Functional surface area measures

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Outline

- Surface area measures
  - Surface area measures for convex bodies
  - Surface area measures for log-concave functions
  - Proof Sketch
- L^p-Minkowski theorem, $0 < p < 1$
  - L^p surface area measures
  - Our Theorem and Proof Sketch

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  - Surface area measures for convex bodies
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  - L^p surface area measures
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Notation

- $K, L \subseteq \mathbb{R}^n$ will denote convex bodies (compact, non-empty interior)
- $|K|$ denotes the volume of K, but $|x|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$.
- $h_K : \mathbb{R}^n \to \mathbb{R}$ denotes the support function,

$$h_K(y) = \max_{x \in K} \langle x, y \rangle$$

.

- The Minkowski sum $K + tL$ is defined implicitly by

$$h_{K+tL} = h_K + th_L,$$

or explicitly by

$$K + tL = \{x + ty : \ x \in K \text{ and } y \in L\}.$$

Surface area measures

Theorem

For every K there exists a unique Borel measure S_K on the unit sphere S^{n-1} such that for every L we have

$$\lim_{t \to 0^+} \frac{|K + tL| - |K|}{t} = \int_{S^{n-1}} h_L \mathrm{d}S_K.$$

S_K is called the surface area measure of K. It has an explicit description: for $A \subseteq \mathbb{R}^n$ we have

$$S_K(A) = \mathcal{H}^{n-1}\left(\nu_K^{-1}(A)\right),$$

where $\mathcal{H}^{n-1}$ is the (n-1)-dimensional Hausdorff measure and $\nu_K : \partial K \to S^{n-1}$ is the Gauss map (defined $\mathcal{H}^{n-1}$-a.e.). In other words $S_K = (\nu_K)_\sharp\left(\mathcal{H}^{n-1}|_{\partial K}\right)$.

Log-concave functions

- A function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda$$

  for all $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$.
- We will always assume our log-concave functions are upper semi-continuous and that $0 < \int f < \infty$.
- *Examples*: $f = \mathbb{1}_K$ for a convex body K, $f(x) = e^{-|x|^2/2}$.
- We want to consider log-concave functions as "generalized convex bodies". This proved to be extremely useful in the past.
- For this we need "volume" (easy, take $\int f$), "support function" and "addition".

Addition and Support functions

Theorem (R. '13)

Assume we associate to every log-concave function f a convex support function h_f such that

1. $f \le g$ *if and only if* $h_f \le h_g$.
2. $h_{\mathbb{1}_K} = h_K$.
3. $h_{f \oplus g} = h_f + h_g$ *for some addition* $\oplus$.

Then $h_f(x) = \frac{1}{C} \cdot (-\log f)^* (Cx)$, *where*

$$\phi^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \phi(x))$$

is the Legendre transform. Also,

$$(f \oplus g)(x) = (f \star g)(x) = \sup_{y \in \mathbb{R}^n} f(y) g(x - y)$$

is the sup-convolution. The corresponding scalar multiplication is $(t \cdot f)(x) = f\left(\frac{x}{t}\right)^t$.

Functional surface area measure

Definition

For a log-concave function $f = e^{-\phi} : \mathbb{R}^n \to \mathbb{R}$, it surface area measure S_f is a Borel measure on $\mathbb{R}^n$ defined by

$$S_f = (\nabla\phi)_\sharp (f \mathrm{d}x)$$

This is well defined, since ϕ is differentiable $f\mathrm{d}x$-a.e.

Examples

If $f = e^{-|x|^2/2}$ then $\nabla\phi = Id$, so $S_f = f\mathrm{d}x$.
If $f = e^{-\max\{\langle x,v_1\rangle,\langle x,v_2\rangle,\ldots,\langle x,v_m\rangle\}}$ then $S_f = \sum_{i=1}^m c_i\delta_{v_i}$, where

$$c_k = \int \mathbb{1}_{\left\{f=e^{-\langle x,v_k\rangle}\right\}} f\mathrm{d}x.$$

If $f = \mathbb{1}_K$ then $S_f = |K| \cdot \delta_0$.

Also note that $S_f(\mathbb{R}^n) = \int f$, which is the "volume" of f, not its "surface area".

First variation

Why should we think of S_f as a surface area measure? Because "sometimes"

$$\lim_{t\to 0^+} \frac{\int (f \star (t \cdot g)) - \int f}{t} = \int h_g \mathrm{d}S_f.$$

Theorem (Colesanti-Fragalà)

This holds for $f = e^{-\phi}$, $g = e^{-\beta}$ *if*

- $\phi, \beta : \mathbb{R}^n \to \mathbb{R}$ *are finite and* C^2_+.
- $\lim_{|x|\to\infty} \frac{\phi(x)}{|x|} = \lim_{|x|\to\infty} \frac{\beta(x)}{|x|} = +\infty$.
- $\phi^* - c\beta^*$ *is convex for small enough* $c > 0$.

Theorem (R.)

This holds for $f = e^{-|x|^2/2}$ *(and all* g*).*

Sub-differential may be easier

Klartag and Cordero-Erausquin proved a very related result. To explain it we define two functionals on convex functions:

- $F(\psi) = -\log \int e^{-\psi^*}$.
  The Prékopa–Leindler inequality is *exactly* the statement that F is convex.
- $\ell_f(\psi) = \int \psi \mathrm{d}S_f$, where f is a fixed log-concave function.
  Obviously ℓ is linear.

The identity

$$\lim_{t \to 0^+} \frac{\int (f \star (t \cdot g)) - \int f}{t} = \int h_g \mathrm{d}S_f$$

can be written compactly as $\nabla F(h_f) = -\frac{\ell_f}{\int f}$.

Since F is convex we can ask an easier question: Is it true that $-\frac{\ell_f}{\int f} \in \partial F(h_f)$?

Essential Continuity

Definition

A log-concave function f is called essentially continuous if

$$\mathcal{H}^{n-1}\left(\{x \in \mathbb{R}^n : \ f \text{ is not continuous at } x\}\right) = 0.$$

Write $K = \text{support}\,(f) = \overline{\{x : \ f(x) > 0\}}$. f is always continuous outside of ∂K, and for $x \in \partial K$ we have by upper semi-continuity

$$\lim_{\substack{y \to x \\ y \in K}} f(y) = f(x).$$

Therefore f is essentially continuous if and only $f \equiv 0$ $\mathcal{H}^{n-1}$-a.e. on ∂K.

Theorem (Klartag–Cordero)

$-\frac{\ell_f}{\int f} \in \partial F\left(h_f\right)$ if and only if f is essentially continuous.

Main Theorem

The Klartag-Cordero result is not comparable to Colesanti-Fragalà. The assumptions are much weaker (and optimal!), but the conclusion is also weaker.

Theorem (R.)

Assume f is essentially continuous. Then for all g

$$\lim_{t \to 0^+} \frac{\int (f \star (t \cdot g)) - \int f}{t} = \int h_g \mathrm{d}S_f .$$

Moreover, this equality for $g = \mathbb{1}_{B_2^n}$ also implies that f is essentially continuous.

This theorem is stronger than both Klartag-Cordero and Colesanti-Fragalà and implies both.
Perhaps more importantly, it gives a nice explanation for the importance of essential continuity.

Proof Sketch

Unraveling notation, we have convex functions $\psi = h_f$, $\alpha = h_g$ and we want to show

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+} \int e^{-(\psi + t\alpha)^*} = \int \alpha\left(\nabla \phi\right) e^{-\phi},$$

where $\phi = \psi^* = -\log f$. We follow the following steps, which doesn't use essential continuity:

1. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+} e^{-(\psi + t\alpha)^*(x)} = \alpha\left(\nabla \phi(x)\right) e^{-\phi(x)}$$

   if ϕ is finite and differentiable at x (so $e^{-\phi}\mathrm{d}x$-a.e.). This is fairly standard.

After step 1 we "just" need to differentiate under the integral. Surprisingly, this is the interesting part.

Proof Sketch – Contd.

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} \int e^{-(\psi+t\alpha)^*} = \int \alpha\left(\nabla\phi\right) e^{-\phi}$$

1. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} e^{-(\psi+t\alpha)^*(x)} = \alpha\left(\nabla\phi(x)\right) e^{-\phi(x)}$$

   if ϕ is finite and differentiable at x (so $e^{-\phi}\mathrm{d}x$-a.e.). This is fairly standard.

2. Reduce to the case that $\alpha(x) \leq m\,|x| + c$ for some $m, c > 0$. This is done by clever approximation and uses Prékopa–Leindler.
3. Reduce to the case $\alpha(x) = m\,|x| + c$. This is a simple measure theoretic argument. For this talk take $m = 1$, $c = 0$.

The Case $\alpha(x) = |x|$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} \int e^{-(\psi+t|x|)^*} = \int |\nabla\phi|\, e^{-\phi}$$

Write $f = e^{-\phi}$ so $\psi = \phi^* = h_f$. On the RHS we have $\int |\nabla f|$.
On the LHS we have

$$f_t(x) = e^{-(\psi+t|x|)^*(x)} = \left[f \star \left(t \cdot \mathbb{1}_{B_2^n}\right)\right](x) = \sup_{y:\ |y-x|\le t} f(y).$$

By layer cake decomposition

$$\int f_t = \int_0^\infty |[f_t > s]|\,\mathrm{d}s = \int_0^\infty |[f > s] + tB_2^n|\,\mathrm{d}s,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} \int f_t = \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} |[f > s] + tB_2^n|\right)\mathrm{d}s$$
$$= \int_0^\infty \mathcal{H}^{n-1}\left([f = s]\right)\mathrm{d}s$$

The Punch Line

The whole theorem reduced to the case $\alpha(x) = |x|$. We computed that the required result in this case is exactly

$$\int_0^\infty \mathcal{H}^{n-1}([f = s])\, \mathrm{d}s = \int |\nabla f|,$$

i.e. the co-area formula!

Theorem

For every log-concave function $f : \mathbb{R}^n \to [0, \infty)$ one has

$$\int_0^\infty \mathcal{H}^{n-1}([f = s])\, \mathrm{d}s = \int |\nabla f| + \int_{\partial(support(f))} f \,\mathrm{d}\mathcal{H}^{n-1}.$$

The proof uses the divergence theorem for Lipschitz domains and the co-area formula for BV functions.

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Alexandrov Bodies and Functions

- Given $\psi : S^{n-1} \to (-\infty, \infty]$, the Alexandrov body of ψ is the largest convex body K with $h_K \leq \psi$. Explicitly
$$K = \left\{ x : \ \langle x, \theta \rangle \leq \psi(\theta) \text{ for all } \theta \in S^{n-1} \right\}.$$
- Similarly, given $\psi : \mathbb{R}^n \to (-\infty, \infty]$ we define its Alexandrov function $f = [\psi]$ to be the largest log-concave function with $h_f \leq \psi$. Explicitly $f = e^{-\psi^*}$.

Fact

Let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be lower semi-continuous and $f = [\psi]$ be its Alexandrov function. Then $h_f = \psi$ at S_f-almost every point.

L^p-addition

- Fix $0 < p < 1$. For bodies K and L *containing the origin*, $K +_p t \cdot L$ is the Alexandrov body of $\left(h_K^p + th_L^p\right)^{1/p}$.
- We then have

$$\lim_{t \to 0^+} \frac{|K +_p t \cdot L| - |K|}{t} = \frac{1}{p} \int h_L^p h_K^{1-p} \mathrm{d}S_K.$$

- For log-concave functionsf, g with $h_f, h_g \geq 0$ we define $f \star_p t \cdot g$ to be the Alexandrov function of $\left(h_f^p + th_g^p\right)^{1/p}$.
- Under technical conditions we then have

$$\lim_{t \to 0^+} \frac{\int (f \star_p (t \cdot g)) - \int f}{t} = \frac{1}{p} \int h_g^p h_f^{1-p} \mathrm{d}S_f$$

p-surface area measures

Definition

The *p-surface area measure* $S_{K,p}$ of a convex body K containing the origin is $\mathrm{d}S_{K,p} = h_K^{1-p}\mathrm{d}S_K$.

Definition

The *p-surface area measure* $S_{f,p}$ of a log-concave function f with $h_f \geq 0$ is $\mathrm{d}S_{f,p} = h_f^{1-p}\mathrm{d}S_f$.

We are interested in the p-Minkowski existence theorem: Given $0 < p \leq 1$ and a measure μ, find a log-concave function f with $S_{f,p} = \mu$.

L^p-Minkowski theorem for symmetric bodies

Theorem (Lutwak)

Let μ be an even finite Borel measure on S^{n-1} which is not supported on any hyperplane. Then for every $0 < p \neq n$ there exists a symmetric convex body K with $S_{K,p} = \mu$. For $p = n$ there exists a symmetric convex body K with $S_{K,p} = c \cdot \mu$ for some $c > 0$.

- Uniqueness is much harder and is related to the L^p-Brunn-Minkowski inequality.
- The non-even case is also much harder.

Sketch of the proof.

Let K be the minimizer of $I(K) = |K|^{-\frac{p}{n}} \cdot \int_{S^{n-1}} h_K^p \mathrm{d}\mu$. The condition $\nabla I(K) = 0$ is exactly $S_{K,p} = c \cdot \mu$. For $p \neq n$ we can use homogeneity to make $c = 1$.

Minkowski theorem for log-concave functions ($p = 1$)

Theorem (Cordero-Klartag)

Let μ be a centered probability Borel measure which is not supported on any hyperplane. Then there exists a unique essentially continuous log-concave function f with $S_f = \mu$.

Sketch of the proof of existence.

Let f be the minimizer of

$$I(f) = \int_{S^{n-1}} h_f \mathrm{d}\mu - \log \int f.$$

The condition $\nabla I(f) = 0$ is exactly $S_f = c \cdot \mu$.
Since $S_{cf} = c \cdot S_f$ we can again make $c = 1$.

L^p-Minkowski Theorem for Log-Concave Functions

Theorem (R.)

Fix $0 < p < 1$. *Let* μ *be an even finite Borel measure with finite first moment that is not supported on any hyperplane. Then there exists an even log-concave function* f *with* $h_f \geq 0$ *such that* $S_{f,p} = c \cdot \mu$ *for some* $c > 0$.

- The main issue is lack of invariance: In general $S_{c \cdot f,p}$ is not proportional to $S_{f,p}$ for any notion of dilation.
- Therefore f cannot be found by solving an *unconstrained* optimization problem.
- Instead, we solve the *constrained* problem

$$\min \int h_f^p \, \mathrm{d}\mu \text{ subject to } \int f = a$$

and use "Lagrange multipliers".

Some more details

Define

$$D = \left\{ f : \begin{array}{l} f \text{ is even, log-concave} \\ \text{and } h_f \geq 0 \end{array} \right\}.$$

And define $I(f) = \int h_f^p \mathrm{d}\mu$ and $J(f) = \int f$. Then we:

1. Show that I attains a minimum under the constraint $J = a$.
2. Show that if a is large enough the minimizer f belongs to the *interior* of D, i.e. $h_f(0) > 0$.
3. Prove the "Lagrange multiplier" condition $\nabla I = c \cdot \nabla J$.
4. Compute both sides and deduce that $S_{f,p} = c \cdot \mu$.

Thank you