(Non-)optimal point sets for numerical integration

based on joint work with D. Krieg and F. Pillichshammer

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The problem setting

$D \ldots$ some bounded domain in $\mathbb{R}^d$, a manifold, etc.

$\mu \ldots$ (absolutely continuous to) the uniform measure on D

$\mathcal{F}(D) \ldots$ a set of bounded functions on D

Given $P = \{x_1, \ldots, x_n\} \subset D$ approximate $\int_D f(x) \mathrm{d}\mu(x)$ for every $f \in \mathcal{F}(D)$ by a linear algorithm using $f(x_1), \ldots, f(x_n)$.

Worst-case error of algorithm with weights $w = (w_1, \ldots, w_n)$:

$$e\big(P, w, \mathcal{F}(D)\big) := \sup_{f \in \mathcal{F}(D)} \left| \sum_{i=1}^{n} w_i f(x_i) - \int_D f(x) \mathrm{d}\mu(x) \right|.$$

Quality of P can be e.g. $e\big(P, \mathbf{1/n}, \mathcal{F}(D)\big)$ or

$$e\big(P, \mathcal{F}(D)\big) := \inf_{w \in \mathbb{R}^n} e\big(P, w, \mathcal{F}(D)\big).$$

Example: Discrepancy on $\mathbb{S}^d$

$D \ldots \mathbb{S}^d$, unit sphere of $\mathbb{R}^{d+1}$

$\mu \ldots \sigma$, normalized surface measure on $\mathbb{S}^d$

$\mathcal{F}(D) \ldots \ \{\mathbf{1}_C : C \in \mathcal{C}_d\}$, where $\mathbf{1}_C$ are indicators of caps

$\mathcal{C}_d = \big\{C(z,t) = \{x \in \mathbb{S}^d : x \cdot z \geq t\} : z \in \mathbb{S}^d, t \in [-1,1]\big\}$

Worst-case error of a linear algorithm is weighted discrepancy:

$$e\big(P, w, \mathcal{F}(D)\big) = \sup_{C \in \mathcal{C}_d} \left| \sum_{i=1}^{n} w_i \mathbf{1}_C(x_i) - \int_{\mathbb{S}^d} \mathbf{1}_C(x) \mathrm{d}\sigma(x) \right|$$

Spherical cap discrepancy is $D(P, \mathcal{C}_d) := e\big(P, \mathbf{1}/\mathbf{n}, \mathcal{F}(D)\big)$.

Questions and Motivation

- How good is optimal? That is find $g(n)$ such that $\inf_{\#P=n} D(P, \mathcal{C}_d) \asymp g(n)$ or $\inf_{\#P=n} e\big(P, \mathcal{F}(D)\big) \asymp g(n)$.
- How to find (near-)optimal (weighted) points?
- How to find optimal weights?

Describe the quality of rather general P in terms of "easy to analyze" geometric quantities.

A sum of distances

$\mathcal{F}(D) \ldots B\big(W_2^{(d+1)/2}(\mathbb{S}^d)\big)$, the unit ball of $W_2^{(d+1)/2}(\mathbb{S}^d)$ (Sobolev space of smoothness $(d+1)/2$)

J. S. Brauchart, J. Dick, 2013

For every finite $P \subset \mathbb{S}^d$ we have

$$\sup_{f \in B(W_2^{(d+1)/2}(\mathbb{S}^d)))} \left| \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \int_{\mathbb{S}^d} f(x) \mathrm{d}\sigma(x) \right|$$

$$=$$

$$c_d \left(\int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x-y\|_2 \mathrm{d}\sigma(x) \mathrm{d}\sigma(y) - \frac{1}{n^2} \sum_{i,j=1}^{n} \|x_i - x_j\|_2 \right)^{1/2}$$

Integration of Sobolev functions

$D \ldots$ a bounded convex domain in $\mathbb{R}^d$ and $\mu \ldots$ Lebesgue measure

$\mathcal{F}(D) \ldots B\big(W_p^s(D)\big)$, the unit ball of $W_p^s(D)$

If $P = \{x_1, \ldots, x_n\} \subset D$, the distance function returns for $x \in \mathbb{R}^d$

$$\operatorname{dist}(x, P) = \min_{1 \leq i \leq n} \|x - x_i\|_2.$$

Theorem (D. Krieg, MS, 2020)

Let $1 \leq p \leq \infty$, $1/p + 1/p^* = 1$ and $s > d/p$. For any finite $P \subset D$

$$e\big(P, B\big(W_p^s(D)\big)\big) \asymp \begin{cases} \|\operatorname{dist}(\cdot, P)\|_{L_\infty(D)}^s & \textit{if } p = 1, \\ \|\operatorname{dist}(\cdot, P)\|_{L_{sp^*}(D)}^s & \textit{if } p > 1. \end{cases}$$

The quantization problem

Let X be a random vector in $\mathbb{R}^d$ with distribution μ and $r \geq 1$.

Minimize among $f : \mathbb{R}^d \to \mathbb{R}^d$ measurable with $\# f(\mathbb{R}^d) \leq n$

$$\mathbb{E} \| X - f(X) \|_2^r$$

$$\Updownarrow$$

Minimize among point sets with $\# P \leq n$

$$\int_{\mathbb{R}^d} \operatorname{dist}(x, P)^r \mathrm{d}\mu(x) = \| \operatorname{dist}(\cdot, P) \|_{L_r(\mu)}^r .$$

Theorem 2: "sp^*-Quantization error of $P \asymp e\big(P, B\big(W_p^s(D)\big)\big)$ "

Application: random points

Let $X_1, X_2, \dots$ be i.i.d. uniform on D and $P_n = \{X_1, \dots, X_n\}$.

$$\mathbb{E}\,\|\mathrm{dist}(\cdot, P_n)\|_{L_\gamma(D)} \asymp \begin{cases} n^{-1/d} & \text{if } 0 < \gamma < \infty, \\ n^{-1/d}(\log n)^{1/d} & \text{if } \gamma = \infty. \end{cases}$$

Corollary 1

Random points are asymptotically optimal for weighted numerical integration in $W_p^s(D)$ iff $p > 1$.

Open questions

Say $D = [0,1]$ and $0 < x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$. Then we have a discretization of the form

$$\|\mathrm{dist}(\cdot, P)\|_{L_\gamma([0,1])}^{\gamma} \asymp \sum_{i=0}^{n} |x_{i+1} - x_i|^{\gamma+1}.$$

- Discrete version in higher dimensions?
- Characterization for $e\big(P, \mathbf{1}/\mathbf{n}, B\big(W_p^s(D)\big)\big)$?
- Relation of $\|\mathrm{dist}(\cdot, P)\|_{L_\gamma(D)}$ to (weighted) discrepancy?