An Introduction to Homomesy through Promotion and Rowmotion on Order Ideals

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This talk is being recorded.

Main Topics

1. Posets and toggles
2. Homomesy on order ideals of $[a] \times [b]$
3. Homomesy on order ideals of $[2] \times [b] \times [c]$
4. Refined homomesy
5. Homomesy for actions with infinite orbits

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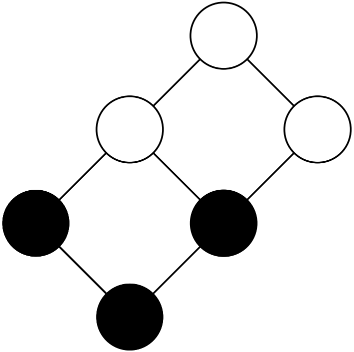
What is a toggle?

Define a toggle t_e for each e in P.



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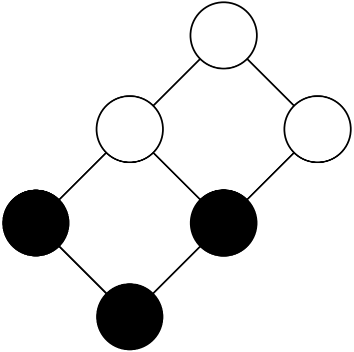

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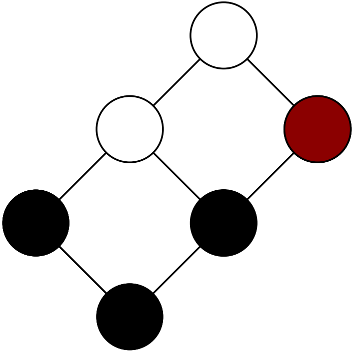

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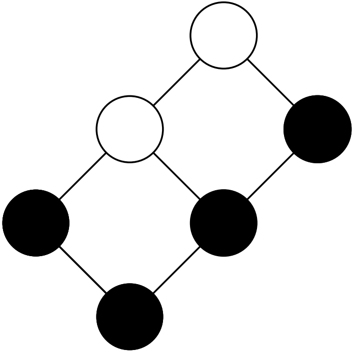

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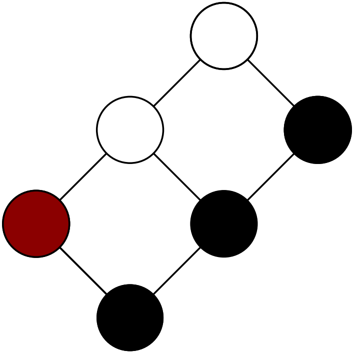

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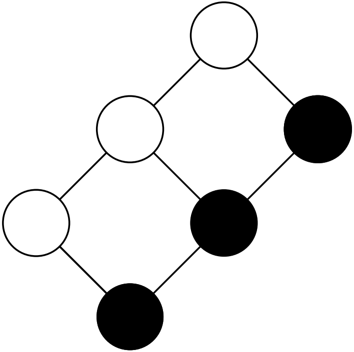

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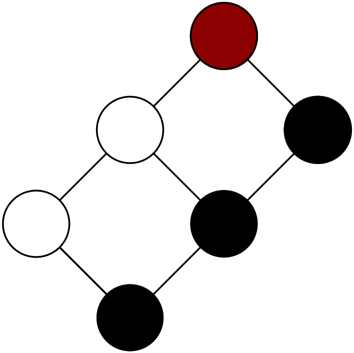

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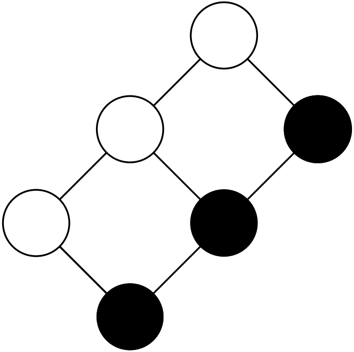


If an element cannot be toggled in (or out) of an order ideal, nothing happens.

The rowmotion action

We can define an action *rowmotion* in two ways.

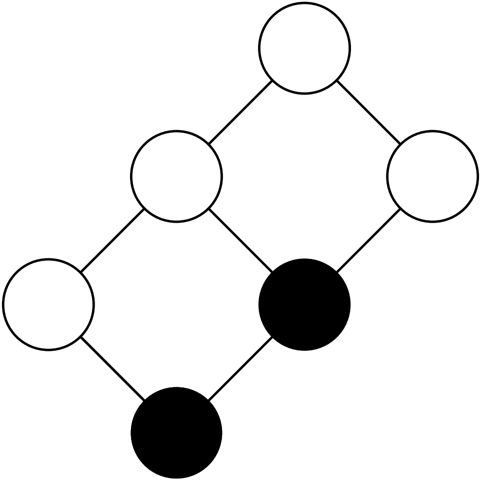
Definition

Let P be a poset and I an order ideal of P. $\mathrm{Row}(I)$ is the order ideal generated by the minimal elements of P not in I.

Theorem (Cameron and Fon-der-Flaass, 1995)

Rowmotion can be performed on a finite poset by toggling from top to bottom.

Rowmotion example



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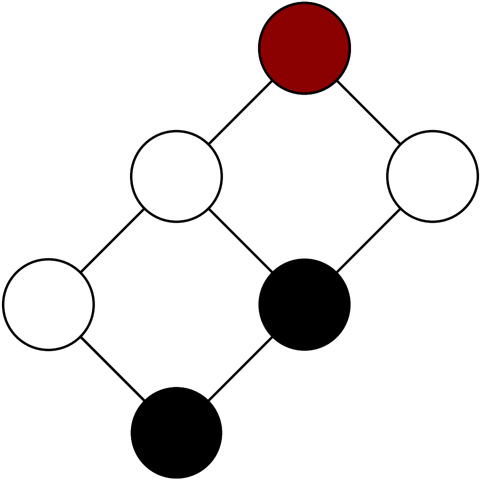

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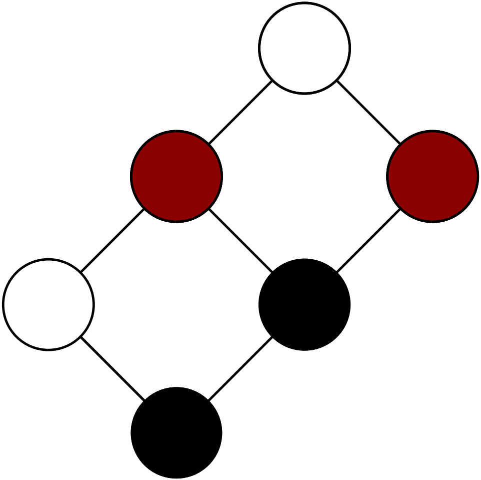

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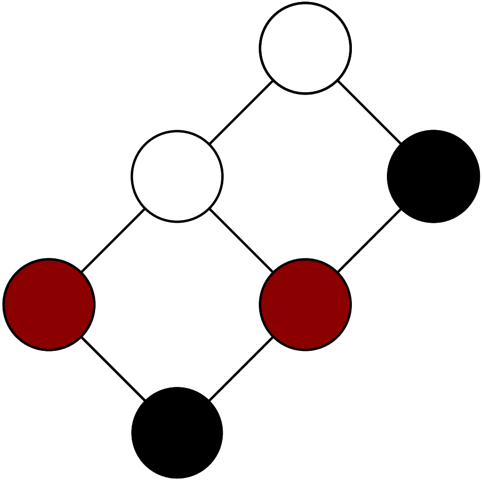

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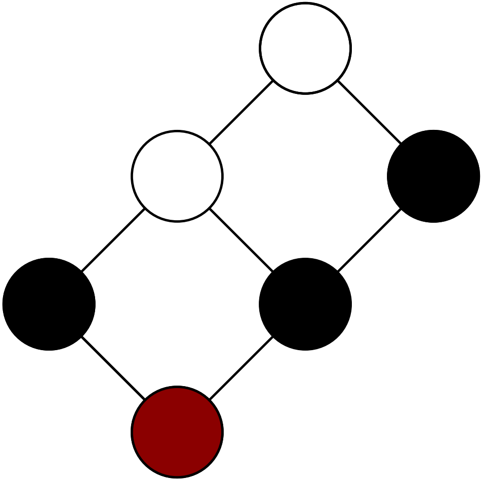

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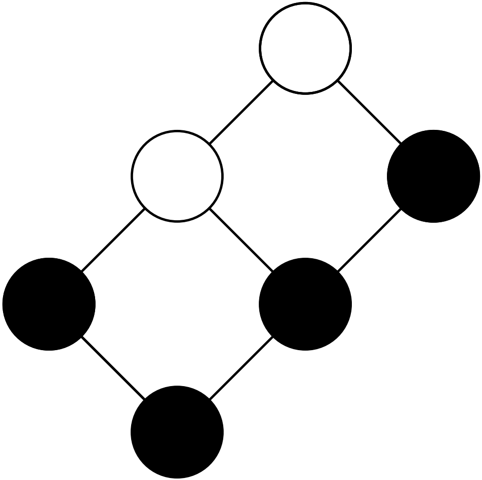
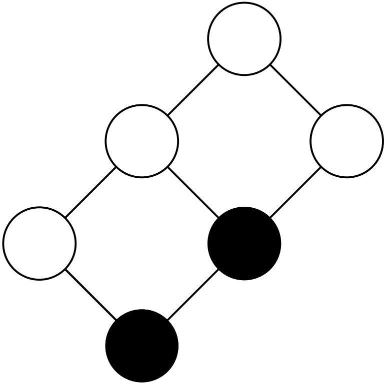


Promotion

- Rowmotion toggles our poset from top to bottom.
- We can define, analogously, *promotion* which toggles our poset from left to right.



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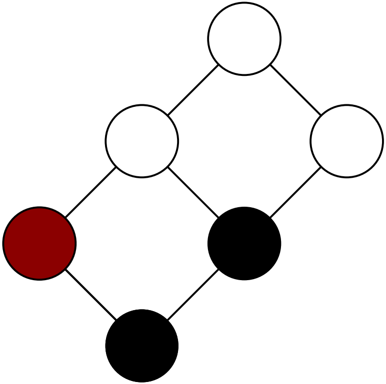

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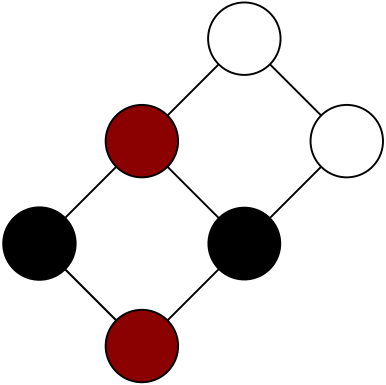

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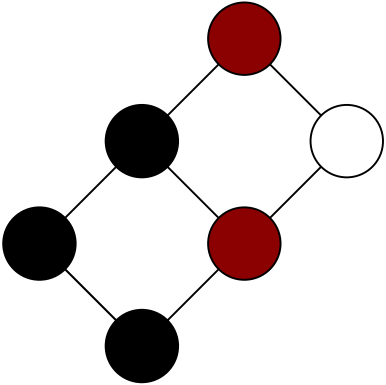

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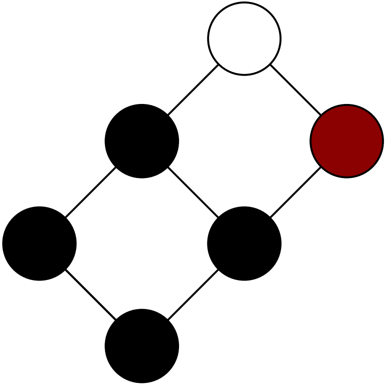

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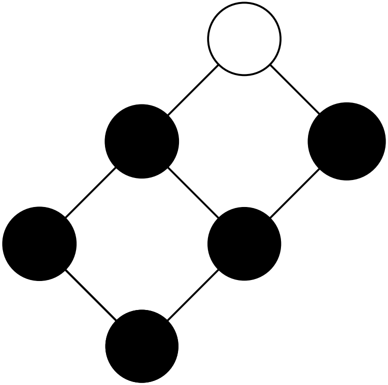
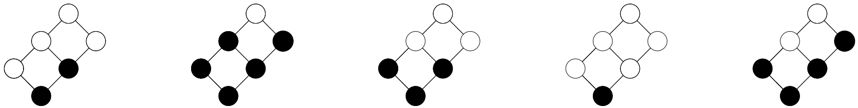


If we continue to apply promotion, we eventually return to the order ideal at which we started, giving us an orbit of order ideals under the action.



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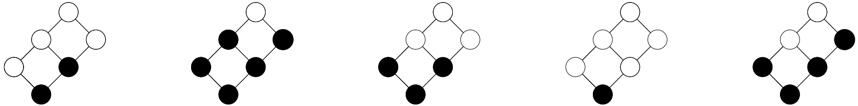
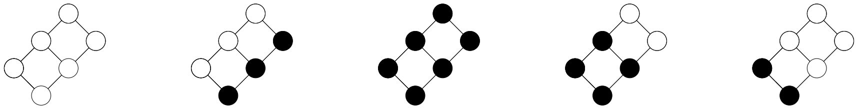
Observe: the average cardinality of our example orbit under promotion is 3.



$$\frac{2+5+3+1+4}{5} = 3$$

If we check another orbit, the average cardinality is also 3.



$$\frac{0+3+6+4+2}{5} = 3$$

Notice that for the poset $[3] \times [2]$, the average cardinality of an order ideal over **all** order ideals is 3.

Homomesy in the two-dimensional product of chains

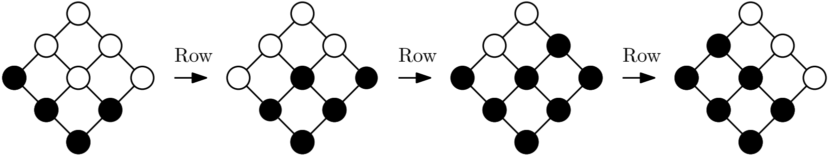
If **every** orbit average of a statistic is the same as the global average of that statistic, we say we have homomesy.

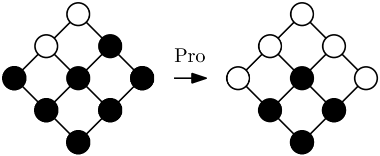
Theorem (Propp and Roby, 2015)

Order ideals of $[a] \times [b]$ under promotion with cardinality statistic exhibit homomesy with average value $ab/2$.

What about rowmotion?

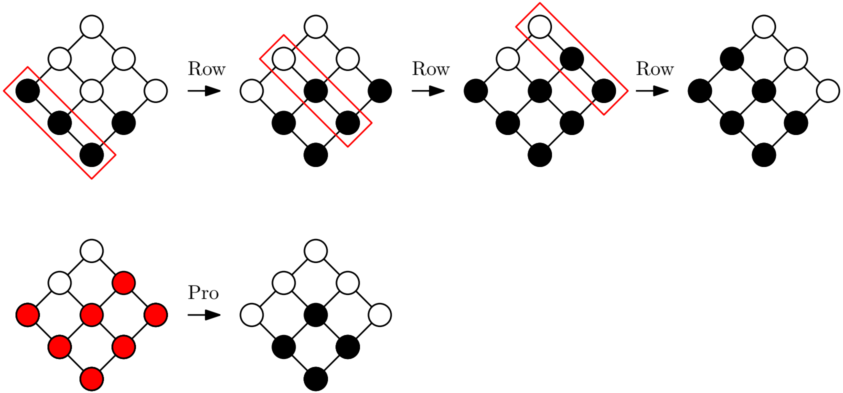
Recombination





These are two partial orbits, the top is under rowmotion, the bottom is under promotion.

Recombination



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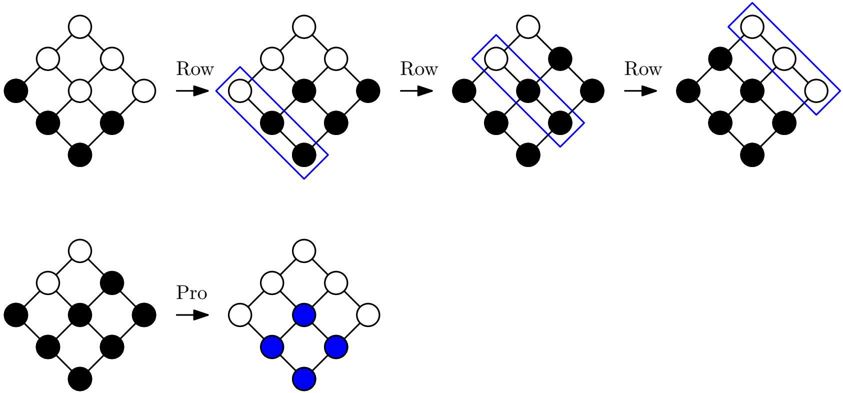

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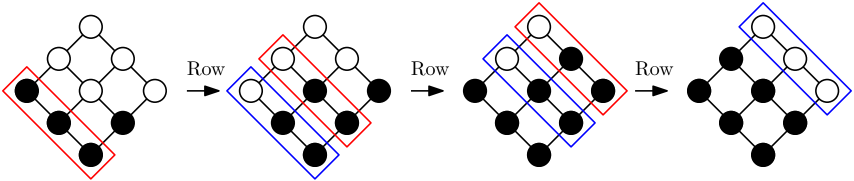

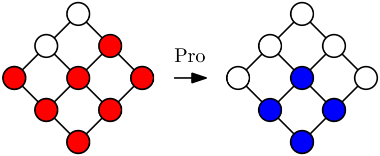

Recombination

The previous proof technique is called *recombination*.

Theorem (Einstein and Propp, 2014)

Recombination gives a bijection between order ideals of a product of chains poset under rowmotion and promotion.

Because recombination preserves cardinality, this gives a slick proof for the following result.

Theorem (Propp and Roby, 2015)

Order ideals of $[a] \times [b]$ under rowmotion with cardinality statistic exhibit homomesy with average value $ab/2$.

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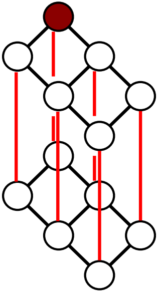
Promotion on a higher dimensional product of chains

Definition (Dilks, Pechenik, Striker, 2017)

Let $P = [a_1] \times \cdots \times [a_n]$ and let $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$. Instead of toggling from left to right, we sweep through P with a hyperplane in a direction given by v. We call this Pro_v.

Example: Toggle order of $\mathrm{Pro}_{(1,1,1)}$



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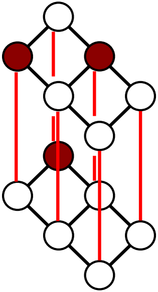

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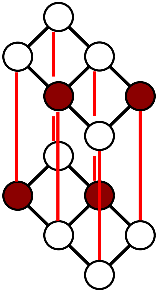

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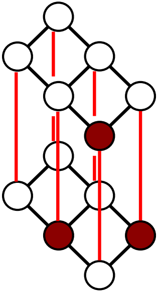

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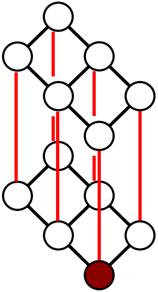

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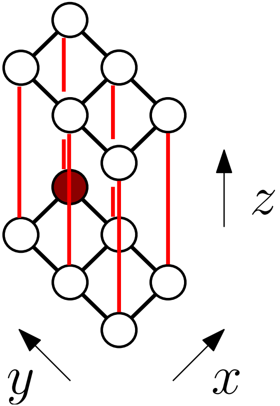
Observe: $\text{Pro}_{(1,1,1)}$ is Row.

Another Example: Toggle order of $\mathrm{Pro}_{(1,1,-1)}$

Toggling elements on the hyperplane $x + y - z = 4$



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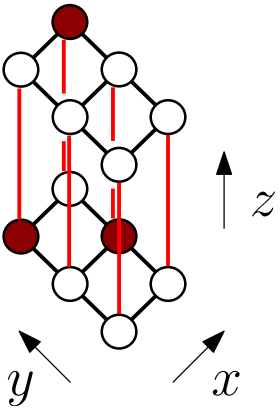
Toggling elements on the hyperplane $x + y - z = 3$



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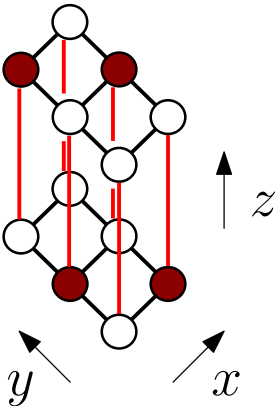
Toggling elements on the hyperplane $x + y - z = 2$



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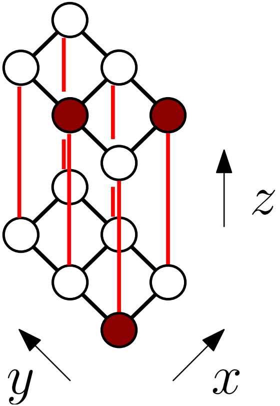
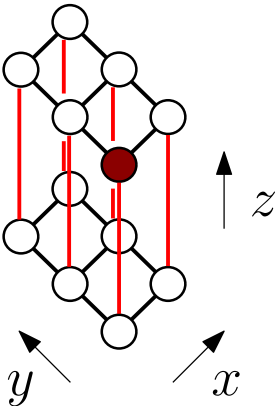
Toggling elements on the hyperplane $x + y - z = 1$



Another Example: Toggle order of $\mathrm{Pro}_{(1,1,-1)}$

Toggling elements on the hyperplane $x + y - z = 0$

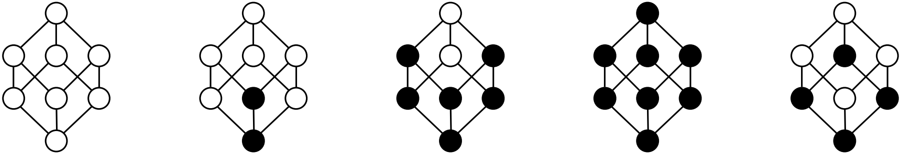


Homomesy in the product of three chains

We will start by focusing on one particular Pro_v.

Theorem (V., 2019)

Let $v = (1, 1, -1)$. Order ideals of $[2] \times [b] \times [c]$ under Pro_v with cardinality statistic exhibit homomesy with average value bc.



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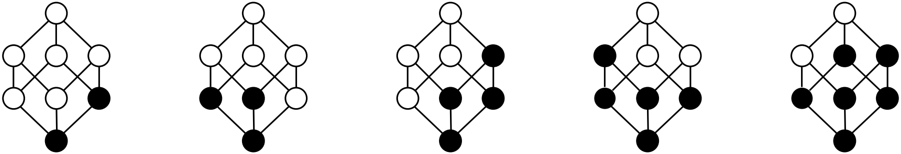

$$\frac{2+3+4+5+6}{5} = 4$$

Homomesy in the product of three chains

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Theorem (V., 2019)

Let $v = (1, 1, -1)$*. Order ideals of* $[2] \times [b] \times [c]$ *under* Pro_v *with cardinality statistic exhibit homomesy with average value* bc*.*

To prove this result on $v = (1, 1, -1)$, we use *increasing tableaux*.

Increasing tableaux

Definition

An increasing tableau is a filling of a partition shape with positive integers such that the rows and columns are strictly increasing.

Example:

1	2	4
2	4	5
6		

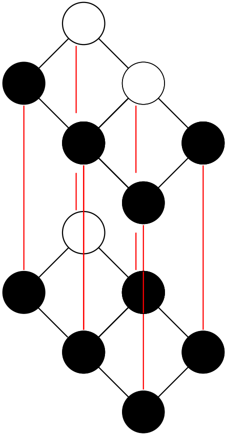
A useful bijection

Theorem (Dilks, Pechenik, Striker, 2017)

There exists a bijection between order ideals of $[a] \times [b] \times [c]$ *and increasing tableaux of shape* $a \times b$ *and entries at most* $a + b + c - 1$.

Corollary

There exists a bijection between order ideals of $[2] \times [b] \times [c]$ *and increasing tableaux of shape* $2 \times b$ *and entries at most* $b + c + 1$.



$\longleftrightarrow$

0	1	2
2	2	2

$\longleftrightarrow$

1	3	5
4	5	6

$\mathrm{Pro}_{(1,1,-1)}$ on order ideals of $[a] \times [b] \times [c]$ corresponds to an action K-promotion on increasing tableaux.

K-promotion

1	2	4	6
4	5	6	7

Switch 1's to 2's and 2's to 1's, if possible.

K-promotion

1	2	4	6
4	5	6	7

Switch 2's to 3's and 3's to 2's, if possible.

K-promotion

1	3	4	6
4	5	6	7

Switch 2's to 3's and 3's to 2's, if possible.

K-promotion

1	3	4	6
4	5	6	7

Switch 3's to 4's and 4's to 3's, if possible.

K-promotion

1	3	4	6
3	5	6	7

Switch 3's to 4's and 4's to 3's, if possible.

K-promotion

1	3	4	6
3	5	6	7

Switch 4's to 5's and 5's to 4's, if possible.

K-promotion

1	3	5	6
3	4	6	7

Switch 4's to 5's and 5's to 4's, if possible.

K-promotion

1	3	5	6
3	4	6	7

Switch 5's to 6's and 6's to 5's, if possible.

K-promotion

1	3	5	6
3	4	6	7

Switch 6's to 7's and 7's to 6's, if possible.

K-promotion

1	3	5	6
3	4	6	7

The result is K-Pro(T).

A K-Promotion result

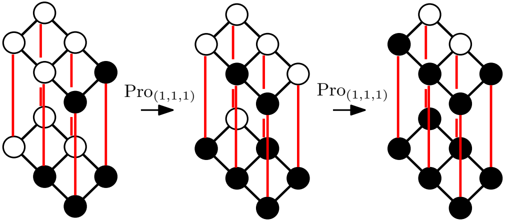
Theorem (Bloom, Pechenik, Saracino, 2016)

Increasing tableaux of shape $2 \times n$ and entries at most q under K-promotion with statistic the sum of the entries exhibits homomesy.

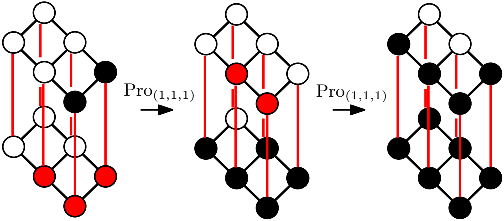
Theorem (V., 2019)

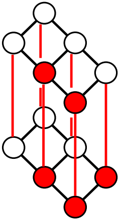
Let $v = (1, 1, -1)$. Order ideals of $[2] \times [b] \times [c]$ under Pro_v with cardinality statistic exhibit homomesy with average value bc.

Does recombination work in higher dimensions? We'll look at an example. Below is a partial orbit under $\mathrm{Pro}_{(1,1,1)}$.

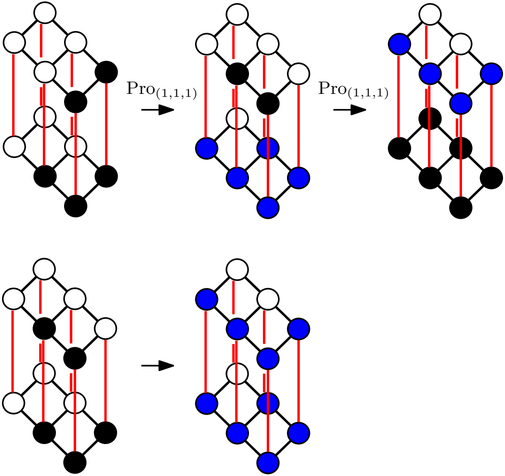


Recombination

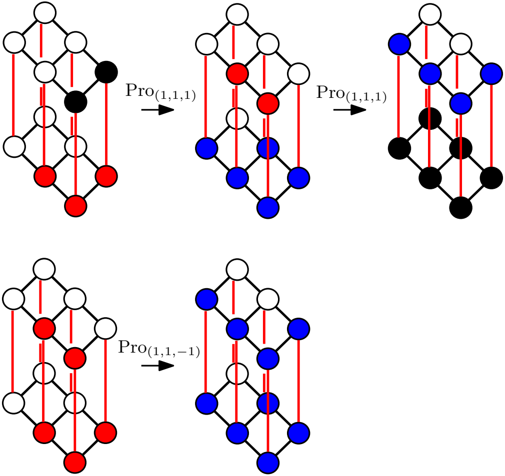




Recombination



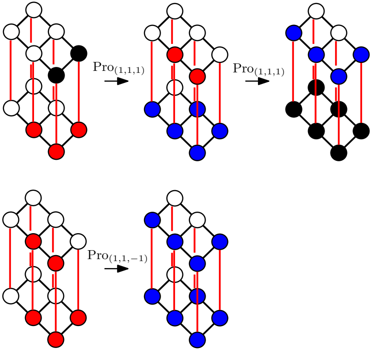
Recombination



General recombination result

Theorem (V., 2019)

Let v and u be n-dimensional vectors with entries ± 1 such that v and u differ in one component. Then we can perform recombination to get from Pro_v to Pro_u.



Homomesy in the product of three chains

Theorem (V., 2019)

Let $v = (1, 1, -1)$*. Order ideals of* $[2] \times [b] \times [c]$ *under* Pro_v *with cardinality statistic exhibit homomesy with average value* bc*.*

Using recombination, we obtain homomesy results for all v.

Theorem (V., 2019)

Order ideals of $[2] \times [b] \times [c]$ *under* Pro_v *with cardinality statistic exhibit homomesy with average value* bc*.*

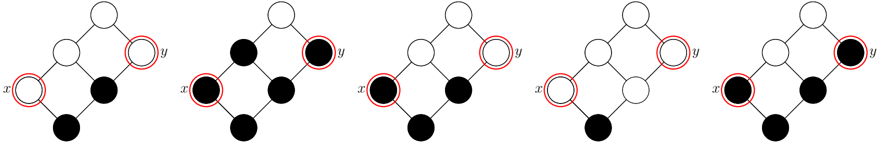
Homomesy nonexamples in the product of chains

- Order ideals of $[3] \times [3] \times [4]$ under Pro_v with cardinality statistic are not homomesic.
- Order ideals of $[2] \times [2] \times [2] \times [3]$ under Pro_v with cardinality statistic are not homomesic.
- Order ideals of $[2] \times [2] \times [2] \times [2] \times [2]$ under Pro_v with cardinality statistic are not homomesic.

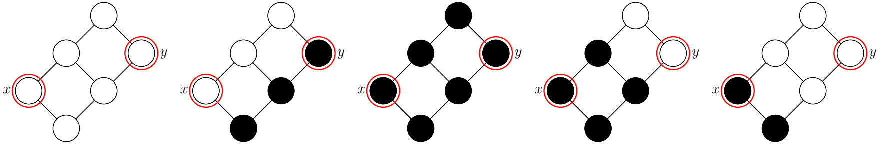
Main Topics

1. Posets and toggles
2. Homomesy on order ideals of $[a] \times [b]$
3. Homomesy on order ideals of $[2] \times [b] \times [c]$
4. Refined homomesy
5. Homomesy for actions with infinite orbits

Refined homomesy example on $[3] \times [2]$



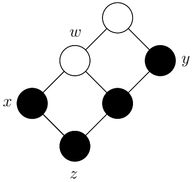
$$\frac{0+2+1+0+2}{5} = 1$$



$$\frac{0+1+2+1+1}{5} = 1$$

Refined homomesy on $[a] \times [b]$

- In a product of chains, x and y are *antipodal* if x can be obtained from y by rotating 180° about the center.
- The $x - y$ *file* contains all elements (x, y) with constant value $x - y$.



x and y are antipodal, w and z are in the same file.

Refined homomesy on $[a] \times [b]$

Theorem (Propp and Roby, 2015)

Let g denote the cardinality of two antipodal elements in $[a] \times [b]$. Order ideals of $[a] \times [b]$ under rowmotion (or promotion) with statistic g exhibit homomesy.

Theorem (Propp and Roby, 2015)

Let h denote the cardinality of elements in a file of $[a] \times [b]$. Order ideals of $[a] \times [b]$ under rowmotion (or promotion) with statistic h exhibit homomesy.

Antipodal refined homomesy on $[2] \times [b] \times [c]$

Theorem (V., 2019)

Let g denote the cardinality of two antipodal elements in $[2] \times [b] \times [c]$. Order ideals of $[2] \times [b] \times [c]$ under Pro_v with statistic g exhibit homomesy.

Tableaux result

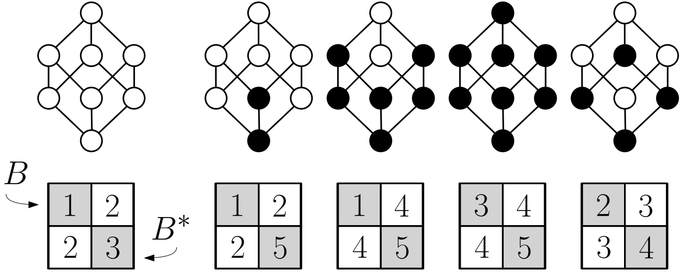
- Let $T \in \mathrm{Inc}^q(\lambda)$ with fixed box B. Let $\mathit{Dist}(B)$ denote the set of values box B attains in an orbit of K-Pro.
- Let $\mathit{arDist}(B)$ denote the alphabet reversal of $\mathit{Dist}(B)$, the set of values $q+1-b$ for every $b \in \mathit{Dist}(B)$.

Theorem (Pechenik)

Let $T \in \mathrm{Inc}^q(2 \times a)$, fix B and B^* such that B^* is the box 180° rotated from B. Then $\mathit{Dist}(B) = \mathit{arDist}(B^*)$.

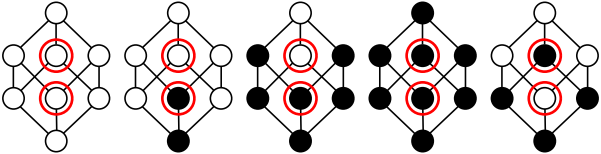
We will look at an example orbit of order ideals of $[2] \times [2] \times [2]$ under $\mathrm{Pro}_{(1,1,-1)}$ and the corresponding orbit of $\mathrm{Inc}^5(2 \times 2)$ under K-promotion.

Example



$Dist(B) = \{1, 1, 1, 3, 2\}$, $Dist(B^*) = \{3, 5, 5, 5, 4\}$,
$arDist(B^*) = \{3, 1, 1, 1, 2\}$

Example



1	2
2	3

1	2
2	5

1	4
4	5

3	4
4	5

2	3
3	4

$Dist(B) = \{1, 1, 1, 3, 2\}$, $Dist(B^*) = \{3, 5, 5, 5, 4\}$,
$arDist(B^*) = \{3, 1, 1, 1, 2\}$

Main Topics

1. Posets and toggles
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A more general homomesy definition

Rowmotion on a finite poset is a bijective action with finite orbits. With infinite posets, this is not necessarily the case. We need to modify the previous definition of homomesy.

Definition (Roby)

Given a set S, an action $\tau : S \to S$, and a statistic f, then (S, τ, f) exhibits homomesy if there exists c such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = c$$

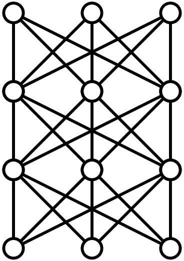
is independent of the starting point $x \in S$.

Ordinal sums of antichains

Definition

Let P_n denote the *n-element antichain*.

We consider ordinal sums of P_n. For example, the following is the poset $\bigoplus_{i=1}^{4} P_3 = P_3 \bigoplus P_3 \bigoplus P_3 \bigoplus P_3$.



Ordinal sums of antichains

We have a homomesy result for finite ordinal sums and for infinite ordinal sums.

Theorem (V.)

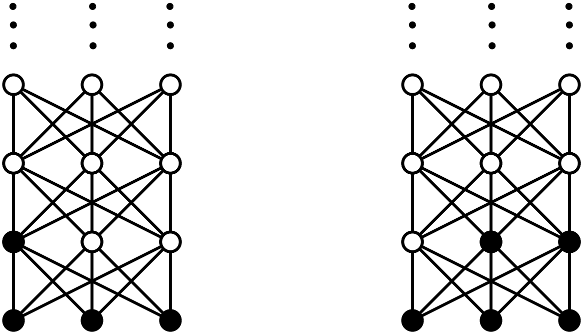
If k is odd, order ideals of $\bigoplus_{i=1}^{k} P_n$ under rowmotion with signed cardinality statistic are $n/2$-mesic.

Theorem (V.)

Order ideals of $\bigoplus_{i\in\mathbb{N}} P_n$ under rowmotion with signed cardinality statistic are $n/2$-mesic.

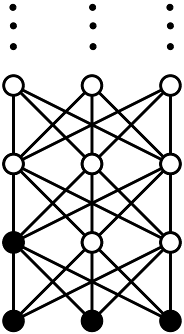
Ordinal sums of antichains example

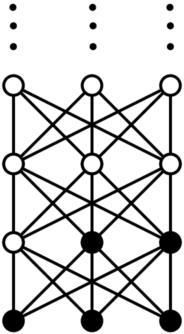
Consider $\bigoplus_{i\in\mathbb{N}} P_n$. If we start with an order ideal that is not generated by n elements of the same rank, we obtain an orbit of size two under rowmotion.



Ordinal sums of antichains example

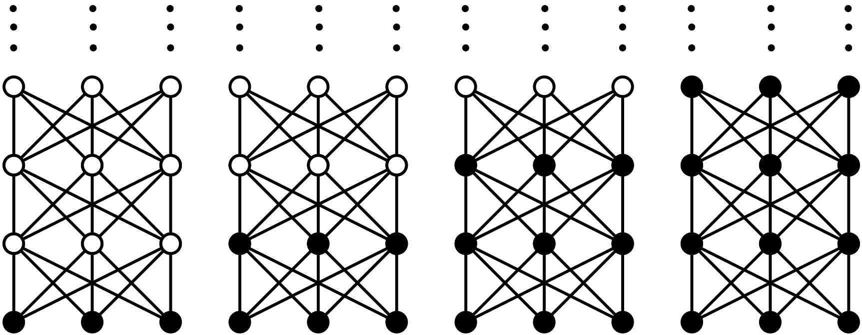
The order ideal on the left has signed cardinality 3-1=2, whereas the order ideal on the right has signed cardinality 3-2=1. Therefore, the average over the orbit is $3/2$.





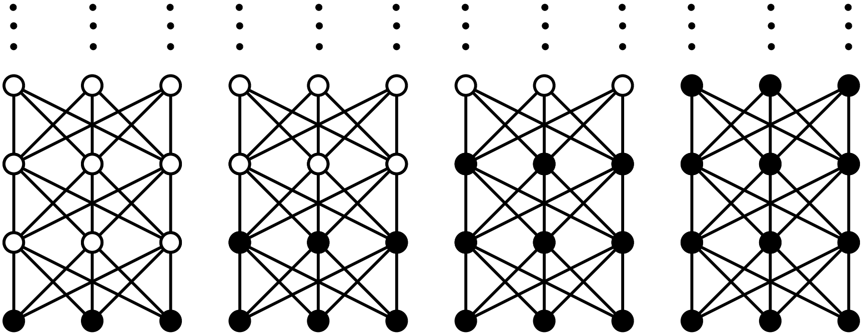
Ordinal sums of antichains example

If we start with an order ideal that is generated by n elements of rank k, applying rowmotion results in the order ideal generated by n elements of rank $k+1$.



Ordinal sums of antichains example

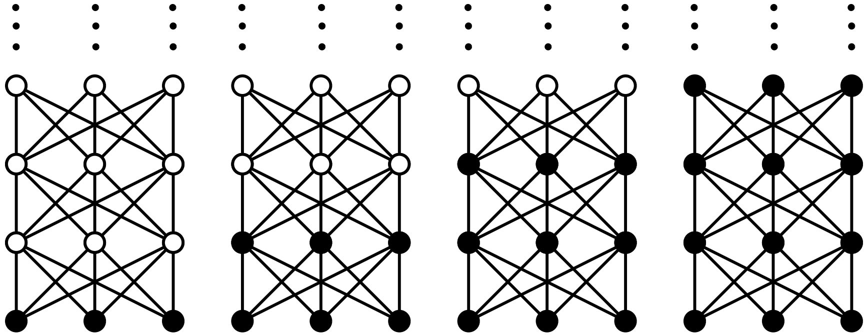
The signed cardinalities of the order ideals are 3, 0, 3, and 0 respectively.



Ordinal sums of antichains example

If f is the signed cardinality statistic and N is even,

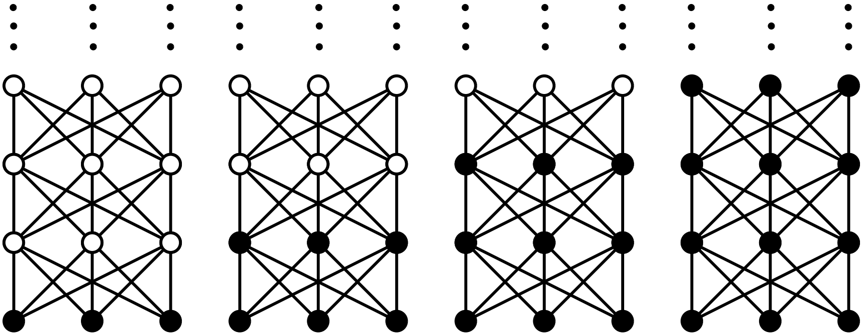
$$\frac{1}{N}\sum_{j=0}^{N-1} f(\mathrm{Row}^j(I)) = \frac{3N}{2N} = \frac{3}{2}$$



Ordinal sums of antichains example

If f is the signed cardinality statistic and N is odd,

$$\frac{1}{N}\sum_{j=0}^{N-1} f(\mathrm{Row}^j(I)) = \frac{3+0+3+\cdots+0+3}{N} = \frac{3(N+1)}{2N}$$



Thanks!

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