Proof complexity of systems of (non-deterministic) decision trees and branching programs

Authors:
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Institute:
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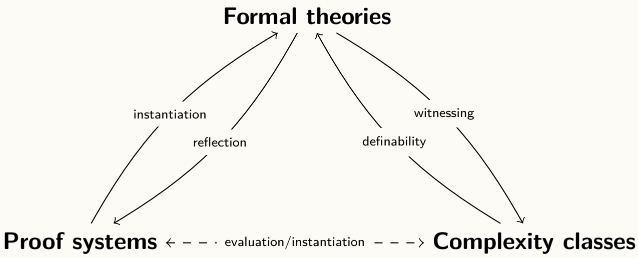
The Bounded Arithmetic Correspondence



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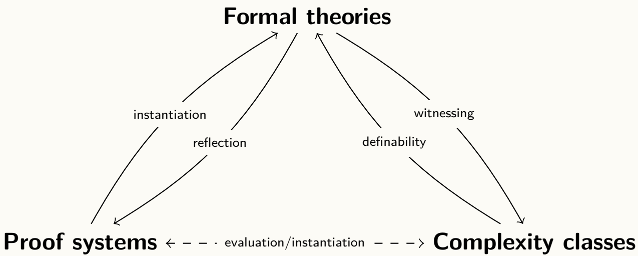

A proof theoretic version of other "uniform-nonuniform" correspondences.

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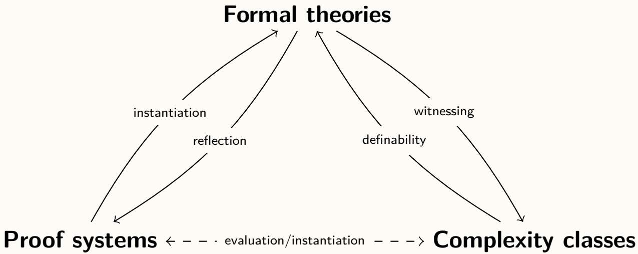

A proof theoretic version of other "uniform-nonuniform" correspondences. E.g., the bottom dashed arrow, for **P**, is the correspondence between **P** machines and polynomial-size circuits.

Theories and Proof Systems

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Let $\phi(x)$ be a Σ^b_0. Then we can write, in a natural way, a propositional formula $[\![\phi]\!]_{n,\alpha}$ on the variables x_1, ..., x_n saying that A is true (α is an assignment to all other free variables).

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THEOREM

If $\mathsf{S}^1_2 \vdash \forall x\ \phi(x)$, *then* $[\![\phi]\!]_{n,\alpha}$ *has a polynomial size proof in extended Frege.*

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Let us illustrate this relation on the triplet extended Frege, S_2^1, and class **P**. So it is possible to prove the following theorem.

THEOREM

If $\mathsf{S}_2^1 \vdash \forall x\, \phi(x)$*, then* $[\![\phi]\!]_{n,\alpha}$ *has a polynomial size proof in extended Frege. Moreover,* S_2^1 *proves the reflection principle for extended Frege.*

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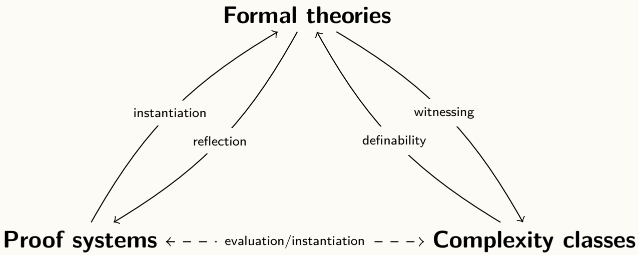

Theories and Complexity Classes

DEFINITION

A function $f : \mathbb{N} \to \mathbb{N}$ is Σ_1^b-definable by a theory R iff there is a Σ_1^b formula $A(x, y)$ such that

- $R \vdash \forall x \, \exists y \leq t \; A(x, y)$ for some term t,
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THEOREM

S_2^1 can Σ_1^b-define any polynomial time function. Moreover, if f is Σ_1^b-definable by S_2^1, then f is polynomial time computable.

The Bounded Arithmetic Correspondence

Formal Theories	Propositional Proof Systems	Complexity Class	References
PV, S^1_2	$e\mathcal{F}$	$\mathbf{P}$	[Coo75, Bus86]
PSA, U^1_2	G	**PSPACE**	[Dow78, Bus86]
T^i_2, S^{i+1}_2	G_i, G^*_{i+1}	$\mathbf{P}^{\Sigma^p_i}$	[KP90, KT92, Bus86]
VNC^0	$\mathcal{F}$	**ALogTime**	[CM05, CN10, Ara00]
VL	GL^*	**L**	[Per05, CN10]
VNL	GNL^*	**NL**	[Per09, CN10]

Previous Works

Proof systems corresponding to **L** and **NL** have been considered in the past:

- Perron gives systems based on logical characterisations of **L** and **NL**, namely CNF(2) and ΣKrom respectively. [Per05, Per09]
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- Inspired by Cook's approach, we build a *bona fide* inference system based on **branching programs**.
- In particular, we treat decision trees, the tree-like branching programs, and recover dag-like ones by extension.

Branching Programs

A *branching program* (BP) is a dag where:

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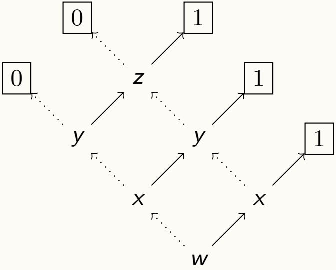

Key

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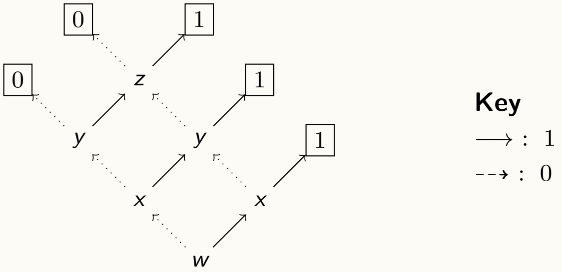

Can also consider *nondeterministic branching programs* (NBPs) and tree-like ones, *decision trees* (DTs) or both (NDTs).

A proof system for tree-like programs

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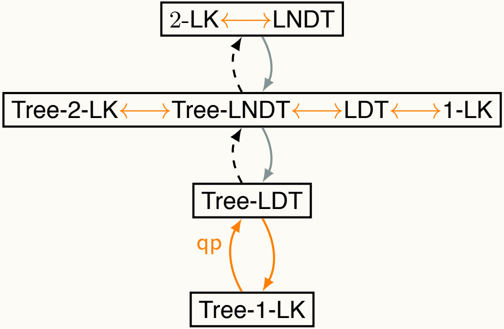
The system LNDT extends LDT by standard rules for $\vee$:

$$\vee\text{-}l\ \frac{\Gamma, A \to \Delta \quad \Gamma, B \to \Delta}{\Gamma, A \vee B \to \Delta} \qquad \vee\text{-}r\ \frac{\Gamma \to A, B, \Delta}{\Gamma \to A \vee B, \Delta}$$

L(N)DT Proofs



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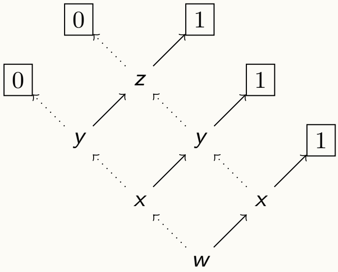
A proof of eLDT or eLNDT is just like that of LDT or LNDT, but comes equipped with a set of axioms of the form $e_n \leftrightarrow A_n(e_i)_{i<n}$. The conclusion of such a proof must not contain extension variables.

Example



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e_{00} &\leftrightarrow e_{10}\, w\, e_{11} \\
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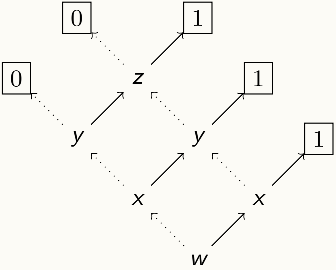

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- Here e_{ij} names the jth node, left-right, of the ith row, bottom-up.
- The entire program is now expressed by e_{00}.

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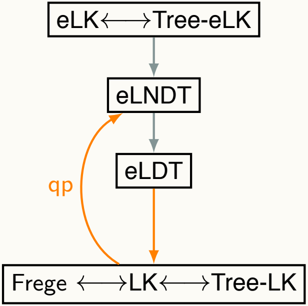
NB: these proofs are crucially dag-like!

Results



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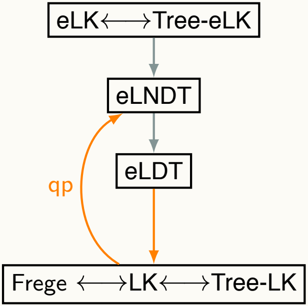

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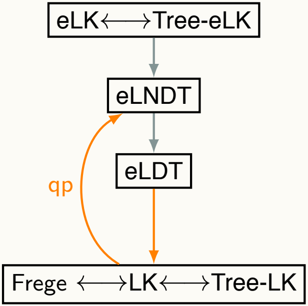

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- Results follow by direct simulations, under equivalence of isomorphic (N)BPs.
- We rely on Buss' qp-size formulas for st-connectivity and their small proofs in LK to evaluate NBPs and prove truth conditions. [Bus15].

VL and VNL

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\begin{aligned}
&(\forall x{\leq}a)(\exists y{\leq}a)A(x, y) \to \\
&\qquad\qquad (\exists X{\preccurlyeq}\langle b, a\rangle)[X(0, 0) \wedge \\
&\qquad\qquad\qquad (\forall z{\leq}b)(\forall y{\leq}a)(X(z, y) \to (\forall y'{<}y)\neg X(z, y')) \wedge \\
&\qquad\qquad\qquad (\forall z{<}b)(\exists y{\leq}a)(\exists y'{\leq}a)(X(z, y) \wedge X(z{+}1, y') \wedge A(y, y'))]
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VNL has an axiom saying that there is a function that gives distance from any fixed vertex.

$$\exists X \leq \langle a, a\rangle (\forall i \leq a\; X(0, i) \leftrightarrow (i = 0)) \wedge$$
$$\big(\forall w, x \leq a\; (X(x, w+1) \leftrightarrow [\exists y \leq a\; X(y, w) \wedge \phi(y, x)])\big)$$

Cook-style Translation

Let ϕ be a Σ_0^b formula with one free second-order variable X.

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THEOREM

- *If **VL** $\vdash \exists X\, \phi(X)$, then there is a eLDT proof of $[\![\phi]\!]_{n,\alpha}$.*
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The problem of this approach is that the sequents my have Σ_1^b formulas since the axiomatizations of VL and VNL have Σ_1^b axioms.

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In case of VL this actually works, but in case of VNL there is a problem...
Negation of the reachability has no clean representation as a eLNDT. To avoid this, we need to prove some analogue of Immerman–Szelepcsényi's theorem.

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Thank you!

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