

# INTEGRABILITY AND NEAR-INTEGRABILITY IN MECHANICS AND GEOMETRY (16w5017)

Boris Khesin (University of Toronto),  
Sergei Tabachnikov (Pennsylvania State University)  
Vadim Zharnitsky (University of Illinois at Urbana-Champaign)

June 12 – June 17, 2016

## 1 Introduction

The notion of integrability is one of cornerstones of the modern theory of dynamical systems and it lies at the intersection of several mathematical domains: differential geometry, mechanics, Lie groups, quantum groups, analysis and differential equations. While integrability is an exceptional property, usually related to the existence of explicit or hidden symmetries for the dynamical system considered, once integrability is established, the corresponding dynamics can be described rather straightforwardly. Furthermore, near-integrable systems may often partially share some of key properties of the integrable ones (e.g., the existence of invariant tori in the KAM theory).

The goal of the workshop was to bring together researchers working on various problems related to complete integrability or to near-integrable dynamics to exchange ideas and methods and to foster new collaborations. The areas of research represented at the workshop included celestial mechanics, mathematical billiards, fluid dynamics, discrete differential geometry, classical mechanics, computer visualization, etc.

Below we briefly describe the topics discussed at the workshop, grouped according to the main themes.

## 2 Integrable and nearly integrable billiard dynamics

One of the ‘Holy Grails’ in the study of mathematical billiards is the Birkhoff conjecture that states that the only plane billiards with completely integrable dynamics are the ones bounded by ellipses.

*M. Bialy’s* (Tel Aviv University) talk concerned recent progress in the proof of the algebraic version of this conjecture where the integral is assumed to be polynomial in momentum. In a series of recent papers with A. Mironov, they obtained strong results toward this conjecture and its variants in the spherical and hyperbolic geometries, or in the presence of magnetic field, and it is expected

that a full proof of the algebraic Birkhoff conjecture will be obtained using methods of algebraic geometry in collaboration with A. Glutsyuk.

*V. Kaloshin* (University of Maryland, joint work with A. Avila and J. De Simoi) described a proof of a local version of the Birkhoff conjecture, namely, that it holds for perturbations of ellipses of small eccentricity. The method of proof gives an insight into deformational spectral rigidity of planar axis symmetric domains and gives a partial answer to a question of P. Sarnak (joint work with J. De Simoi and Q. Wei).

*V. Dragovic* (University of Texas Dallas) presented a class of nonconvex billiards whose boundary is composed of arcs of confocal conics with reflex angles. Such systems are not integrable, but carry strong traces of integrability. In particular, a connection with interval exchange transformation was established together with the Keane-type conditions for minimality. This is joint work with M. Radnovic.

*R. Montgomery* (University of California Santa Cruz), jointly with A. Knauf and J. Fejoz, also studied a billiard-like system motivated by scattering in  $n$ -body problem. The data for their “point billiard process” consist of a Euclidean vector space endowed with a finite collection of linear subspaces called “collision subspaces”. The solutions, or “billiard trajectories” move in straight lines away from the collision subspaces, and upon hitting a subspace, they reflect off according to the standard law of reflection. Among the basic questions studied are: Are the itineraries of the trajectories always finite? What is the structure of the space of all trajectories having a fixed itinerary? The answer to the latter is as follows: this space of trajectories is a Lagrangian relation on the space of oriented lines in the ambient Euclidean space.

*R. Schwartz* (Brown University) reported on another variant of billiard system, the outer billiard on polygons whose study goes back to J. Moser. Namely, Schwartz described a combinatorial model, called the plaid model, that, for each rational parameter, produces a finite union of embedded polyhedral surfaces in a cube. When the surfaces are sliced in one direction, the resulting curves encode the dynamics of outer billiards on kites. When the surfaces are sliced in other directions, they give the same families of curves as those produced by P. Hooper’s Truchet tile system.

### 3 Discrete complete integrability and discrete differential geometry

The emerging field of discrete differential geometry provides new methods of proving complete integrability of discrete dynamical systems (for example, by utilizing the theory of cluster algebras) and serves as a source of new examples of such systems.

*V. Ovsienko* (University of Reims) surveyed recent work on the pentagram map, a discrete Liouville integrable system of geometric origin. Discovered by R. Schwartz about 25 years ago, this system and its various generalizations continues to be an active area of research.

*R. Kenyon* (Brown University) discussed integrable structures on spaces of polygonal tilings. Special cases include the pentagram maps and their generalizations, dimer integrable systems of Goncharov and Kenyon, and resistor network integrable systems.

*S. Ramassamy* (Brown University) studied circle patterns with the combinatorics of the square grid, that is, planar embedding of the square grid such that every face admits a circumcircle. Using Miquel’s six circles theorem, one defines a dynamical system on the space of such circle patterns on a flat torus. Computer experiments indicate that this is an integrable system. Ramassamy discussed work in progress in this direction.

*G. Mari-Beffa* (University of Wisconsin) related, by reduction, Semenov-Tian-Shansky's twisted Poisson structure on the projective group with the moduli space of projective polygons defined by the discrete projective curvatures (the pentagram map is a discrete evolution on this space). A second structure is defined by reducing a right invariant tensor, which is proven to be compatible with the first reduction for dimensions 2 and 3, and conjectured to be for any dimension. The pair are Hamiltonian structures for integrable discretizations of  $W_n$  algebras in any dimension. Thus, one can write a very simple realization of this integrable system as an evolution of projective polygons (joint work with J. P. Wang).

*M. Shapiro* (Michigan State University, joint work with A. Felikson and P. Tumarkin) studied quiver mutations that play an important role in definition of cluster algebras and also appeared independently as Seiberg duality in mathematical physics. He presented classification of quivers with finite mutation class and discussed its application.

*K. Stephenson* (University of Tennessee, joint work with P. Bowers) presented a study of aperiodic tiling of the plane generated by subdivision rules by putting conformal, rather than Euclidean, structure on the tiles. Conformal tiling is determined by combinatorics alone, and is not limited by the rigid geometric constraints of classical tilings, thus it brings up new issues in tiling theory. Numerous computer experiments suggest that aggregates of conformal tiles may converge in shape to their classical Euclidean counterparts. This raises an interesting issue: how can rigid Euclidean shapes be encoded in abstract combinatorics?

## 4 Fluid dynamics and waves

Several talks concerned integrability and near-integrability in the framework of fluid dynamics.

In 1966, V. Arnold suggested a group-theoretic framework for ideal hydrodynamics. In this approach, the motion of an incompressible fluid on a Riemannian manifold is described as the geodesic flow of a right-invariant metric on the group of volume-preserving diffeomorphisms. *A. Izosimov* (University of Toronto) showed how Arnold's approach could be extended to incorporate certain discontinuous fluid motions, known as vortex sheets. This is done by replacing groups and algebras in Arnold's approach by certain groupoids and algebroids (joint work with B. Khesin).

*P. Olver's* (University of Minnesota) talk concerned an interesting phenomenon: the evolution, through spatially periodic linear dispersion, of rough initial data leads to surprising quantized structures at rational times, and fractal, non-differentiable profiles at irrational times. The Talbot effect, named after an optical experiment by one of the founders of photography, was first observed in optics and quantum mechanics, and leads to intriguing connections with exponential sums arising in number theory. Olver discussed ramifications of these phenomena and recent progress on the analysis, numerics, and extensions to nonlinear wave models, both integrable and non-integrable.

*U. Pinkall* (Technical University of Berlin) presented intriguing computer experiments in near-integrable fluid simulation. This is part of the work done by the geometry group at TU Berlin in the framework of discretization in geometry and dynamics with application to mathematical visualization.

## 5 Integrable and near-integrable systems from mechanics and geometry

Classical mechanics and classical geometry continue to serve as sources of interesting dynamical systems that exhibit a variety of behaviors, from regular to chaotic.

*G. Bor* (CIMAT) presented a study of the bicycle kinematics and the geometry of bicycles tracks in dimension three. The bicycle is modeled as a directed segment of fixed length that can move in such a way that the velocity of its rear end is always aligned with the segment. The work (joint with *M. Levi*, *R. Perline* and *S. Tabachnikov*) is a combination of theoretical results with computer experiments.

*M. Levi* (Penn State University) described counterintuitive behavior of particles in rotating potentials, as well as some related effects such as the curious Coriolis-like force acting on binaries in an ambient gravitational field. In particular, the Gaussian curvature plays an unexpected role in this connection.

*M. Arnold* (University of Texas Dallas) studied the dynamics of a simple piecewise linear map of a strip, which can be thought as a toy model for various economical problems. It turns out that even such a simple model possess non-trivial dynamics. The main result is a classification of the attractors for the different values of the parameters of this model.

*R. Perline* (Drexel University) presented a new class of integrable surfaces associated with Bertrand curves. These surfaces are foliated by constant-torsion curves evolving according to a novel integrable geometric flow. Curves transverse to the constant-torsion curves (orbit curves) are Bertrand curves on the surface. These surfaces interpolate two known integrable systems. Tools from soliton theory are used to generate surface solutions using Bäcklund transformations.

*I. Izmistiev* (University of Fribourg) described an extension of the classical Ivory theorem on the gravitational potential of homeoids and its generalization to algebraic surfaces, due to *V. Arnold*, to the spherical and the hyperbolic spaces. The methods of study are mostly geometrical. This is a joint work with *S. Tabachnikov*.

## 6 Topics in the theory of completely integrable systems

Several talks addressed various aspects of the classical and modern theory of completely integrable systems.

*Yu. Fedorov* (Universitat Politècnica de Catalunya) revisited the celebrated *S. Kovalevskaya* top. He showed how the original *Kovalevskaya* curve of separation can be obtained, by a simple one-step transformation, from the spectral curve of the Lax representation found by *Bobenko*, *Reyman*, and *Semenov-Tian-Shansky*. The algorithm works for the general constants of motion of the top and is based on *W. Barth's* description of Prym varieties via pencils of genus 3 curves. This also allows to derive existing and new curves of separation for the *Kovalevskaya* gyrostat in one and two force fields.

Two particle systems are said to be dual if the maps that linearize their dynamics are inverses. Bispectrality, on the other hand, describes the situation in which a function satisfies two eigenvalue equations with the roles of spacial and spectral variables exchanged. Although bispectrality does not initially appear to be dynamical in nature, it turns out that duality manifests itself as bispectrality on both the classical and quantum levels. (For example, the *Calogero-Moser* system is self-dual since

its action-angle map is an involution, and this can be seen either in a symmetry of the eigenfunction of its quantum Hamiltonian or through the appearance of this particle system in the pole dynamics of bispectral solutions of the KP Hierarchy.) *A. Kasman* (College of Charleston) reviewed both old and recent results on this subject and presented a list of associated open problems.

*C. Roger* (University of Lyon) studied hierarchies of integrable PDEs in super-dimension  $(2|1)$ . In particular, he described a Miura transform in dimension  $(2|1)$  for non stationary Schrödinger type operators. He discussed the construction of an algebra of pseudodifferential symbols in dimension  $(2|1)$ ; that generalizes the algebra used in construction of hierarchies from isospectral deformations of stationary Schrödinger type operators.

*M. Dunajski* (University of Cambridge) presented a solution of an outstanding open problem of Sylvester concerning binary sextics. This is based on a recent joint work with R. Penrose. As an application he constructed a conformal structure on a solution space to the unique 7th order integrable ODE with sub-maximal group of contact symmetries (joint work with V. Sokolov).

*A. Pedroza* (Universidad de Colima) explained how a Hamiltonian diffeomorphism can be lifted to the symplectic one-point blow up. Then he consider loops of Hamiltonians diffeomorphisms and showed that the rank of fundamental group of the group of Hamiltonian diffeomorphisms of the symplectic one-point blow up was positive.

Finally, a stand-alone talk on applications of geometry in theoretical computer science was given by *J. Landsberg* (Texas A&M University). Complexity theory deals with determining when there does or does not exist a faster algorithm than the standard for some basic task such as multiplying matrices. In the case of matrix multiplication, Strassen shocked the world in 1969 by finding an algorithm faster than the standard one and computer scientists now make the astonishing conjecture that as the size of the matrices increase, it becomes almost as easy to multiply two matrices as it is to add them. The famous P v. NP problem essentially conjectures that there is no fast solution to the traveling salesperson problem. Recently algebraic geometry and representation theory have led to advances regarding these conjectures that were surveyed in the talk.

The workshop was characterized by a high level of informal interaction between the participants and an intense exchange of ideas. The material for most of the talks can be found among recent *arXiv* preprints. Below we mention several “classical” books on the most common topics of the workshop

## References

- [1] V. Arnold, B. Khesin. *Topological Methods in Hydrodynamics*. Springer, New York, 1998.
- [2] V. Arnold, V. Kozlov, A. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, EMS, Springer, 2006.
- [3] A. Bobenko, Yu. Suris. *Discrete Differential Geometry. Integrable Structure*. Amer. Math. Soc., Providence, RI, 2008.
- [4] V. Dragović, M. Radnović, *Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics*, Birkhäuser Basel, Frontiers in Mathematics, 2011.
- [5] M. Gekhtman, M. Shapiro, A. Vainshtein, *Cluster Algebra and Poisson Geometry*, AMS, Mathematical Surveys and Monographs, vol.167, 2010.

- [6] T. Kappeler, J. Pöschel, *KdV & KAM*, Springer, A Series of Modern Surveys in Mathematics, 2003.
- [7] J. Moser, *Various Aspects of Integrable Hamiltonian Systems*, in *Dynamical Systems*, (C.I.M.E. Summer School, Bressanone, 1978), Progr. Math. 8, Birkhäuser, Boston, Mass., 1980, pp. 233-289.
- [8] S. Tabachnikov, *Geometry and Billiards*, Amer. Math. Soc., Providence, RI, 2005.