Volume of intersections of convex bodies with their symmetric images and efficient coverings

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10 February 2020

Levi (1955), Hadwiger (1957), Gohberg–Markus (1960) covering/illumination conjecture

Let K be a convex body in $\mathbb{R}^n$.

How many translates of int$K$ are needed to cover K?

How many external light sources are needed to illuminate ∂K?

$$N(K) = I(K)$$

CONJECTURE. $N(K) \leq 2^n$ with equality iff K is a cube (up to an affine map).

$N(K) \leq 2^n$?

$\star$Many partial results (but open for $n \geq 3$)

$\star$Asymptotic bounds: Rogers-Shephard-Zong

$$N(K) \leq \frac{|K-K|}{|K|}\vartheta(K)$$

$$\vartheta(K) = \text{covering density of } \mathbb{R}^n \text{ by translates of } K$$
$$\leq n\log n + \log\log n + 5n + 1$$

If K is symmetric

$$N(K) \leq 2^n(n\log n + \log\log n + 5n + 1)$$

In general

$$N(K) \leq \binom{2n}{n}(n\log n + \log\log n + 5n + 1) = \frac{1+o(1)}{\sqrt{\pi}}4^n\sqrt{n}\log n$$

$N(K) \le 4^n \sqrt{n} \log n$

Our contribution

THM. $N(K) \le C \cdot 4^n e^{-c\sqrt{n}}$

Artstein-Avidan–Slomka, Naszódi:

$$N(K) \le C \cdot \underbrace{\inf_{x \in \mathbb{R}^n} \frac{|K|}{|K \cap (x - K)|}}_{\text{Kövner-Besicovitch measure of assymmetry}} \cdot 2^n n \log n$$

THM. If $\mathrm{bar}(K) = 0$, then $\frac{|K|}{|K \cap -K|} \le 2^n e^{-c\sqrt{n}}$.

$$\delta(K) = \inf_{x \in \mathbb{R}^n} \frac{|K|}{|K \cap (x - K)|} \qquad \delta_0(K) = \frac{|K|}{|K \cap (2\mathrm{bar}(K) - K)|}$$

V. Milman-Pajor: $\delta_0(K) \leq 2^n$

THM. $\delta_0(K) \leq 2^n e^{-c\sqrt{n}}$.

CONJ. $\delta_0(K) \leq \delta_0(\text{simplex}) = \delta(\text{simplex}) = (1 + o(1))\frac{e\sqrt{3}}{2}\left(\frac{e}{2}\right)^n$.

Warm-up: $\delta(K) \leq 2^n$

- $|K| = 1,\ f = \mathbf{1}_K$
- $\|f \star f\|_\infty = \sup_x \int \mathbf{1}_K(y)\mathbf{1}_K(x-y)\mathrm{d}y = \sup_x |K \cap (x - K)|$
- $|2K|\|f \star f\|_\infty \geq \int_{2K} f \star f = \int_{\mathbb{R}^n} f \star f = |K|^2$

Goal: $\delta_0(K) \leq 2^n e^{-c\sqrt{n}}$

Different proxy: entropy of a random vector X with density f

$$\mathcal{S}(X) = -\int_{\mathbb{R}^n} f \log f = \mathbb{E}\big[-\log f(X)\big].$$

LM 1. If X, Y i.i.d. Unif(K), then

$$\log \delta_0(K) \leq \mathcal{S}(X+Y) - \mathcal{S}(X)$$

Proof.

- bar(K)=0, i.e. $\mathbb{E}X = 0$; $|K| = 1$, i.e. $\mathcal{S}(X) = \log|K| = 0$
- Want: $\log \frac{1}{|K \cap -K|} \leq \mathcal{S}(X+Y)$
- $-\log f \star f$ is convex (Prékopa-Leindler)
- $\mathcal{S}(X+Y) = \mathbb{E}\big[-\log(f \star f)(X+Y)\big] \geq$
  $-\log(f \star f)(\mathbb{E}(X+Y)) = -\log(f \star f)(0) = -\log|K \cap -K|$

LM 1. $\log \delta_0(K) \le \mathcal{S}(X+Y) - \mathcal{S}(X)$

LM 2. $\mathcal{S}(X) \le \mathbb{E}\big[-\log h(X)\big] + \log\left(\int h\right),\ h\colon \mathbb{R}^n \to [0,+\infty)$

COR. $\delta_0(K) \le \mathbb{E}\big[-\log h(X+Y)\big] + \log\left(\frac{1}{|K|}\int_{2K} h\right)$

RMK. $h = 1$ gives $\delta_0(K) \le \log(2^n)$

How to improve?

- Let $\mathbb{E}X = 0$, $\mathrm{Cov}(X) = Id$ (X is isotropic)
- Optimal $h =$ density of $X + Y$ which is "almost" Gaussian
- $h(x) = e^{-\lambda |x|^2/4}$, $\lambda > 0$
- $\mathbb{E}\big[-\log h(X+Y)\big] = \frac{\lambda}{4}\mathbb{E}|X+Y|^2 = \frac{\lambda n}{2}$
- $\log\left(\frac{1}{|K|}\int_{2K} h\right) = \log\left(\frac{2^n}{|K|}\int_K h(2x)\right) = \log(2^n) + \log \mathbb{E}e^{-\lambda|X|^2}$
- $\log \delta_0(K) \le \log(2^n) + \frac{\lambda n}{2} + \log \mathbb{E}e^{-\lambda |X|^2}$

$\log \delta_0(K) \leq \log(2^n) + \frac{\lambda n}{2} + \log \mathbb{E} e^{-\lambda |X|^2}$

LM 3. $\frac{\lambda n}{2} + \log \mathbb{E} e^{-\lambda |X|^2} \leq \log(Ce^{-c\sqrt{n}})$ for $\lambda = \frac{c'}{\sqrt{n}}$

Proof. Thin-shell (Guédon and E. Milman):

$$\mathbb{P}\left(||X| - \sqrt{n}| > t\sqrt{n}\right) \leq Ce^{-ct^3 \cdot \sqrt{n}}, \quad t \in (0,1)$$

so

$$\begin{aligned} \mathbb{E} e^{-\lambda |X|^2} &= \mathbb{E} e^{-\lambda |X|^2} \mathbf{1}_{\{|X| < (1-t)\sqrt{n}\}} + \mathbb{E} e^{-\lambda |X|^2} \mathbf{1}_{\{|X| \geq (1-t)\sqrt{n}\}} \\ &\leq \mathbb{P}\left(|X| - \sqrt{n} < -t\sqrt{n}\right) + e^{-\lambda (1-t)^2 n} \\ &\leq C \underbrace{e^{-ct^3 \cdot \sqrt{n}}}_{A} + \underbrace{e^{-\lambda (1-t)^2 n}}_{B} \end{aligned}$$

Choose λ such that $A = B$, i.e. $\lambda = \frac{ct^3}{(1-t)^2} \frac{1}{\sqrt{n}}$.

Choose $t = \frac{1}{2}$ such that $\frac{\lambda n}{2} = \frac{ct^3}{2(1-t)^2} \sqrt{n}$ is absorbed.

FINAL REMARKS

- If we knew $\exists\lambda_0 > 0 \;\; \frac{\lambda_0 n}{2} + \log \mathbb{E} e^{-\lambda_0 |X|^2} \leq \log(Ce^{-cn})$, then $\sup_K L_K = O((\log n)^2)$
- If K is ψ_α with constant b_α, then $\delta_0(K) \leq C \cdot 2^n e^{-cb_\alpha^{-\alpha} n^{\alpha/2}}$
- Concentration due to Arias-De-Reyna, Ball, Villa and $h(x) = e^{-\lambda \|x\|_K}$ give exponential improvements for K with positive modulus of convexity

$\max\{\mathcal{S}(X+Y) - \mathcal{S}(X) : \; X, Y \text{ i.i.d. log-concave r.v.s}\} = ?$

THANK YOU