx = 0$$

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- If μ is an isotropic log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \|f_\mu\|_\infty^{\frac{1}{n}}.$$

- If C is a centered convex body of volume 1 in $\mathbb{R}^n$ then we say that a direction $\xi \in S^{n-1}$ is a ψ_α-direction (where $1 \leqslant \alpha \leqslant 2$) for C with constant $\varrho > 0$ if

$$\|\langle \cdot, \xi \rangle\|_{L_q(C)} \leqslant \varrho\, q^{1/\alpha} \|\langle \cdot, \xi \rangle\|_{L_2(C)},$$

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- From log-concavity it follows that every $\xi \in S^{n-1}$ is a ψ_1-direction for any C or μ with an absolute constant ϱ: there exists $\varrho > 0$ such that

$$\|\langle \cdot, \xi\rangle\|_{L_q(\mu)} \leqslant \varrho\, q \|\langle \cdot, \xi\rangle\|_{L_2(\mu)}$$

  for all $n \geqslant 1$, all centered log-concave probability measures μ on $\mathbb{R}^n$ and all $\xi \in S^{n-1}$ and $q \geqslant 2$.

Upper bounds

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- It follows that if $g_{\mathbf{t}}$ is the density of $\nu_{\mathbf{t}}$ then $f_{\mathbf{t}}(x) = L_C^n g_{\mathbf{t}}(L_C x)$ is the density of an isotropic log-concave probability measure $\mu_{\mathbf{t}}$ on $\mathbb{R}^n$.

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- From Lemma 1 we have a bound for the isotropic constants of all these measures:

$$L_{\mu_{\mathbf{t}}}=\|f_{\mathbf{t}}\|_\infty^{\frac{1}{n}}=L_C\|g_{\mathbf{t}}\|_\infty^{\frac{1}{n}}\leqslant eL_C$$

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for all $\mathbf{t}\in\mathbb{R}^s$ with $\|\mathbf{t}\|_2=1$.
- We also have

$$\|\mathbf{t}\|_{C^s,K}=\int_{\mathbb{R}^n}\|x\|_K\,d\nu_{\mathbf{t}}(x)=L_C^{-n}\int_{\mathbb{R}^n}\|x\|_Kf_{\mathbf{t}}(x/L_C)\,dx=L_C\int_{\mathbb{R}^n}\|y\|_Kd\mu_{\mathbf{t}}(y).$$

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- In this case
$$\|\mathbf{t}\|_{C^s,K} = \|\mathbf{t}\|_2 L_C \, I_1(\mu_{\mathbf{t}}, K),$$
where $\mu_{\mathbf{t}}$ is an isotropic, compactly supported log-concave probability measure depending on $\mathbf{t}$ and
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$$I_1(\mu, K) = \int_{\mathbb{R}^n} \|x\|_K d\mu(x).$$
- Note that if μ is isotropic and K is a symmetric convex body of volume 1 in $\mathbb{R}^n$ then
$$\int_{O(n)} I_1(\mu, U(K))\, d\nu(U) = \int_{\mathbb{R}^n} \int_{O(n)} \|x\|_{U(K)} d\nu(U)\, d\mu(x)$$
$$= M(K) \int_{\mathbb{R}^n} \|x\|_2 d\mu(x) \approx \sqrt{n} M(K).$$

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where $\mu_{\mathbf{t}}$ is an isotropic, compactly supported log-concave probability measure depending on $\mathbf{t}$ and
$$I_1(\mu, K) = \int_{\mathbb{R}^n} \|x\|_K d\mu(x).$$
- Note that if μ is isotropic and K is a symmetric convex body of volume 1 in $\mathbb{R}^n$ then
$$\begin{aligned}\int_{O(n)} I_1(\mu, U(K))\, d\nu(U) &= \int_{\mathbb{R}^n} \int_{O(n)} \|x\|_{U(K)} d\nu(U)\, d\mu(x) \\ &= M(K) \int_{\mathbb{R}^n} \|x\|_2 d\mu(x) \approx \sqrt{n} M(K).\end{aligned}$$
- It follows that $\int_{O(n)} \|\mathbf{t}\|_{U(C)^s,K} \approx (L_C \sqrt{n} M(K))\, \|\mathbf{t}\|_2$.

- Since $\|\mathbf{t}\|_{C^s,K} = \|\mathbf{t}\|_{(TC)^s,TK}$ for any $T \in SL(n)$, we may restrict our attention to the case where C is isotropic.
- In this case
$$\|\mathbf{t}\|_{C^s,K} = \|\mathbf{t}\|_2 L_C\, I_1(\mu_{\mathbf{t}}, K),$$
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$$= M(K) \int_{\mathbb{R}^n} \|x\|_2 d\mu(x) \approx \sqrt{n} M(K).$$
- It follows that $\int_{O(n)} \|\mathbf{t}\|_{U(C)^s,K} \approx (L_C \sqrt{n} M(K))\, \|\mathbf{t}\|_2$.
- Therefore, our goal is to obtain a constant of the order of $L_C \sqrt{n} M(K)$ in our upper estimate for $\|\mathbf{t}\|_{C^s,K}$.

Bounds for $M(K_{\rm iso})$

- In particular, in the case $C = K$ we may assume that K is isotropic, and an optimal upper bound would be $O(L_K\sqrt{n}M(K_{\rm iso}))$.

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  proved in [G. - E. Milman].
- There, it is also shown that in the case where K is a ψ_2-body with constant ϱ one has

$$M(K_{\rm iso}) \leqslant \frac{c\sqrt[3]{\varrho}(\log n)^{1/3}}{\sqrt[6]{n} L_K}.$$

A general upper bound

G.-Chasapis-Skarmogiannis

Let C be an isotropic convex body in $\mathbb{R}^n$ and K be a symmetric convex body in $\mathbb{R}^n$. Then,

$$\|\mathbf{t}\|_{C^s,K} \leqslant c \max\left\{\sqrt[4]{n}, \sqrt{\log(1+s)}\right\} L_C\sqrt{n}M(K)\|\mathbf{t}\|_2$$

for every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$, where $c > 0$ is an absolute constant.

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- For the proof one has to estimate

$$I_1(\mu_{\mathbf{t}}, K) = \int_{\mathbb{R}^n} \|x\|_K d\mu_{\mathbf{t}}(x)$$

where $\mu_{\mathbf{t}}$ is an isotropic, compactly supported log-concave probability measure depending on the unit vector $\mathbf{t}$.

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  where $\mu_{\mathbf{t}}$ is an isotropic, compactly supported log-concave probability measure depending on the unit vector $\mathbf{t}$.
- This is done with an argument that resembles Bourgain's proof of the bound $L_K = O(\sqrt[4]{n}\log n)$ and makes use of Talagrand's comparison theorem.

ψ_2-case

Let C be an isotropic convex body in $\mathbb{R}^n$, which is a ψ_2-body with constant ϱ, and K be a symmetric convex body in $\mathbb{R}^n$. Then for any $s \geqslant 1$ and every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$,

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$$\|\mathbf{t}\|_{C^s,K} \leqslant \big(c\,L_C\,C_2(X_K)\sqrt{n}M(K)\big)\,\|\mathbf{t}\|_2$$

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where $C_2(X_K)$ is the cotype-2 constant of the space with unit ball K.

- This is a consequence of our representation of $\|\mathbf{t}\|_{C^s,K}$ and of a result of E. Milman: If μ is a finite, compactly supported isotropic measure on $\mathbb{R}^n$ then, for any symmetric convex body K in $\mathbb{R}^n$,

$$I_1(\mu, K) \leqslant c\, C_2(X_K) \sqrt{n} M(K).$$

- In particular, for any symmetric convex body K of volume 1 in $\mathbb{R}^n$ we have that

$$\int_K \cdots \int_K \Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K dx_s \cdots dx_1 \leqslant \big(cL_K C_2(X_K)\sqrt{n} M(K_{\rm iso}) \big) \, \|\mathbf{t}\|_2,$$

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Unconditional case

There exists an absolute constant $c > 0$ with the following property: if K and $C_1, \dots, C_s$ are isotropic unconditional convex bodies in $\mathbb{R}^n$ then, for every $q \geqslant 1$,

$$\left(\int_{C_1} \dots \int_{C_s} \Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K^q dx_1 \dots dx_s \right)^{1/q} \leqslant cn^{1/q}\sqrt{q} \cdot \max\{\|\mathbf{t}\|_2, \sqrt{q}\|\mathbf{t}\|_\infty\} \leqslant cn^{1/q} q \, \|\mathbf{t}\|_2,$$

for every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$. In particular,

$$\|\mathbf{t}\|_{\mathcal{C},K} \leqslant c\sqrt{\log n} \cdot \max\{\|\mathbf{t}\|_2, \sqrt{\log n}\|\mathbf{t}\|_\infty\} \leqslant c \log n \, \|\mathbf{t}\|_2.$$

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- This is essentially proved in [G.-Hartzoulaki-Tsolomitis]. The proof makes use of the comparison theorem of Bobkov and Nazarov.

- Let K be a symmetric convex body in $\mathbb{R}^N$. For any $\mathbf{x} = (x_1, \ldots, x_N) \in \oplus_{i=1}^N \mathbb{R}^n$ we denote by

$$T_{\mathbf{x}} = [x_1 \cdots x_N]$$

the $n \times N$ matrix whose columns are the vectors x_i, and consider the convex body $T_{\mathbf{x}}(K)$ in $\mathbb{R}^n$.

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Examples

- If $K = B_1^N$ then

$$T_{\mathbf{x}}(K) = \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}.$$

Expected volume of random convex sets

- Let K be a symmetric convex body in $\mathbb{R}^N$. For any $\mathbf{x} = (x_1, \ldots, x_N) \in \oplus_{i=1}^N \mathbb{R}^n$ we denote by

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$$T_{\mathbf{x}}(K) = \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}.$$

- If $K = B_\infty^N$ then

$$T_{\mathbf{x}}(K) = \sum_{i=1}^N [-x_i, x_i].$$

Expected volume of random convex sets

- The question that we study is to estimate the expected volume of $T_{\mathbf{x}}(K)$ when $x_1, \ldots, x_N$ are independent random points distributed according to an isotropic log-concave probability measure μ.

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Paouris-Pivovarov

Let $N \geqslant n$ and $f_1, \ldots, f_N$ be probability densities on $\mathbb{R}^n$ with $\|f_i\|_\infty \leqslant 1$ for all $i = 1, \ldots, N$. Then,

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |T_{\mathbf{x}}(K)| \prod_{i=1}^{N} f_i(x_i)\, dx_N \cdots dx_1 \geqslant \int_{D_n} \cdots \int_{D_n} |T_{\mathbf{x}}(K)|\, dx_N \cdots dx_1,$$

where D_n is the (centered at the origin) Euclidean ball of volume 1.

- The theorem of Paouris and Pivovarov shows that for a lower bound it is useful to examine the case $\mu = \mu_{D_n}$, where μ_{D_n} is the uniform measure on D_n.

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G.-Skarmogiannis

For any $N \geqslant n$ and any convex body K in $\mathbb{R}^N$ we have

$$c_1\sqrt{N/n}\,\mathrm{vrad}(K) \leqslant \left(\mathbb{E}_{\mu_{D_n}^N}\,|T_x(K)|^{1/n}\right) \leqslant \left(\mathbb{E}_{\mu_{D_n}^N}\,|T_x(K)|\right)^{1/n} \leqslant c_2\sqrt{N/n}\,w(K),$$

where $c_1, c_2 > 0$ are absolute constants.

Expected volume of random convex sets

$K = B_\infty^N$

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$$\mathbb{E}_{\mu^N}\left(|\mathrm{conv}\{\pm x_1,\dots,\pm x_N\}|\right)^{1/n} \approx \frac{\sqrt{\log(2N/n)}}{\sqrt{n}} \leqslant \sqrt{N/n}\,w(B_1^N).$$

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Unconditional K

Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any unconditional isotropic convex body K in $\mathbb{R}^N$ we have

$$\mathbb{E}_{\mu^N}\left(|T_x(K)|\right)^{1/n} \leqslant c\sqrt{N/n}\,\mathrm{vrad}(K)\,\sqrt{\log(2N/n)}.$$

A general upper bound

Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any $N \geqslant n$ and any symmetric convex body K in $\mathbb{R}^N$ we have

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where $c > 0$ is an absolute constant.

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where $c > 0$ is an absolute constant.

- Our starting point is the formula

$$|T_{\mathsf{x}}(K)| = \sqrt{\det(T_{\mathsf{x}}T_{\mathsf{x}}^*)}\,|P_{E_{\mathsf{x}}}(K)|,$$

where $E_{\mathsf{x}} = \ker(T_{\mathsf{x}})^{\perp} = \mathrm{Range}(T_{\mathsf{x}}^*)$.

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- By the Cauchy-Binet formula

$$\det(T_{\mathrm{x}} T_{\mathrm{x}}^*) = \sum_{|S|=n} \det((T_{\mathrm{x}}|_S)(T_{\mathrm{x}}|_S)^*).$$

  and

$$\mathbb{E}_{\mu^N}\big(\det((T_{\mathrm{x}}|_S)(T_{\mathrm{x}}|_S)^*)\big) = n!\, \det(\mathrm{Cov}(\mu)).$$

- Assuming that μ is isotropic we have that $\det(\mathrm{Cov}(\mu)) = 1$. It follows that

$$\mathbb{E}_{\mu^N}\big(\det(T_{\mathsf{x}} T_{\mathsf{x}}^*)\big) = \binom{N}{n} n!\, \det(\mathrm{Cov}(\mu)) \leqslant N^n.$$

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- Then,

$$\begin{aligned}\mathbb{E}_{\mu^N}\Big(|T_{\mathbf{x}}(K)|\Big) &\leqslant \Big(\mathbb{E}_{\mu^N}\Big(\det(T_{\mathbf{x}}T_{\mathbf{x}}^*)\Big)\Big)^{1/2}\Big(\mathbb{E}_{\mu^N}|P_{E_{\mathbf{x}}}(K)|^2\Big)^{1/2}\\ &\leqslant N^{n/2}\Big(\mathbb{E}_{\mu^N}|P_{E_{\mathbf{x}}}(K)|^2\Big)^{1/2}.\end{aligned}$$

Expected volume of random convex sets

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- Then we use the fact that if K is a centrally symmetric convex body in $\mathbb{R}^N$ then for any $1 \leqslant n < N$ and any $E \in G_{N,n}$ we have that

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- This follows in a standard way from Sudakov's inequality.

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Isotropic K

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where $c > 0$ is an absolute constant.

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where $c > 0$ is an absolute constant.

- This time we use a classical inequality of Rogers and Shephard:

$$|P_{E_{\mathbf{x}}}(K)| \leqslant \binom{N}{n} |K \cap E_{\mathbf{x}}^{\perp}|^{-1}$$

for all $\mathbf{x}$.

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for all $\mathbf{x}$.

- Since K is isotropic, we also know that

$$|K \cap E_{\mathbf{x}}^{\perp}|^{1/n} \geqslant \frac{c}{L_K}.$$

Random ball polyhedra

- Let f be a probability density on $\mathbb{R}^n$ with $\|f\|_\infty \leqslant 1$, fix $N \geqslant 1$ and an N-tuple $\mathbf{r} = (r_1, \ldots, r_N)$ of positive real numbers. Consider a sequence $x_1, \ldots, x_N$ of independent random points in $\mathbb{R}^n$ distributed according to f, and define the random ball-polyhedron

$$B(\mathbf{x}, \mathbf{r}) := \bigcap_{i=1}^{N} B(x_i, r_i)$$

which is the intersection of the Euclidean balls $B(x_i, r_i) = x_i + r_i B_2^n$.

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- Paouris and Pivovarov showed that if $z_1, \ldots, z_N$ is a sequence of independent random points in $\mathbb{R}^n$ distributed according to the uniform measure on the Euclidean ball D_n of volume 1 then, for any $1 \leqslant j \leqslant n$ and for any $r_1, \ldots, r_N > 0$,

$$\mathbb{E}_{\mu^N} V_j\Big(\bigcap_{i=1}^{N} B(x_i, r_i)\Big) \leqslant \mathbb{E}_{\mu_{D_n}^N} V_j\Big(\bigcap_{i=1}^{N} B(z_i, r_i)\Big),$$

where V_j denotes the j-th intrinsic volume.

- Let f be a probability density on $\mathbb{R}^n$ with $\|f\|_\infty \leqslant 1$, fix $N \geqslant 1$ and an N-tuple $\mathbf{r} = (r_1, \ldots, r_N)$ of positive real numbers. Consider a sequence $x_1, \ldots, x_N$ of independent random points in $\mathbb{R}^n$ distributed according to f, and define the random ball-polyhedron

$$B(\mathbf{x}, \mathbf{r}) := \bigcap_{i=1}^{N} B(x_i, r_i)$$

which is the intersection of the Euclidean balls $B(x_i, r_i) = x_i + r_i B_2^n$.

- Paouris and Pivovarov showed that if $z_1, \ldots, z_N$ is a sequence of independent random points in $\mathbb{R}^n$ distributed according to the uniform measure on the Euclidean ball D_n of volume 1 then, for any $1 \leqslant j \leqslant n$ and for any $r_1, \ldots, r_N > 0$,

$$\mathbb{E}_{\mu^N} V_j\Big(\bigcap_{i=1}^{N} B(x_i, r_i)\Big) \leqslant \mathbb{E}_{\mu_{D_n}^N} V_j\Big(\bigcap_{i=1}^{N} B(z_i, r_i)\Big),$$

where V_j denotes the j-th intrinsic volume.

- In fact, they showed that the same holds true for any function $\varphi : \mathcal{K}^n \to [0, \infty)$ which is quasi-concave with respect to Minkowski addition, monotone and invariant under orthogonal transformations. The intrinsic volumes satisfy the above - the quasi-concavity is a consequence of the Aleksandrov-Fenchel inequality.

- Question: to estimate the expected volume

$$\mathbb{E}\left|\bigcap_{i=1}^{N} B(x_i, r_i)\right|$$

where $x_1, \ldots, x_N$ are independent random points uniformly distributed in a convex body K of volume 1 in $\mathbb{R}^n$, and $r_1, \ldots, r_N > 0$.

Random ball polyhedra

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- More generally, to estimate the expected volume

$$\mathbb{E}\left|\bigcap_{i=1}^{N} (x_i + r_i C)\right|$$

  where $x_1, \ldots, x_N$ are independent random points uniformly distributed in a convex body K of volume 1 in $\mathbb{R}^n$, C is any symmetric convex body in $\mathbb{R}^n$, and $r_1, \ldots, r_N > 0$.

Skarmogiannis

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ and $x_1,\dots,x_N$ be independent random points uniformly distributed in K. Then, for any symmetric convex body C in $\mathbb{R}^n$ and any $r_1,\dots,r_N>0$,

$$c_{n,N}|K+rC|\prod_{i=1}^N|K\cap r_iC|\leqslant \mathbb{E}_{\mu_K^N}\left(\Big|\bigcap_{i=1}^N(x_i+rC)\Big|\right)\leqslant |K+rC|\prod_{i=1}^N|K\cap r_iC|,$$

where $r=\min\{r_1,\dots,r_N\}$ and $c_{n,N}=nB(n,nN+1)$ where $B(a,b)$ is the Beta function.

Lemma

Let K, C be centrally symmetric convex bodies in $\mathbb{R}^n$. Assume that $|K| = 1$. For any $r_1, \ldots, r_N > 0$,

$$\mathbb{E}_{\mu_K^N}\left(\left|\bigcap_{i=1}^{N}(x_i + r_i C)\right|\right) = \int_{K+(\min_i r_i)C} \prod_{i=1}^{N} |(K-y)\cap r_i C|\, dy.$$

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Let $r_1, \ldots, r_N > 0$. We write

$$\mathbb{E}_{\mu_K^N}\left(\left|\bigcap_{i=1}^{N}(x_i + r_i C)\right|\right) = \int_K \cdots \int_K \int_{\mathbb{R}^n} \mathbf{1}_{\bigcap_{i=1}^{N}(x_i+r_iC)}(y)\, dy\, dx_N \cdots dx_1$$

$$= \int_K \cdots \int_K \int_{\mathbb{R}^n} \prod_{i=1}^{N} \mathbf{1}_{x_i+r_iC}(y)\, dy\, dx_N \cdots dx_1$$

$$= \int_{\mathbb{R}^n} \int_K \cdots \int_K \prod_{i=1}^{N} \mathbf{1}_{y+r_iC}(x_i)\, dx_N \cdots dx_1\, dy = \int_{\mathbb{R}^n} \prod_{i=1}^{N} \left(\int_K \mathbf{1}_{y+r_iC}(x_i)\, dx_i\right) dy$$

$$= \int_{\mathbb{R}^n} \prod_{i=1}^{N} |K \cap (y + r_i C)|\, dy = \int_{\mathbb{R}^n} \prod_{i=1}^{N} |(K-y)\cap(r_i C)|\, dy.$$

Random ball polyhedra

Lower bound

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ and $x_1, \ldots, x_N$ be independent random points uniformly distributed in K. Then, for any symmetric convex body C in $\mathbb{R}^n$ and any $r_1, \ldots, r_N > 0$,

$$\mathbb{E}_{\mu_K^N}\left(\Big|\bigcap_{i=1}^{N} B(x_i, r)\Big|\right) \geqslant nB(n, nN+1)\,|K + rC| \prod_{i=1}^{N} |K \cap r_i C|,$$

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- Let ϱ denote the radial function of $K+rC$ on S^{n-1}. Then,

$$\mathbb{E}_{\mu_K^N}\Big(\Big|\bigcap_{i=1}^N(x_i+rC)\Big|\Big)=n\omega_n\int_{S^{n-1}}\int_0^{\varrho(\xi)}t^{n-1}\prod_{i=1}^N u_i^n(t\xi)\,dt\,d\sigma(\xi).$$

Random ball polyhedra

- Since each u_i is concave, we have

$$u_i(t\xi) \geqslant (1 - t/\varrho(\xi))u_i(0) + (t/\varrho(\xi))u_i(\varrho(\xi)\xi) \geqslant (1 - t/\varrho(\xi))u_i(0).$$

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- Therefore,

$$\begin{aligned}
&\mathbb{E}_{\mu_K^N}\left(\Big|\bigcap_{i=1}^N (x_i + rC)\Big|\right) \\
&\geqslant n\omega_n \prod_{i=1}^N u_i^n(0) \int_{S^{n-1}} \int_0^{\varrho(\xi)} t^{n-1}\left(1 - \frac{t}{\varrho(\xi)}\right)^{nN} dt\, d\sigma(\xi) \\
&= n\omega_n \prod_{i=1}^N |K \cap r_i C| \int_{S^{n-1}} \int_0^1 \varrho^n(\xi) s^{n-1}(1-s)^{nN}\, ds\, d\sigma(\xi) \\
&= n \prod_{i=1}^N |K \cap r_i C| \cdot \omega_n \int_{S^{n-1}} \varrho^n(\xi)\, d\sigma(\xi) \cdot \int_0^1 s^{n-1}(1-s)^{nN}\, ds \\
&= nB(n, nN+1)|K + rC| \prod_{i=1}^N |K \cap r_i C|.
\end{aligned}$$

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Random ball polyhedra

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- One can check that

$$\lim_{r\to\infty}\frac{1}{|K+rC|\cdot|K\cap rC|^N}\mathbb{E}_{\mu_K^N}\left|\bigcap_{i=1}^N(x_i+rC)\right|=1$$

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- Also,

$$\lim_{r\to 0^+}\frac{1}{|K+rC|\cdot|K\cap rC|^N}\mathbb{E}_{\mu_K^N}\left|\bigcap_{i=1}^N(x_i+rC)\right|=1$$

and $|K+rC|\cdot|K\cap rC|^N\sim|rC|^N$ as $r\to 0^+$.