New Directions in Rational Points

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1 Overview of the field

Rational points are solutions of Diophantine equations, that is, polynomial equations with integer or rational coefficients. This is one of the oldest areas of mathematics and an area to which many of the great contemporary mathematical projects are directed. Historically, progress in number theory was due to fruitful confluence of algebraic, analytic, and geometric ideas, and, more recently, cohomological methods. This is also true for rational points, where many results are obtained by combining analytic number theory and arithmetic geometry. The area of rational points has grown in prominence during the last 100 years through the work of Poincaré, Mordell, Weil, Cassels, Tate, Manin, Faltings, and many other mathematicians. It continues to see further ground-breaking developments.

In the last decade, several hitherto unrelated lines of research have brought massive changes to the area of rational points. Here is a brief and highly selective overview of recent breakthroughs.

1. Arithmetic statistics: in recent years, partly inspired by groundbreaking works of Bhargava, many researchers have been studying arithmetic phenomena (existence of rational points, everywhere local solubility, the Brauer–Manin obstruction) for families of varieties. New conjectures are being made which generate further research (conjectures of Poonen–Voloch, Serre, and Loughran). Deeper analytic methods have been applied to prove the Hasse principle for almost all Fano varieties of dimension at least 3 (Browning–le Boudec–Sawin), and the Schinzel hypothesis on average with applications to rational points in families (Skorobogatov–Sofos).

2. Developments in p-adic methods make them highly efficient for dealing with rational points. A recent example is a new p-adic proof of the Mordell conjecture by Lawrence and Venkatesh (Fields medal 2018). Work of Bright–Newton created new tools for computing the Brauer–Manin obstruction in the cases that could not be treated by previously known methods.

3. Work of Harpaz and Wittenberg revolutionized the fibration and descent methods with novel applications to the inverse Galois problem. They also stated new conjectures, particular cases of which have been proved by Browning–Schindler and Shute.

4. Novel forms of the Hardy–Littlewood–Ramanujan circle method: this most powerful analytic technique to study Diophantine equations originated in the work of Hardy and Ramanujan on the partition

function almost exactly a century ago. Several new forms of the method have been developed and applied to study rational points: Duke–Friedlander–Iwaniec, Heath-Brown (Delta symbol method), Browning–Vishe (a version for number fields and function fields), Browning–Sawin (an algebraic version), Bilu–Browning (a motivic version).

The key objective of this conference was to bring together a diverse group of well-established as well as early career experts working on rational points across the aforementioned research directions to discuss new developments and enhance existing interaction as well as to create conditions for future collaborations.

2 Recent developments and open problems

2.1 First problem session

At the end of the first day of the conference we held an open problem session, where participants were invited to contribute and describe open problems. The following problems were put forward.

Question (Daniel Loughran) Prove an asymptotic for

$$\#\{a,b \in \mathbb{Z}\cap[-x,x] \mid \gcd(a,b)=1,\ a^4+2b^4 \text{ is a sum of two integer squares}\}$$

as $x \to \infty$. The expected growth is a constant multiple of $x^2/\sqrt{\log x}$. This is motivated by geometric questions for fibrations over $\mathbb{P}^1$: see [17], where the upper bound $O(x^2/\sqrt{\log x})$ is established.

Following discussion:

- (Tim Browning) Asymptotics are open even for degree 3 homogeneous polynomials.
- (Daniel Loughran) For degree 4 it would be nice to know that we have growth at least $x^{1-\delta}$ for some $\delta \in (0,1)$.
- (Efthymios Sofos) The lower bound of the form $x^2/(\log x)^c$ is known but not recorded in the literature: let r_0 be the indicator of the sums of two squares and r be the number of representations as a sum of two squares. We have

$$x^4 \ll \left(\sum_{\substack{a,b\in\mathbb{Z}\cap[-x,x]\\ \gcd(a,b)=1}} r(a^4+2b^4)\right)^2 \leq \sum_{\substack{a,b\in\mathbb{Z}\cap[-x,x]\\ \gcd(a,b)=1}} r^2(a^4+2b^4) \sum_{\substack{a,b\in\mathbb{Z}\cap[-x,x]\\ \gcd(a,b)=1}} r_0(a^4+2b^4),$$

  where the first inequality follows from [5] and the second by Cauchy. The sum of r^2 is $O(x^2\log x)$ by [4]. Replacing Cauchy with Hölder's inequality with exponents $q,p>1$ and taking p arbitrarily large shows that the log-exponent can approach $\log 2 = 0.693\ldots$. Specifically, for any fixed positive ϵ one has

$$\sum_{\substack{a,b\in\mathbb{Z}\cap[-x,x]\\ \gcd(a,b)=1}} r_0(a^4+2b^4) \gg \frac{x^2}{(\log x)^{(\log 2)+\epsilon}}.$$

Question (Martin Bright) Let X be a $K3$ surface over $\mathbb{Q}$ and let p be an odd prime of good reduction. Is it true that the p-torsion of $(\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}))[p]$ is trivial? This holds in known examples. There is related work of Ieronymou–Skorobogatov, Gvirtz and Pagano.

Question (Felipe Voloch) Let X be a variety defined over $\mathbb{Q}$ (or any global field) and assume we are given a dominant map $X \to \mathbb{P}^n$ defined over $\mathbb{Q}$, whose generic fibre is a curve of genus ≥ 2. The spirit of the question is to investigate what are the right geometric conditions so that 'most' fibres have no $\mathbb{Q}$-point. Assume that there exists $\epsilon > 0$ such that for all large enough T one has

$$\#\{b \in \mathbb{P}^n(\mathbb{Q}) \mid X_b(\mathbb{Q}) \neq \emptyset,\ H(b) \leq T\} \geq \epsilon\#\{b \in \mathbb{P}^n(\mathbb{Q}) \mid H(b) \leq T\}.$$

The question is whether the dominant map has a section defined over $\mathbb{Q}$. This is a special case of the conjecture in the work of Poonen–Voloch corresponding to random curves, see [23]. *Following discussion:*

- (Felipe Voloch) Using Vojta's conjecture the answer is positive for $\mathbb{P}^1$.
- (Felipe Voloch) Using the ABC conjecture the answer is positive for the family $ax^d + by^d = cz^d$, where $(a : b : c) \in \mathbb{P}^2(\mathbb{Q})$ is random. The idea is to use the ABC conjecture to bound the height of a point.
- (Domenico Valloni) What happens in the case of isotrivial families?
- (Alexei Skorobogatov) Maybe the section is needed after base change.
- (Daniel Loughran) What about genus 1?
- (Tim Browning) False as shown by M. Bhargava in the case of plane cubics.
- (Daniel Loughran) True for genus 0.
- (Felipe Voloch) Need to understand this in greater generality: one should ask this in fact for families of random varieties of general type.
- (Felipe Voloch) What happens when we replace the base $\mathbb{P}^n$ by a Fano variety and we restrict to the fibers above points that lie outside an accumulating set in the sense of Manin's conjecture?

Question (Julia Brandes) An old problem of Manin (see page 3 in [19]) asks whether a smooth cubic hypersurface over $\mathbb{Q}$ can be generated by a *finite* secant and tangent process. Papanikolopoulos–Siksek [22] proved that if the dimension is ≥ 48 then one point suffices to generate all. Brandes–Dietmann [3] later proved that one point suffices for all hypersurfaces with dimension ≥ 29. The question is: what is the least n_0 so that all rational points on any smooth cubic hypersurface of dimension $\geq n_0$ are generated by a single point? *Following discussion:*

- (Colliot-Thélène) A bit unclear since the original question of Manin is still open.
- (Colliot-Thélène) Using general results from work of Colliot-Thélène–Sansuc–Swinnerton-Dyer in Crelle's journal, Siksek [26] proved that for any integer n one can find a cubic surface for which at least n points are necessary to generate all points.

Question (Julia Brandes) Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be a non-singular polynomial of degree d. Assume that n is large relative to d. Denote the Hessian by Hx and consider $\det(Hx)$, which is a polynomial of degree $n(d-2)$ in n variables. The question is: what is the growth of the function

$$\#\{x \in \mathbb{Z}^n \cap [-B, B] \mid F(x) = \det(Hx) = 0\}?$$

This is motivated by problems with lines in cubic hypersurfaces. It is also desirable to give upper bounds for this counting function. Probably there are very few solutions and one might even ask whether for 'generic' F the only common solution is $x = 0$.

Question (Marta Pieropan) For $m \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{Z}$ we say that x is m-full if $p^m \mid x$ for every prime $p \mid x$, an example being integers of the form $x = y_0^m y_1^{m+1} \cdots y_{m-1}^{n+m-1}$. Now assume that we are given $s, m, m_1, \ldots, m_s \in \mathbb{Z}_{\geq 1}$ with

$$m + 1 - \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) \leq 0.$$

The question is to find a non-trivial example of linear forms $L_0, \ldots, L_s \in \mathbb{Z}[x_0, \ldots, x_n]$ in 'general' position (i.e. pairwise coprime) such that

$$\#\{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \mid \gcd(x_0, \ldots, x_n) = 1,\ L_i(x_0, \ldots, x_n) \text{ is } m_i\text{-full } \forall i\} < \infty$$

Following comments:

- (Marta Pieropan) Assuming the orbifold Mordell conjecture one expects this to be true when $m_1 = \cdots = m_s = 2, m = 1$ and $s \geq 5$. Cases with $s = 4$ are open.

Question (Marta Pieropan) Assume that we are given a set $\mathcal{A} \subset \mathbb{Z}$ whose density we can estimate as

$$\#\{a \in \mathcal{A} \mid |a| \leq T\} = f(T) + O(g(T)),$$

where $f(T)$ has the form $c_0 T^c (\log T)^e$ for some real constants c_0, c, e and $\lim_{T\to\infty} g(T)/(T^c(\log T)^e) = 0$. For an arbitrary polynomial $F \in \mathbb{Z}[x_1, \dots, x_n]$ define

$$N(T; \mathcal{A}, F) = \#\{\mathbf{x} \in \mathbb{Z}^n \cap [-T, T]^n \mid F(\mathbf{x}) \in \mathcal{A}\}.$$

The question is under what conditions on $\mathcal{A}, f, g, n$ and $\deg(F)$ can one prove asymptotics for $N(T; \mathcal{A}, F)$ as as $T \to \infty$? *Following comments:*

- (Marta Pieropan) This is interesting even for special $\mathcal{A}$ such as the squares, m-full, square-free or prime numbers.
- (Marta Pieropan) One can ask what happens if we add or remove a low-density set to $\mathcal{A}$. To be precise: with the same $\mathcal{A}, f, g, F$ as above assume that we are given $\mathcal{B} \subset \mathbb{Z}$ with the property

  $$\#\{b \in \mathcal{B} \mid |b| \leq B\} = o(f(T)).$$

  Assume that we can prove asymptotics for $N(T; \mathcal{A}, F)$. Then what can we say for

  $$N(T; \mathcal{A} \cup \mathcal{B}, F) \text{ or } N(T; \mathcal{A} \setminus \mathcal{B}, F)?$$

- (Efthymios Sofos) For Birch polynomials and $\mathcal{A}$ being the primes or square-free numbers this is known [10]. In upcoming work the same authors extend this to arbitrary sets $\mathcal{A}$ having a Siegel–Walfisz property, i.e. being equidistributed in arithmetic progressions of 'small' modulus.

Question (Tim Browning)

1. Work by Green–Tao–Ziegler [15] on prime values of linear polynomials $L_1(\mathbf{t}), \dots, L_n(\mathbf{t})$ led to
2. work of Lilian Matthiesen [20] on norm values of linear polynomials $L_1(\mathbf{t}), \dots, L_n(\mathbf{t})$ led to
3. applications to Brauer–Manin obstruction for equations of type $\mathrm{N}_{K/\mathbb{Q}}(\mathbf{x}) = \prod_{i=1}^n L_i(\mathbf{t})$, see [16] and [6], for example.

A first question is to complete a similar sequence that starts from a single quadratic form in $n \geq 8$ variables. The analog of the n variables taking prime values has been completed by Green [14]. Another question is to find interesting applications for the Brauer–Manin obstruction.

Question (Tim Browning) What can we say about the arithmetic of log-Calabi–Yao threefolds? For example

$$x_1^2 + x_2^2 + x_3^2 x_4^2 = 4x_1x_2x_3x_4 \subseteq \mathbb{A}^4.$$

Can we prove strong approximation or Brauer–Manin obstruction results? *Following comments:*

- (Tim Browning) A problematic issue is that the compactification is typically very singular.
- Recent work of Gamburd–Magee–Ronan [13] established the asymptotic for the number of integral points of size B in the equation above. The asymptotic is $c_0(\log B)^\beta$ where β is an interesting constant in the interval $(2.4, 2.5)$. This was studied earlier by Baragar [1]. A question here is whether

  $$\beta \in \mathbb{Q}?$$

- A further question is whether the standard heuristics from circle method can help with interpreting β. These heuristics would come from estimating the major arcs only; this would furnish the asymptotic $\approx \sigma_\infty(B) \prod_{p \leq B} \sigma_p$, where σ_∞, σ_p are the densities of the real and p-adic solutions respectively. For the equation at hand one can easily show that $\sigma_p = 1 + O(p^{-3/2})$, hence the product over the primes converges. Thus the exponent β should be related to $\sigma_\infty(B)$ only. Estimating $\sigma_\infty(B)$ for varieties with smooth compactification has been done by Chambert-Loir–Tschinkel [7] but the equations at hand have highly singular compactifications.

Question (Joni Teräväinen, remotely) Let $\Omega(n)$ be the total number of prime factors of an integer n, counting multiplicity. Prove that there exists an irreducible polynomial $C \in \mathbb{Z}[t]$ such that $(-1)^{\Omega(C(n))}$ changes sign infinitely often as n runs through the positive integers. This is related to Chowla's conjecture – there is recent work for products of linear and quadratic polynomials: see the results and references in [29]. For quadratic polynomials the problem is related to integer solutions of Pell equations.

2.2 Second problem session

On the Thursday evening there was a second open problem session, where one answer and several new problems were put forward.

Answer (Alexei Skorobogatov, solution to the question of Martin Bright, with some help from Yuri Zarhin) For a K3 surface over $\mathbb{Q}$ with good reduction at a prime $p \neq 2$ the question was whether $\big(\mathrm{Br}(X)/\mathrm{Br}_0(X)\big)[p] = 0$, where $\mathrm{Br}_0(X) = \mathrm{Im}[\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(X)]$. The answer is negative. Here is an example. Let $X = \mathrm{Kum}(E_1 \times E_2)$, where E_1 and E_2 are elliptic curves over $\mathbb{Q}$. By Theorem 2.4 and Proposition 3.3 of [27], the group $\mathrm{Br}(X)[p]/\mathrm{Br}_1(X)[p]$ is canonically isomorphic to

$$\mathrm{Hom}_{\mathbb{Q}}(E_1[p], E_2[p])/(\mathrm{Hom}_{\overline{\mathbb{Q}}}(E_1, E_2)/p)^{\Gamma},$$

where $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let E_1 be a CM elliptic curve and E_2 a non-CM elliptic curve. Then $\mathrm{Hom}_{\overline{\mathbb{Q}}}(E_1, E_2) = 0$. Take $p = 3$. Let E_1 be given by $y^2 = x^3 - x$. It has CM and good reduction at 3. For E_2 take the elliptic curve

$$y^2 = x^3 + (27t^4 - 18t^2 - 1)x + 4t(27t^4 + 1),$$

studied by Rubin and Silverberg [24, Theorem 4.4]. For every $t \in \mathbb{Q}$ the Galois modules $E_1[3]$ and $E_2[3]$ are isomorphic. (In this case the relevant modular curve has genus 0.) If t is integral at 3, then E_2 has good reduction at $p = 3$. Then X has good reduction at 3 and $\big(\mathrm{Br}(X)/\mathrm{Br}_0(X)\big)[3] \neq 0$.

A similar method works for $p = 5$, because the relevant modular curve also has genus 0. The situation for $p > 5$ is less clear because the genus of the modular curve becomes greater than 1.

Question (Peter Koymans) Fix a finite group G and a number field k. How many G-extensions K/k satisfy the Hasse Norm Principle? One can order K by discriminant or by the product of ramified primes. There is previous work by Macedo (D_4, see his Ph.D. thesis [18]), Sebastian Monnet (S_4, see the preprint [21]), Frei–Loughran–Newton (abelian G, see [11] and [12]), but we need more examples.

Question (Tim Santens) Let $A, B, C \in \mathbb{Z} \setminus \{0\}$. Does the Hasse principle hold for

$$Ax^2 + By^3 + Cz^5 = 0,\ x, y, z \in \mathbb{Z}, \gcd(x, y, z) = 1?$$

The context for this question is that for Fermat equations of shape $Ax^r + By^p + Cz^q = 0$ with

$$\frac{1}{r} + \frac{1}{p} + \frac{1}{q} > 1$$

we expect infinitely many integer solutions. It may be helpful to think of Brauer–Manin obstruction for integral points on a stacky curve. There is some work in this direction from Darmon [8], Granville–Darmon [9] and Beukers [2]. There is also the more recent work [25], in which the Brauer group is trivial.

Question (Sho Tanimoto) Let X be a geometrically uniruled smooth projective variety defined over a number field F and let L be a big and nef divisor on X. Let

$$\alpha(X, L) = \inf\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\},$$

where $\overline{\mathrm{Eff}}^1(X)$ is the cone of pseudo-effective divisors of X. Weak Manin's conjecture states that there exists a Zariski open $U \subseteq X$ such that for all fixed $\epsilon > 0$ one has

$$\#N(U(F), H_L, B) := \#\{P \in U(F) \mid H_L(P) \leq B\} = O(B^{\alpha(X,L)+\epsilon}).$$

A weird example is the following: Let S be a smooth projective rational surface defined over a number field F with infinitely many (-1)-curves defined over F. Then there exists a projective bundle threefold X over S such that $-K_X$ is big, but not nef. Moreover let T be the pullback of a (-1)-curve. Then we have $\alpha(T, -K_X|_T) > \alpha(X, -K_X) = 1$. Thus the closed set version of weak Manin's conjecture for only big divisors fails. A natural question is whether the thin set version of such a conjecture holds or not. This leads to the following question:

> Is the set of rational points on (-1)-curves thin on S?

My guess is no, and there is a potential counterexample by Oguiso. He constructed a smooth projective rational surface with infinitely many elliptic fibrations.

Question (Colliot-Thélène) Consider the Fermat surface W of degree p given by the homogeneous equation $ax^p + by^p + cz^p + dt^p = 0$ with $a, b, c, d \in \mathbb{Q}^\times$. Assume that $W(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset$. Is there a finite extension $L/\mathbb{Q}$ of degree coprime to p such that $W(L) \neq \emptyset$? This is a concrete case of the general question whether for arbitrary smooth projective varieties over a number field the Brauer–Manin obstruction is the only obstruction to the existence of a zero-cycle of degree one.

In the case $p = 3$, and with moderate extra assumptions on the coefficients, assuming finiteness of Sha of elliptic curves over number fields, Swinnerton-Dyer used his technique "very small Sha implies Sha=0" to give a positive answer [28]). A variant of the technique was applied to various Kummer surfaces in works of P. Swinnerton-Dyer, A. Skorobogatov, Y. Harpaz, and recently A. Morgan (see his talk at this conference).

Still assuming finiteness of various Sha's of abelian varieties, the aim here would be to find an element $\lambda_0 \in \mathbb{Q}^\times$ such that $ax^p + by^p = \lambda_0 = cz^p + dt^p$ has solutions in all completions of $\mathbb{Q}$ (this part should be easy) and such that the p-torsion of the Tate-Shafarevich groups of the Jacobians of the curves $ax^p+by^p = \lambda_0$ and $\lambda_0 = cz^p + dt^p$ be so "small" that it should vanish. As in Swinnerton-Dyer's case, one should consider suitable isogenies rather than mere multiplication by p on the Jacobians.

3 Presentation highlights

Levent Alpöge (Harvard), *The average size of 2-Selmer groups in the families $y^2 = x^3 + B^k$*

> Let $k \in \mathbb{Z}$. I will compute the average size of the 2-Selmer group of $y^2 = x^3 + B^k$ over $\mathbb{Q}$ over $B \in \mathbb{Z}$.

Subham Bhakta (IISER Trivandrum), *Arithmetic statistics of modular degree*

> Given an elliptic curve E over Q of conductor N, there exists a surjective morphism $X_0(N) \to E$ defined over Q. The modular degree m_E of E is the minimum degree of all such modular parametrizations. Watkins conjectured that the rank of E(Q) is less than or equal to $\nu_2(m_E)$. In this talk, we shall delve into various analytic approaches, aiming to deepen our understanding and address this intriguing conjecture.

Tim Browning (IST Austria), *Generalised quadratic forms over totally real number fields*

> We introduce a new class of generalised quadratic forms over totally real number fields, which is rich enough to capture the arithmetic of arbitrary systems of quadrics over the rational numbers. We explore this connection through a version of the Hardy–Littlewood circle method over number fields. This is joint work with Lillian Pierce and Damaris Schindler.

Jean-Louis Colliot-Thélène (Paris-Saclay), *Arithmetic on the intersection of two quadrics*

> I shall report on the Hasse principle for rational points on intersections of two quadrics in 7-dimensional projective space. The non-singular case was obtained by R. Heath-Brown in 2018 and revisited by me in 2022. The singular case was handled in 2023 by A. Molyakov.

Brendan Creutz (Canterbury), *Degrees of points on varieties over Henselian fields*

Let X/k be a variety over the field of fractions of a Henselian discrete valuation ring R. For example, k could be the field of p-adic numbers. I will explain how one can compute the set of all degrees of closed points on X from data pertaining only to the special fiber of a suitable model of X over R. In the case of curves over p-adic fields this gives an algorithm to compute the degree set, which yields some surprising possibilities. This is joint work with Bianca Viray.

Julian Demeio (Bath), *The Grunwald Problem for solvable groups*

Let K be a number field. The Grunwald problem for a finite group (scheme) G/K asks what the closure of the image of $H^1(K,G) \to \prod_{v\in M_K} H^1(K_v,G)$ is. For a general G, there is a Brauer–Manin obstruction to the problem, and this is conjectured to be the only one. In 2017, Harpaz and Wittenberg introduced a technique that managed to give a positive answer (BMO is the only one) for supersolvable groups. I will present a new fibration theorem over quasi-trivial tori that, combined with the approach of Harpaz and Wittenberg, gives a positive answer for all solvable groups.

The fibration theorem presents two difficulties: lifting local points and avoiding Brauer–Manin obstruction on the fibers. To overcome the first, we employ ideas of Shafarevich used in his solution of the Inverse Galois Problem for solvable groups. To overcome the second, one first reduces to a desirable subcase using a base-change method due to Harpaz and Wittenberg. One then proceeds with the core computation of the "triple variation" of the Brauer–Manin obstruction on the fibers in terms of some Redéi symbols (which may be thought as "triple pairings" of algebraic numbers) and concludes by following a general combinatorial principle first noted by Alexander Smith in the context of class groups and Selmer groups.

This is work in progress. Partial results (including the "lifting local points" part and the appearance of "triple pairings" in the BMO) were also obtained independently by Harpaz and Wittenberg.

Christopher Frei (Graz), *Linear equations in Chebotarev and Artin primes*

We show that the von Mangoldt functions for primes restricted to a fixed Chebotarev class or (conditionally on GRH) with a fixed primitive root are not correlated with nilsequences in a quantitative sense. Via Green–Tao–Ziegler nilpotent machinery, this yields asymptotic formulae for the number of solutions to non-degenerate systems of linear equations in such primes. This is joint work with Magdaléna Tinková.

Jakob Glas (IST Austria), *Rational points on del Pezzo surfaces of low degree*

We establish upper bounds for the number of rational points of bounded anti-canonical height on del Pezzo surfaces of degree at most 5 over global fields. The approach uses hyperplane sections and uniform upper bounds for the number of rational points of bounded height on elliptic curves. The results are unconditional in positive characteristic and for number fields rely on a conjecture relating the rank of an elliptic curve to its conductor. This is joint work with Leonhard Hochfilzer.

Wataru Kai (Tohoku), *Linear patterns of prime elements in number fields*

I discuss the number field analogue of a result by Green–Tao–Ziegler (2012) on linear patterns of prime numbers. This combined with techniques developed originally by Colliot-Thélène, Sansuc, Swinnerton-Dyer, Harari and others proves a Hasse principle type result for rational points on varieties over number fields fibered over $\mathbb{P}^1$, as was done over $\mathbb{Q}$ by Harpaz–Skorobogatov–Wittenberg in 2014 using the Green–Tao–Ziegler theorem.

Peter Koymans (ETH Zürich), *Averages of multiplicative functions over integer sequences*

In this talk we are interested in the average value of a multiplicative function when summed over a sequence that behaves well in small arithmetic progressions. As an application of our techniques, we obtain the size of the average 6-torsion, get tail bounds for the number of prime divisors of discriminants of S_5-extensions and count rational points on some varieties. This is joint work with Stephanie Chan, Carlo Pagano and Efthymios Sofos.

Dan Loughran (Bath), *The leading constant in Malle's conjecture*

A conjecture of Malle predicts an asymptotic formula for the number of field extensions with given Galois group and bounded discriminant. Malle conjectured the shape of the formula but not the leading constant. We present a new conjecture on the leading constant motivated by a version for algebraic stacks of Peyre's constant from Manin's conjecture. This is joint work with Tim Santens.

Adam Morgan (Glasgow), *On the Hasse principle for Kummer varieties*

Conditional on finiteness of relevant Shafarevich–Tate groups, Harpaz and Skorobogatov established the Hasse principle for Kummer varieties associated to 2-coverings of a principally polarised abelian variety A, under certain large image assumptions on the Galois action on $A[2]$. However, their method stops short of treating the case where the image is the full symplectic group, due to the possible failure of the Shafarevich–Tate group to have square order in this case. I will present recent work which overcomes this obstruction by combining second descent ideas in the spirit of Harpaz and Smith with new results on the 2-parity conjecture.

Simon Rydin Myerson (Warwick), *A two-dimensional delta method and applications to quadratic forms*

We develop a two-dimensional version of the delta symbol method and apply it to establish quantitative Hasse principle for a smooth pair of quadrics defined over $\mathbb{Q}$ in at least 10 variables. This is joint work with Pankaj Vishe (Durham) and Junxian Li (Bonn).

Margherita Pagano (Leiden), *The role of primes of good reduction in the Brauer-Manin obstruction to weak approximation*

A way to study rational points on a variety is by looking at their image in the p-adic points. Some natural questions that arise are the following: is there any obstruction to weak approximation on the variety? Which primes might be involved in it? I will explain how primes of good reduction can play a role in the Brauer–Manin obstruction to weak approximation, with particular emphasis on the case of K3 surfaces. I will then explain how the reduction type (in particular, ordinary or non-ordinary good reduction) plays a role.

Raman Parimala (Emory), *A Hasse principle for twisted moduli spaces*

The existence of rational points on certain twisted moduli spaces of rank two stable vector bundles on curves over number fields has consequences for the existence of rational points on large dimensional quadrics over number fields. We explain a connection of this problem to a Hasse principle for the existence of large dimensional Grassmannian spaces in the intersection of two quadrics in the case of hyperelliptic curves. (Joint work with Jaya Iyer)

Marta Pieropan (Utrecht), *Points of bounded height on certain subvarieties of toric varieties*

In joint work with Damaris Schindler we develop a new version of the hyperbola method for counting rational points of bounded height that generalizes the work of Blomer and Brüdern for products of projective spaces. The hyperbola method transforms a counting problem into an optimization problem on certain polytopes. For rational points on subvarieties of toric varieties, the polytopes have a geometric meaning that reflects Manin's conjecture, and the same holds for counts of Campana points of bounded height. I will present our results as well as some general heuristics.

Raman Preeti (IIT Bombay), *R-equivalence in adjoint classical groups*

Let E be a field and G be an adjoint classical group defined over E. Let $G(E)$ denote the group of E-rational points of G and let $G(E)/R$ denote the R-equivalence classes. We discuss the triviality of $G(E)/R$ over fields with low virtual cohomological dimension.

Soumya Sankar (Utrecht), *Counting points on stacks and elliptic curves with a rational N-isogeny*

Stacks are ubiquitous in algebraic geometry and in recent years there has been increased interest in studying the arithmetic of stacks and using their structure to answer more classical questions in number theory. The classical problem of counting elliptic curves with a rational N-isogeny can be phrased in terms of counting rational points on certain moduli stacks of elliptic curves. I will talk about the recent progress that has been made on this problem within this context, as well as some open questions in connection with the stacky Batyrev–Manin–Malle conjecture. The talk assumes no prior knowledge of stacks and is based on joint work with Brandon Boggess.

Tim Santens (KU Leuven), *Manin's conjecture for integral points on toric varieties*

Due to work of Manin and his collaborators we now have a good conjectural understanding of the distribution of rational points on Fano varieties. The analogous question of understanding the distribution of integral points on log Fano varieties has proven more challenging.

In this talk I will discuss some of the new phenomena which appear when counting integral points and how one can understand them in terms of universal torsors. I will then explain how one can use universal torsors to count integral points on toric varieties. This corrects an unpublished preprint of Chambert-Loir and Tschinkel.

Alec Shute (Bristol), *Zooming in on quadrics*

Classically, Diophantine approximation is the study of how well real numbers can be approximated by rational numbers with small denominators. However, there is an analogous question where we replace the real line with the real points of an algebraic variety: How well can we approximate a real point with rational points of small height? In this talk I will present joint work with Zhizhong Huang and Damaris Schindler in which we study this question for projective quadrics. Our approach makes use of a version of the circle method developed by Heath-Brown, Duke, Friedlander and Iwaniec.

Sho Tanimoto (Nagoya), *Sections of Fano fibrations over curves*

Manin's conjecture predicts the asymptotic formula for the counting function of rational points on a smooth Fano variety, and it predicts an explicit asymptotic formula in terms of geometric invariants of the underlying variety. When you count rational points, it is important to exclude some contribution of rational points from an exceptional set so that the asymptotic formula reflects global geometry of the underlying variety. I will discuss applications of the study of exceptional sets to moduli spaces of sections of Fano fibrations, and in particular I will explain how exceptional sets explain pathological components of the moduli space of sections. This is based on joint work with Brian Lehmann and Eric Riedl.

Domenico Valloni (EPF Lausanne), *Noether's problem in mixed characteristic*

Let k be a field and let V be a linear and faithful representation of a finite group G. The Noether problem asks whether V/G is a (stably) rational variety over k. It is known that if $p = \mathrm{char}(k) > 0$ and G is a p-group, then V/G is always rational. On the other hand, Saltman and later Bogomolov constructed many examples of p-groups such that V/G is not stably rational over the complex numbers.

The aim of the talk is to study what happens over a discrete valuation ring R of mixed characteristic $(0, p)$. We show for instance that for all the examples found by Saltman and Bogomolov, there cannot exist a smooth projective scheme over R whose special, respectively generic fibre are stably birational to V/G. The proof combines integral p-adic Hodge theory and the study of differential forms in positive characteristic.

4 Scientific progress made

The conference brought together a large group of researchers working on rational points using a wide variety of approaches. The participants were carefully selected to present a wide range of algebraic as well as analytic breakthroughs. We also chose a variety of speakers who work on the boundary between the algebraic and analytic realms, including Christopher Frei (solutions to linear equations in primes), Wataru Kai (linear patterns of primes in number fields), Dan Loughran (leading constant in Malle's conjecture), Marta Pieropan (points of bounded height on certain toric subvarieties) and Subham Bhakta (arithmetic statistics of the modular degree).

The theory of rational points on complete intersections of quadrics holds a central place in Diophantine geometry. The conference started with a beautiful review of the subject by Jean-Louis Colliot-Thélène. A variety of major results from all angles were discussed by Tim Browning (generalised quadratic forms over number fields), Simon Myerson (on a pair of quadrics), Parimala (quadrics over general function fields) and Alec Schute (Diophantine approximation on quadrics).

The key analytic advances presented include a version of circle method over number fields (Browning) partially answering a difficult question, a 2-dimensional delta method (Myerson) which solved a major open problem on the analytic side, rational points on del Pezzo surfaces of low degree using hyperbolic sections (Glas), impressive works on linear equations in primes (Frei and Kai) and Diophantine approximation on quadrics (Schute).

On the arithmetic geometry side we heard beautiful talks by junior mathematicians Alpöge, Morgan, Pagano, Valloni, as well as by more senior Creutz and Tanimoto on a broad spectrum of subjects including Brauer–Manin obstruction and p-adic cohomology methods for rational points.

The allocated time for discussions was used fruitfully by participants engaging in work on curent collaborations and new research projects.

5 Outcome of the meeting

We strictly adhered to the principles of EDI while inviting the participants. More than 33 percent of our invited participants were female and we had roughly the same proportion of BAME invitees. We were proactive in seeking participation from female candidates. Despite several last minute cancellations, about 25 percent of our on-site and online participants were female and roughly a half of our participants were BAME. We also had a number of local students and postdocs who attended this event.

In total there were 22 talks of 50 minutes each. Due to our concerns about the technology and overall experience, we only decided to invite on-site participants to talk at the conference. Originally, we had 7/22 confirmed female speakers but due to late cancellations by Matthiesen, Schindler and Li, we had 5/22 female speakers, 6/22 of our speakers were from BAME background and 14/22 of the speakers were postgraduate students and early career researchers. We specifically made an effort to give opportunity to the younger participants to be able to showcase their results to an international audience.

One problem that was not expected by the co-organisers was that the invited participants from the People's Republic of China could not come because they could not get the Indian visa.

We had roughly 30 online participants. We sought participation from Indian early career researchers via setting up a local page for the conference which was then advertised to a wide range of Indian universities. Partly due to this being the first BIRS conference at Chennai, there were several technical difficulties in the online delivery of talks. The Zoom connection dropped several times, especially in the first few days of the conference. The local organisers reacted promptly and helped greatly, but the quality of experience for our online participants was likely not the same as for our on-site attendees.

We also actively sought to organise short talks involving local PhD students and postdocs so as to give them an opportunity to present their work. However, after consultations with local students, the organisers found that the main interest for the students was to attend the research talks and also the open problem sessions. We decided to allocate sufficient time for problem sessions and discussions, and to run two open problem sessions instead of one as initially planned.

The problem sessions were a big success. We had to organise a second session in order to give sufficient time for willing participants to give problems and for subsequent discussions. We had actively encouraged participants to engage with the session and we think our efforts paid off.

In order to increase inclusivity and to enhance overall experience, we organised a number of events. On Monday, CMI organised a dance performance by an esteemed artist which was attended not only by the participants but also by local students which gave them an opportunity to mix. On Wednesday afternoon an excursion to Mahabalipuram was organised which was attended by a majority of our participants and gave them an opportunity to network as well as to bond. On Thursday afternoon a friendly football game was arranged which was good in boosting team morale.

Reflecting on the workshop, we think that overall conference organisation went relatively smoothly and the local organisers (Aditya Karnataki and Siddhi Pathak) and CMI did a splendid job organising the accommodation, providing visa information as well as transport of the participants. As expected, there were some hiccups early on as this was the first BIRS conference to be held in Chennai but it all worked out relatively smoothly in the end. What turned out to be somewhat inconvenient was that BIRS, Canada seemed to not be aware of logistical issues of organising a conference at Chennai, India (for instance we got contradicting information from BIRS about the visa policy). It also would be very helpful for future events if the local organisers had full access to the BIRS conference page. There was a lot of duplication of the information to be done as the local organisers had no idea who the participants were until we shared this information with them. We would strongly suggest giving CMI access to the BIRS conference pages for the workshops held in CMI.

Overall, the participants really enjoyed their stay at Chennai and had a scientifically invigorating and productive time as well as a culturally rich experience of India.

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