Ask not what algebra can do for biology – ask what biology can do for algebra

Workshop on

Model Theory of Differential Equations, Algebraic Geometry, and their Applications to Modeling

BIRS

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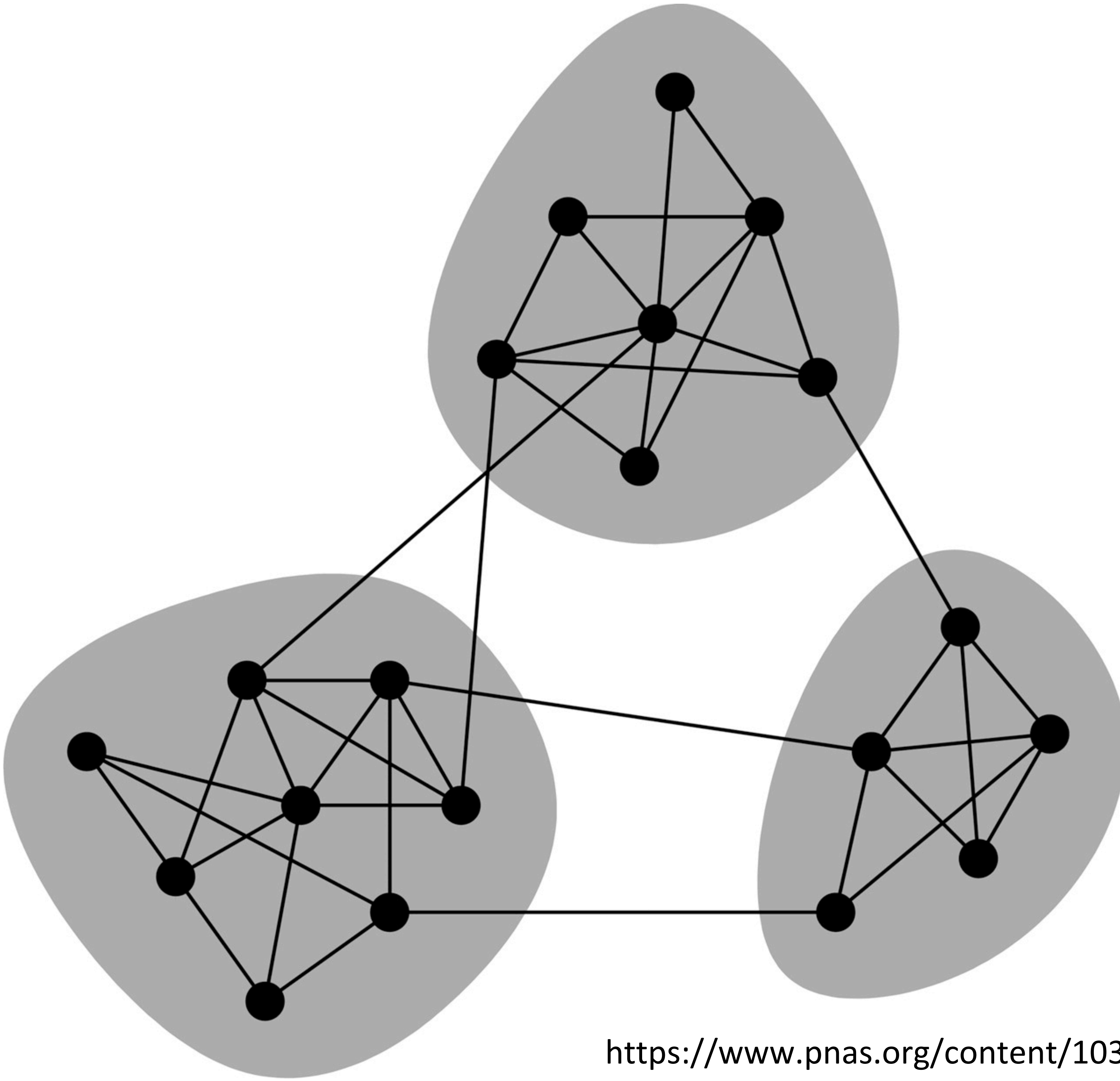
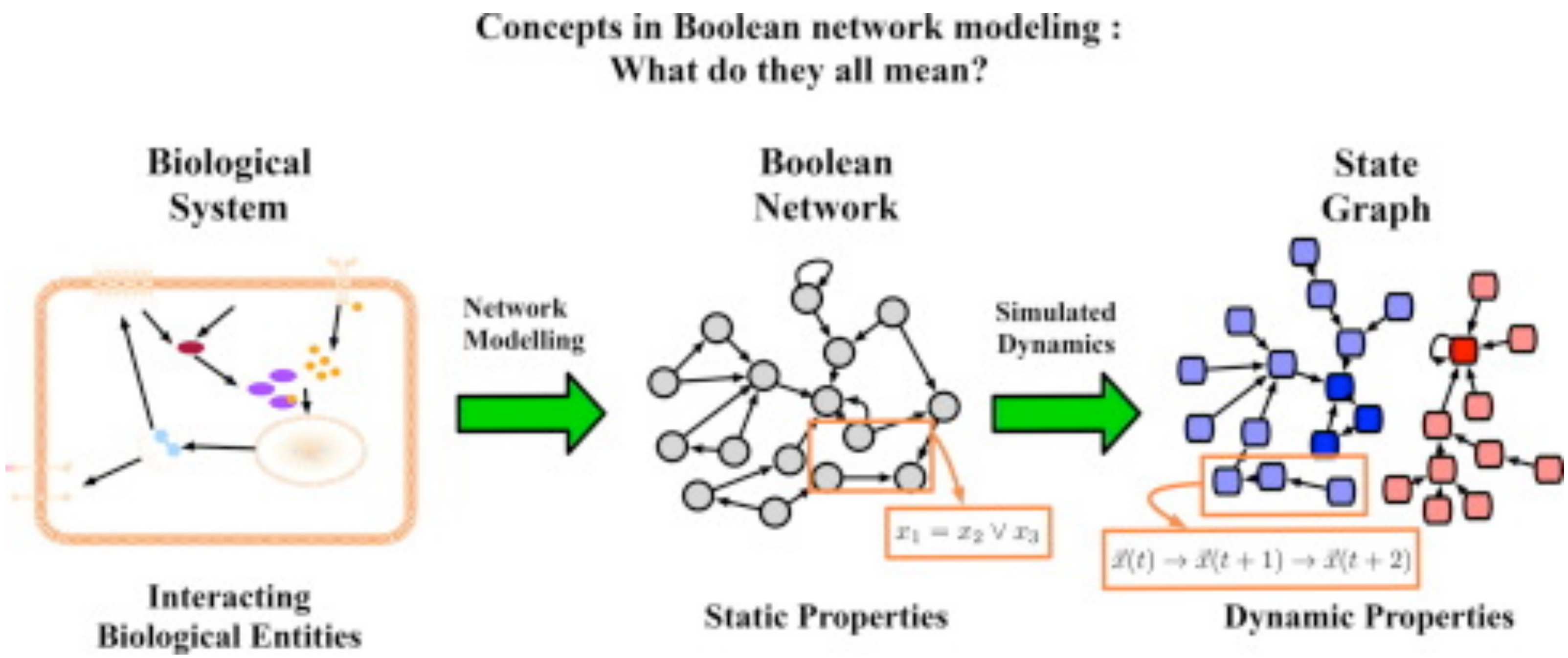
“Nature (likely) has Structure. And many models in natural sciences inherit a part of it. Therefore, understanding and exploiting the structure of a model at hand might be crucial to make the model useful. Algebra and logic offer a variety of tools to work with structures and greatly benefit from new types of structures and structural questions coming from other areas.”

‘Modularity is a widespread property in biological systems.’



https://www.pnas.org/content/103/23/8577



https://www.sciencedirect.com/science/article/pii/S200103701930460X

Definition

A *polynomial dynamical system (PDS)* over a finite field k is a function

$$f = (f_1, \ldots, f_n) : k^n \longrightarrow k^n,$$

with the coordinate functions $f_i : k^n \longrightarrow k$ in the polynomial ring $k[x_1, \ldots, x_n]$.

Iteration of f results in a time discrete dynamical system over the space k^n.

Note: Any function $k^n \longrightarrow k$ can be expressed as a polynomial.

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ORIGINAL ARTICLE

The Dynamics of Conjunctive and Disjunctive Boolean Network Models

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The number of periodic points

Theorem 3 *Let f be a conjunctive Boolean network whose dependency graph is strongly connected and has loop number c. If $c = 1$, then f has the two fixed points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ and no other limit cycles of any length. If $c > 1$ and m is a divisor of c, then the number of periodic states of period m is*

$$|A(m)| = \sum_{i_1=0}^{1} \cdots \sum_{i_r=0}^{1} (-1)^{i_1+i_2+\cdots+i_r} 2^{p_1^{k_1-i_1} p_2^{k_2-i_2} \cdots p_r^{k_r-i_r}},$$

where $m = \prod_{i=1}^{r} p_i^{k_i}$ is the prime factorization of m, that is $p_1, \ldots, p_r$ are distinct primes and $k_i \geq 1$ for all i.

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Brief paper

Dynamics of semilattice networks with strongly connected dependency graph☆

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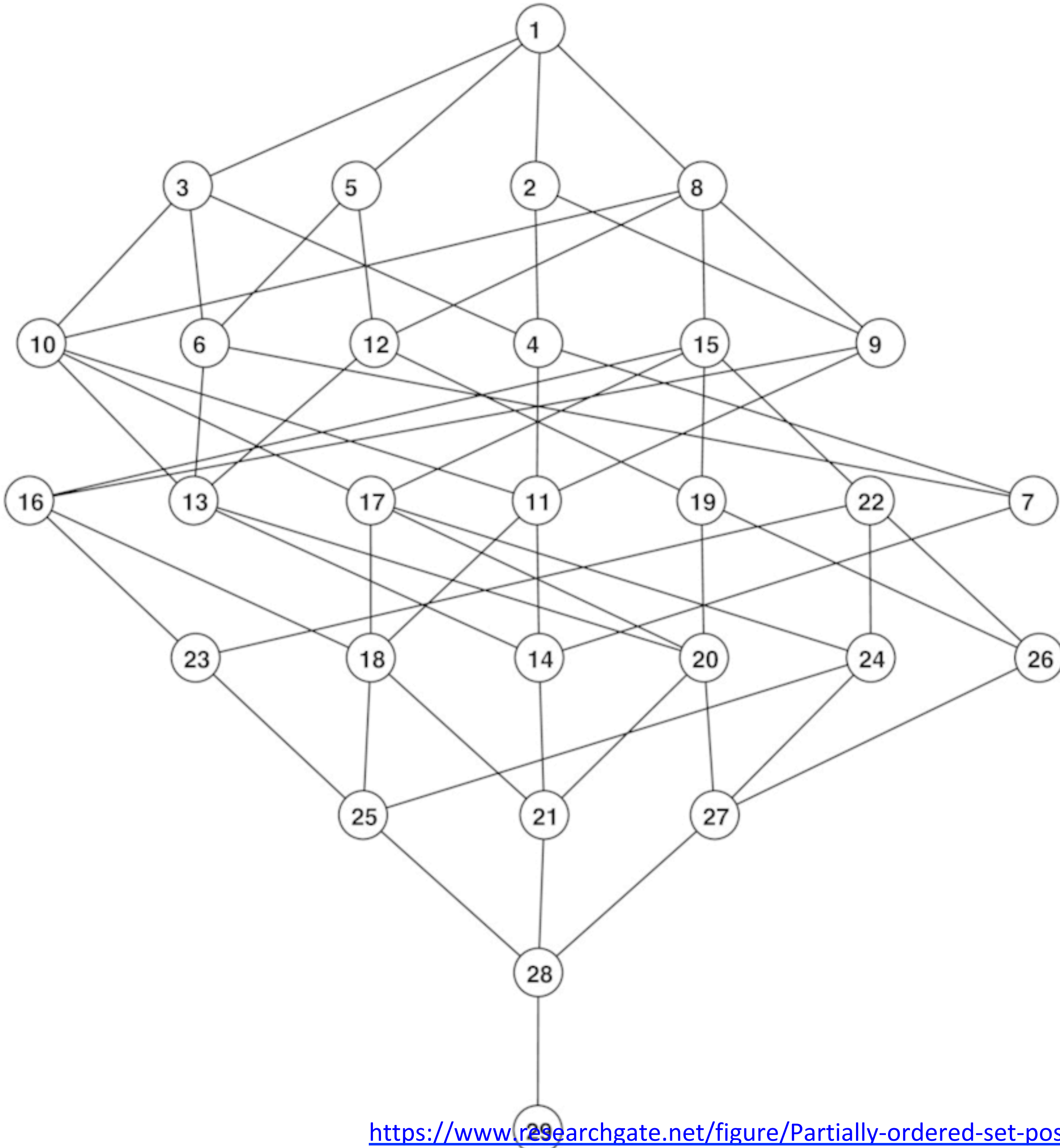
Theorem 6.2. *Consider the function*

$$\mathcal{L}(z_1,\ldots,z_t) := \sum_{\mathcal{J}\subseteq\Omega} (-1)^{|\mathcal{J}|+1} \prod_{j\in\bigcap_{J\in\mathcal{J}} J} z_i\,.$$

Then for any conjunctive Boolean network f with subnetworks $h_1,\ldots,h_t$ and Ω its set of maximal antichains in the poset of f, we have

$$\mathcal{L}\big(\mathcal{C}(h_1),\ldots,\mathcal{C}(h_t)\big) \leq \mathcal{C}(f). \tag{9}$$

Here, the function $\mathcal{L}$ is evaluated using the "multiplication" described in Corollary 3.5. This inequality provides a sharp lower bound on the number of limit cycles of f of a given length.



https://www.researchgate.net/figure/Partially-ordered-set-poset-model-for-cognitive-functioning-of-mild-cognifitive_fig4_236052149

Modularity for dynamic biological systems/models

- Given a model, compute its modules, their attractor structure, and information about the attractor structure of the model itself.

- Characterize the “degrees of freedom” to combine simple models.

A “Hölder Program” for BNs

- Identify a class of “decomposable” BNs.
- Identify a class of decomposable BNs that are “simple” and sufficiently “rich.”
- Define a notion of “quotient” of a BN by a subnetwork.
- Show that each decomposable BN has a filtration by subnetworks so that each successive quotient is a simple network.
- Classify the different ways in which decomposable BNs can be built as extensions of two BNs that are simpler.
- Rigorous definition of “dynamic equivalence” of BNs.
- Develop a category-theoretic foundation for this program.

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PHYSICA D

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Nested canalyzing, unate cascade, and polynomial functions☆

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$$
\begin{aligned}
&f(x_1, x_2, \ldots, x_n) \\
&\quad = (x_1 - a_1)[(x_2 - a_2)[\ldots[(x_{n-1} - a_{n-1})[(x_n - a_n) \\
&\qquad + (b_n - b_{n-1})] + (b_{n-1} - b_{n-2})]\ldots] \\
&\qquad + (b_2 - b_1)] + b_1
\end{aligned}
$$

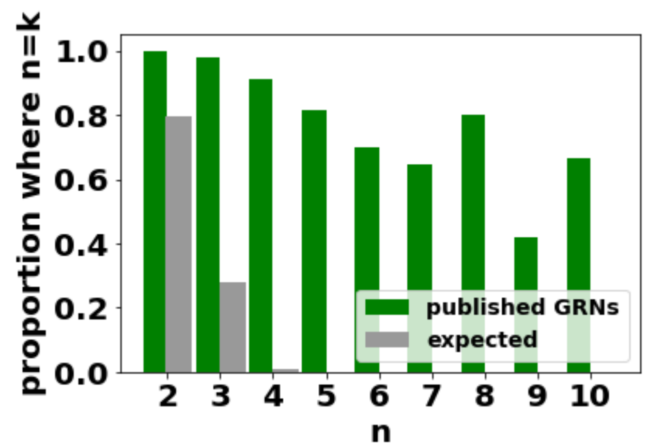
or, equivalently,

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n}(x_i - a_i) \\
&\quad + \sum_{j=1}^{n-1}\left[(b_{n-j+1} - b_{n-j})\prod_{i=1}^{n-j}(x_i - a_i)\right] + b_1.
\end{aligned}
$$

Prevalence of canalization

observed

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	2										
1	0	1688									
2	0	0	744								
3	6	1	0	411							
4	23	2	0	0	251						
5	19	8	2	0	0	133					
6	17	5	6	0	1	0	66				
7	7	4	3	1	1	0	0	27			
8	4	1	0	2	0	0	0	0	26		
9	2	3	3	0	0	3	0	0	0	7	
10	0	0	1	0	1	0	0	0	0	0	4



- ➢ Nested canalizing functions (and therefore? canalizing functions) are overrepresented in GRNs.

Courtesy C. Kadelka

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ADVANCES IN Applied Mathematics

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Decomposition and simulation of sequential dynamical systems

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Future work

- Find a version of the classification of monomial networks in the language of computational algebra.
- Study the properties of nested canalizing polynomials.
- Carry out the Hölder Program for larger classes of networks, for instance, AND-NOT networks.

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