1 Introduction

$G_2$ Geometry is a vibrant and rapidly growing field which interacts with numerous areas in differential and algebraic geometry, as well as with other mathematical research areas such as topology and analysis, and even beyond mathematics to aspects of modern theoretical physics. One of the primary drivers for this expansion of $G_2$ Geometry is the large number of PhD students, postdoctoral researchers and other early career researchers who have joined the field in recent years. As a consequence, we deliberately made the meeting a forum for researchers at an early career stage: the vast majority of the speakers and participants were PhD students, with the rest of the speakers either postdocs or researchers who had recently been postdocs. This gave the meeting a vigorous energy which led to lively discussions, especially since most of the work presented was in progress and was explained at a level that was understandable to the PhD students present. To facilitate these discussions, as well as to provide plenty of opportunities for informal meetings, we ran a successful Open Problem session early in the workshop, and an invaluable Review session towards the end of the workshop. It was clearly evident that there was a great level of interaction between the PhD students and the more senior attendees at the workshop, which was markedly more than at a typical workshop.

2 Overview of the Field

The holonomy group associated to a Riemannian metric is generated by the parallel transport maps around loops in the manifold. The holonomy group is generically the special orthogonal group $SO(n)$ if the manifold is $n$-dimensional, and reductions in the holonomy group are equivalent to various special structures on Riemannian manifolds. For example, on $2m$-dimensional manifolds, a reduction of the holonomy group to the unitary group $U(m)$ corresponds to the metric being Kähler, and a reduction of the holonomy group to $SU(m)$ corresponds to the metric being Calabi–Yau, which implies that the metric is Ricci-flat and Kähler.

The modern study of $G_2$ Geometry began with Berger’s celebrated result in 1955 which classified the possible non-trivial holonomy groups which can occur for a Riemannian $n$-manifold:

- $SO(n)$ \hspace{10pt} (any $n$);
- $U(m)$ \hspace{10pt} ($n = 2m$);
- $SU(m)$ \hspace{10pt} ($n = 2m$);
- $Sp(k)$ \hspace{10pt} ($n = 4k$);
- $Sp(k)Sp(1)$ \hspace{10pt} ($n = 4k$);
- $G_2$ \hspace{10pt} ($n = 7$);
- $Spin(7)$ \hspace{10pt} ($n = 8$).
We see immediately that the only non-trivial possibility for a reduction of the holonomy group in odd dimensions occurs in dimension 7 and the holonomy group in this case must be the exceptional Lie group $G_2$. A Riemannian metric with holonomy $G_2$ is necessarily Ricci-flat, and so finding holonomy $G_2$ metrics gives the only currently known method to construct non-trivial examples of Ricci-flat metrics in odd dimensions.

However, a key point is that Berger’s list of holonomy groups does not say that these groups actually occur as holonomy groups, but only that they can occur. In fact, Berger’s original list contained the additional possibility of $\text{Spin}(9)$ as a holonomy group, but it was shown that this in fact only ever arises in a trivial way: more accurately, any Riemannian manifold with $\text{Spin}(9)$ holonomy must be a (locally) symmetric space. Therefore the first challenge in the field, which continues to drive research in the area, is to find examples of holonomy $G_2$ metrics.

2.1 $G_2$-manifolds

After Bryant first proved the local existence of metrics with holonomy $G_2$ in 1985, Bryant and Salamon soon constructed the first examples of complete metrics with holonomy $G_2$: these metrics are asymptotically conical and play a crucial role in the field. These examples justified the notion of $G_2$-manifold: a manifold endowed with a Riemannian metric whose holonomy is contained in $G_2$.

Then in 1996, Joyce constructed the first compact examples of holonomy $G_2$-manifolds, which was a fundamental breakthrough in the field, and the analytic theory developed by Joyce underpins all known methods to construct compact $G_2$-manifolds. In 2003 Kovalev gave a new construction for compact holonomy $G_2$-manifolds, based on a idea of Donaldson; this construction was later extended by Corti–Haskins–Nordström–Pacini. Based on these constructions, there are now known to be many examples of compact $G_2$-manifolds.

2.2 $G_2$-structures

The key to understanding and constructing $G_2$-manifolds goes via $G_2$-structures: 3-forms on 7-manifolds satisfying a certain positivity condition. A $G_2$-structure determines a metric and an orientation on a 7-manifold, and the condition for the $G_2$-structure to define a metric with holonomy contained in $G_2$ is the so-called torsion-free condition: namely that the 3-form is parallel for the Levi-Civita connection of the metric it defines or, equivalently, that it is closed and co-closed (again, using the metric and orientation that it defines). This is a nonlinear differential equation for the 3-form.

Although the main interest is in torsion-free $G_2$-structures, one can also consider splitting the torsion-free condition into two sub-cases: those which are closed and those which are co-closed. In fact, the co-closed condition is essentially vacuous: on any 7-manifold (compact or otherwise), a $G_2$-structure can be deformed to a co-closed one by the h-principle. By contrast, the closed condition is vital for all known constructions of compact $G_2$-manifolds, and is poorly understood.

2.3 Gauge theory and calibrated geometry

Donaldson–Thomas and Donaldson–Segal pioneered the notion of gauge theory in higher dimensions, and in particular in the setting of $G_2$ geometry. In particular, they defined $G_2$-instantons, which are connections generalising the more familiar anti-self-dual instantons from 4-dimensional geometry. Specifically, $G_2$-instantons are connections whose curvature satisfies the condition that its 2-form part lies pointwise in the Lie algebra $g_2$ of $G_2$, viewed as a subspace of the 2-forms. On $G_2$-manifolds, $G_2$-instantons are automatically Yang–Mills connections. That is, they are critical points of the Yang–Mills functional. The proposal is to try to build enumerative invariants for compact $G_2$-manifolds by “counting” $G_2$-instantons.

There is a close relationship between $G_2$ gauge theory and a ‘dual’ theory of certain submanifolds. On a $G_2$-manifold, the $G_2$-structure and its Hodge dual are calibrations; that is, they are closed differential forms with comass one. The submanifolds calibrated by these calibrations (those submanifolds on which the forms restrict to be the volume form) are called associative and coassociative submanifolds, and they are automatically homologically volume-minimizing. There are also conjectures suggesting that one can build enumerative invariants using calibrated submanifolds.
2.4 Related geometries

There are two close cousins to $G_2$ geometry: SU(3) geometry in 6 dimensions and Spin(7) geometry in 8 dimensions.

Of particular relevance in 6 dimensions are Calabi–Yau 3-folds which have metrics with holonomy SU(3), and nearly Kähler 6-manifolds which have the property that the Riemannian cone on them has a torsion-free $G_2$-structure. In these contexts one has associated problems in gauge theory (namely (pseudo-)Hermitian–Yang–Mills connections) and in calibrated geometry (namely (pseudo-)holomorphic curves and special Lagrangian submanifolds).

In 8 dimensions the most important geometry comes from metrics with holonomy Spin(7), giving Spin(7) manifolds: these include Calabi–Yau 4-folds and hyperkähler 8-folds as special cases. This yields some corresponding geometries in 7 dimensions: for example, nearly $G_2$-manifolds, which have a co-closed $G_2$-structure (called a nearly parallel $G_2$-structure) with the property that the Riemannian cone on them has holonomy contained in Spin(7); and Sasaki–Einstein 7-manifolds, where the Riemannian cone on it is Calabi–Yau. More generally, one can try to understand classes of Spin(7)-structures, which are defined a certain type of very restrictive nondegenerate 4-form on an 8-manifold. In particular, closed Spin(7)-structures are necessarily torsion-free and so define a metric with holonomy contained in Spin(7).

2.5 Physics

Another key direction of interest in $G_2$ geometry comes from theoretical physics. Compact $G_2$-manifolds, and compact 7-manifolds with other types of $G_2$-structures, appear when compactifying String Theory and M-Theory, as well as in the study of anomaly cancellation in heterotic String Theory. In this context, $G_2$-instantons on compact $G_2$-manifolds are important because they minimize the Yang–Mills action, and calibrated submanifolds play a crucial role because they minimize volume.

There are several groups of researchers in theoretical physics actively pursuing $G_2$ geometry, and the physics perspective motivates multiple research directions in $G_2$ geometry for pure mathematics. In particular, the physics viewpoint leads to various predictions which remain conjectural mathematically.

3 Recent Developments and Open Problems

3.1 $G_2$-manifolds

Recently, there have been various successful generalisations of the known constructions of compact $G_2$-manifolds which could lead to further examples, including by Joyce–Karigiannis (who extend the Joyce construction) and Nordström (who extends the Kovalev construction).

In another direction, there has been progress in the rigorous construction of complete non-compact $G_2$-manifolds which had been predicted by physicists. This work by Foscolo–Haskins–Nordström produces infinitely many cohomogeneity one examples which are asymptotically locally conical: the latter are asymptotic to a circle bundle over a Calabi–Yau cone. Foscolo–Haskins–Nordström have also produced infinitely many asymptotically locally conical $G_2$-manifolds which have at most an $S^1$-symmetry. In general, this is contrary to predictions from physics.

The key problem in the study of holonomy $G_2$ metrics remains open:

- which compact 7-manifolds admit holonomy $G_2$ metrics?

Our understanding of this problem is incredibly limited, but there has been some progress on defining topological and analytic invariants of $G_2$ structures by Crowley–Goette–Nordström.

3.2 $G_2$-Laplacian flow

One key problem with current technology for producing compact $G_2$-manifolds is that one must start with a closed $G_2$-structure which is very close to torsion-free and then perturb using Joyce’s analytic technique. An alternative approach to the problem of finding torsion-free $G_2$-structures was suggested by Bryant in 1992: a geometric flow of closed $G_2$-structures called the $G_2$-Laplacian flow. This flow provides the possibility
of making large deformations of closed $G_2$-structures to torsion-free ones. Moreover, it also allows the opportunity to gain further insight into which 7-manifolds that admit closed $G_2$-structures support torsion-free $G_2$-structures.

Despite the $G_2$-Laplacian flow having been introduced almost 30 years ago, the analytic theory has only been developed recently by Bryant–Xu and, most notably, by Lotay–Wei. This has provided the impetus for a major increase in activity in geometric flows of $G_2$-structures, and particularly the $G_2$-Laplacian flow.

Of particular note is that one can obtain surprisingly strong analytic results for the $G_2$-Laplacian flow, which surpass, for example, those of Ricci flow in general dimension. Amongst the most impressive of these results is by Fine–Yau: for a 7-manifold that is a product of a 3-torus and a 4-manifold, so that the 3-tori are associative, the flow exists as long as the torsion (equivalently the scalar curvature) is bounded. Another set of important results (now for a product of a 4-torus and 3-manifold so that the 4-tori are coassociative) by Lambert–Lotay \[6\] was reported on in this meeting.

The key issue in this area is that one needs a closed $G_2$-structure to start the flow and so a major open problem is:

- which compact 7-manifolds admit closed $G_2$-structures?

A related natural problem, which is central to the field, is:

- can a compact 7-manifold admit an exact $G_2$-structure? For example, does the 7-sphere admit a closed (and hence exact) $G_2$-structure?

### 3.3 $G_2$-instantons

An area where there has been a large amount of activity and recent progress is in the study and construction of $G_2$-instantons.

Building on the earlier gluing results of Walpuski, Sá Earp, and Sá Earp–Walpuski for $G_2$-instantons on the Joyce and Kovalev examples of compact $G_2$-manifolds, there has been a great deal of study of the relationship between $G_2$-instantons and associative 3-folds, and the Seiberg–Witten equations with multiple spinors on 3-manifolds. In particular, there have been significant results by Haydys, Walpuski, Haydys–Walpuski, and Doan–Walpuski.

In another direction, Oliveira, Clarke, and Lotay–Oliveira have constructed new examples and have studied the moduli space of cohomogeneity one $G_2$-instantons on cohomogeneity one $G_2$-manifolds, including the Bryant–Salamon $G_2$-manifolds and asymptotically locally conical $G_2$-manifolds. Moreover, Ball–Oliveira have constructed homogeneous $G_2$-instantons on Aloff–Wallach spaces (which are nearly $G_2$-manifolds), and have used them to distinguish between nearly parallel $G_2$-structures on the same Aloff–Wallach space.

In general, the key open problem in the field of $G_2$-instantons, aside from the many analytic issues, is:

- can $G_2$-instantons be used to distinguish between compact $G_2$-manifolds? For example, can they be so used for the known compact $G_2$-manifolds?

### 4 Presentation Highlights

The research presented at the meeting can be broadly be described using 5 main interrelated themes.

- Instantons
- Symmetries
- Special Structures
- Geometric Flows
- Calibrated Submanifolds

Many of the results discussed touched on more than one of these themes.
4.1 Instantons

The presentations on gauge theory in higher dimensions focused on classification results, deformation theory and construction methods for instantons.

4.1.1 DT-instantons on almost complex 6-manifolds

The first talk on instantons was by Goncalo Oliveira, who described joint work with Gavin Ball which focused on the definition of a notion of DT-instantons on any almost complex 6-manifold. This definition generalised the familiar Hermitian–Yang–Mills connections on Kähler manifolds.

The key result presented related to a study of these DT-instantons on the manifold $F_2$ of flags in $\mathbb{C}^3$, which can be viewed as the homogeneous space $\text{SU}(3)/T^2$. It is well-known, since $F_2$ is the twistor space of $\mathbb{CP}^2$, that it has two natural invariant almost complex structures: one which is integrable and one which is not integrable (in fact the former is part of a Kähler structure and the latter is part of a nearly Kähler structure). The main result was a classification theorem for invariant DT-instantons with respect to each of the aforementioned almost complex structures.

4.1.2 $G_2$-instantons on nearly $G_2$-manifolds and SU(3)-instantons on Sasaki–Einstein 7-manifolds

There are many examples of nearly $G_2$-manifolds, and $G_2$-instantons can exist on them. In fact, they are always endowed with a $G_2$-instanton known as the “canonical connection”. Thus it is natural to study these instantons. Sasaki–Einstein 7-manifolds are particular examples of nearly $G_2$-manifolds and Ragini Singhal explained how one can naturally define a notion of SU(3)-instantons on Sasaki–Einstein 7-manifolds, which can be compared to $G_2$-instantons.

Singhal’s focus was on the deformation theory of the two types of instantons, and her main result was progress towards understanding deformations of the canonical connection for homogeneous nearly $G_2$ and Sasaki–Einstein 7-manifolds.

4.1.3 Deformation theory of $G_2$-instantons on asymptotically conical $G_2$-manifolds

Given the recent progress made in constructing asymptotically conical $G_2$-manifolds, and in constructing $G_2$-instantons on them, it is natural to continue to pursue the study of gauge theory on asymptotically conical $G_2$-manifolds. Of particular interest is the question of whether the known $G_2$-instantons, which typically have a large symmetry group (a key aspect of their construction), are unique in some appropriate sense.

Joe Driscoll considered the more general question of deforming asymptotically conical $G_2$-instantons on asymptotically conical $G_2$-manifolds: these are the $G_2$-instantons which converge to a dilation invariant $G_2$-instanton at infinity, which is equivalent to a pseudo-Hermitian–Yang–Mills connection (or SU(3)-instanton) on the nearly Kähler link of the asymptotic cone. A simple but important example of such a $G_2$-instanton is the so-called “standard instanton” on $\mathbb{R}^7$, first constructed by Fairlie and Nuyts, which has gauge group $G_2$ and $\text{SO}(7)$-symmetry.

Driscoll was able to give a local description of the moduli space of asymptotically conical $G_2$-instantons, and his main result was to use this deformation theory to show that the standard instanton on $\mathbb{R}^7$ is locally unique.

4.1.4 $G_2$-instantons on Joyce–Karigiannis $G_2$-manifolds

After providing an overview of the Joyce and Joyce–Karigiannis constructions of compact $G_2$-manifolds and Walpuski’s construction technique for $G_2$-instantons on Joyce’s compact $G_2$-manifolds, Daniel Platt described work in progress towards generalising Walpuski’s construction to provide examples of $G_2$-instantons on the Joyce–Karigiannis $G_2$-manifolds.

Of particular interest is that, unlike in the original construction of Walpuski, one would expect to be able to produce many examples of $G_2$-instantons, thus providing a rich gauge theory. Platt explained how one may be able to achieve these examples by considering the situation where the $G_2$-manifold is a product of a circle with a Calabi–Yau 3-fold, and relating $G_2$-instantons to stable bundles on the Calabi–Yau 3-fold.
4.2 Symmetries

The talks concerning symmetries focused on generalisations of toric geometry, closed $G_2$-structures, and cohomogeneity one methods.

4.2.1 Toric geometry of exceptional holonomy manifolds

Toric geometry has been a very useful tool in the study of Kähler manifolds. Thomas Madsen described his joint work with Andrew Swann [7, 8] where they consider toric geometry of $G_2$ and Spin(7)-manifolds: that is, where the $G_2$ or Spin(7)-manifold admits an action by a 3 or 4-dimensional torus preserving the ambient structure, respectively.

In particular, Madsen explained the definition of a multi-moment map (generalising the standard notion of moment map from symplectic geometry) and how one obtains a trivalent graph in the image of the multi-moment map, which describes the singular orbits of the torus action.

4.2.2 Closed $G_2$-structures with symmetry

Although holonomy $G_2$ metrics on compact manifolds cannot admit continuous symmetries, it is natural to study closed $G_2$-structures with symmetry to see whether one can find interesting examples of compact 7-manifolds with closed $G_2$ structures, in particular to study the question of whether one can have an exact $G_2$-structure on a compact 7-manifold.

Alberto Raffero first described his joint work with Fabio Podestà [10], which gave strong restrictions on the automorphism group of a closed $G_2$-structure on a compact manifold. In particular, the main result is that there are no non-trivial homogeneous or cohomogeneity one compact closed $G_2$-structures.

Raffero then described joint work with Marisa Fernández and Anna Fino [5] which studied left-invariant closed $G_2$-structures on solvable Lie groups. The main result is that, in contrast to the symplectic setting, they find unimodular examples admitting closed $G_2$-structures, and give a classification result for such examples. As a consequence they find an example of an expanding $G_2$-Laplacian soliton. Moreover, they find a unimodular example whose Lie algebra has $b_3 = 0$, so that it has exact left-invariant $G_2$-structures, but they prove it does not admit any compact quotient.

4.2.3 Cohomogeneity one manifolds with exceptional holonomy

Cohomogeneity one techniques have proved to be a powerful tool in geometry, and have been essential in the construction of complete metrics with $G_2$ and Spin(7) holonomy going back to the first examples of such metrics by Bryant and Salamon. Fabian Lehmann provided a detailed overview of cohomogeneity one methods and how they can be used to construct complete metrics with exceptional holonomy.

The main results Lehmann described were one of the constructions of asymptotically locally conical and asymptotically conical $G_2$ manifolds due to Foscolo–Haskins–Nordström, and his own work constructing new examples of asymptotically locally conical and asymptotically conical Spin(7) manifolds.

4.2.4 Toric nearly Kähler manifolds

In a similar spirit to Thomas Madsen’s talk, Kael Dixon studied toric geometry of nearly Kähler 6-manifolds; namely, nearly Kähler 6-manifolds admitting an action of the 3-torus preserving the ambient structure.

One main result was a complete description of the standard homogeneous nearly Kähler $S^3 \times S^3$ in toric terms, which built on work of the speaker in [3]. The other key theorem Dixon presented was a local description of all toric nearly Kähler 6-manifolds in terms of a certain second order nonlinear partial differentiation equation.

4.3 Special Structures

In the study of special structures in $G_2$ geometry and related topics, the presentations focused on balanced Spin(7)-structures and closed $G_2$-structures.
4.3.1 A spinorial approach to balanced Spin(7)-structures

Spin(7)-structures on 8-manifolds can be equivalently be described using nowhere vanishing spinors instead of certain 4-forms. Lucía Martín-Merchán described her work in [9] which gave a spinorial classification of Spin(7)-structures, and in particular identified so-called balanced Spin(7)-structures with harmonic unit spinors.

Martín-Merchán then described her joint work with Giovanni Bazzoni and Vicente Muñoz in [2] which studied 8-manifolds given as the product of a 5 or 6-dimensional nilmanifold with a 3 or 2-dimensional torus, respectively. The main result here was a classification of left-invariant balanced Spin(7)-structures when choosing a 5-dimensional nilmanifold, and examples and a partial classification for the 6-dimensional nilmanifold case.

4.3.2 Quadratic closed G_2-structures

Gavin Ball described his work on what are known as quadratic closed G_2-structures: closed G_2-structures whose torsion (which can be identified with a 2-form) has the property that its exterior derivative is a 3-form that is quadratic in the torsion itself. These generalise the torsion-free G_2-structures (for which the torsion is zero) and the extremally Ricci pinched closed G_2-structures introduced by Bryant.

Ball’s main study was on quadratic closed G_2-structures whose pointwise stabilizer can be identified with one of the two conjugacy classes of U(2) in G_2. This led to: new examples of extremally Ricci pinched closed G_2-structures; Weierstrass formulae which classify some quadratic closed G_2-structures; another classification result involving links to semi-flat T^4-fibrations and maximal spacelike submanifolds contained in a certain quadric in \( \mathbb{R}^{3,3} \); and new examples of G_2-Laplacian solitons including new examples of gradient solitons.

4.4 Geometric Flows

The talks on geometric flows focused on two different areas: the G_2-Laplacian flow with symmetries and a flow of isometric G_2-structures.

4.4.1 G_2-Laplacian flow and spacelike mean curvature flow

Given the recent important on the G_2-Laplacian flow, both in general and in special cases, one is strongly motivated to other situations beyond the general setting where one can get potentially stronger results.

Ben Lambert described joint work with Jason Lotay [6] looking at a special case of the G_2-Laplacian flow, where the 7-manifold is the product of a 4-torus with a 3-manifold, and the closed G_2-structure defines a semi-flat coassociative fibration. In this case, the G_2-Laplacian flow may be identified with the mean curvature flow of spacelike 3-dimensional submanifolds in \( \mathbb{R}^{3,3} \).

The main result Lambert described was when the 3-manifold is \( \mathbb{R}^3 \), where one obtains long-time existence for spacelike mean curvature flow in \( \mathbb{R}^{3,3} \) (and thus the G_2-Laplacian flow) for any initial data. This is a very surprising result since it is the first of its kind for the G_2-Laplacian flow when it is a nonlinear partial differential system which makes no smallness assumption for the initial data or curvature/torsion assumption along the flow.

4.4.2 S^1-invariant G_2-Laplacian flow

In contrast to Ben Lambert’s talk, where one assumes a large amount of symmetry, here Udhav Fowdar described work on the G_2-Laplacian flow with the least amount of continuous symmetry, namely \( S^1 \)-symmetry. Fowdar first showed that the \( S^1 \)-invariant G_2-Laplacian flow is equivalent to a coupled system for SU(3)-structures on a 6-manifold and connections on an \( S^1 \)-bundle over the 6-manifold. This generalised the well-known work of Apostolov–Salamon on \( S^1 \)-quotients of G_2-manifolds. The main results were then a study of the \( S^1 \)-invariant G_2-Laplacian flow and the coupled system on the quotient on two particular examples.

First, for a left-invariant closed G_2-structure studied by Fernández and Bryant, which is on a \( T^4 \)-bundle over \( T^3 \), for which the G_2-Laplacian flow exists for all time, Fowdar showed that the flow of SU(3)-structures
on the $S^1$-quotient has constant symplectic form but the almost complex structure degenerates as time goes to infinity.

Second, Fowdar looked at closed $G_2$-structures on the product of $\mathbb{R}^+\times \mathbb{R}^4$ with a 6-dimensional nilmanifold (endowed with a 1-parameter family of left-invariant structures) which is a $T^2$-bundle over $T^4$. Here, the $S^1$-quotient turns out to be Kähler and the Kähler condition is preserved along the flow, leading to the question of whether or not this is a general phenomenon.

### 4.4.3 A flow of isometric $G_2$-structures

Shubham Dwivedi gave a comprehensive overview of his work with Panagiotis Gianniotis and Spiro Karigiannis in [4] on a flow of $G_2$-structures which preserves the underlying metric that the $G_2$-structures define, hence the term isometric $G_2$-structures. This flow is a gradient flow of the $L^2$-norm of the torsion, restricted to the class of isometric $G_2$-structures.

The main results were Shi-type estimates, an $\epsilon$-regularity theorem, a control on the size of the singular set at a finite time singularity of the flow, and the fact that Type I singularities are modelled by self-shrinkers.

### 4.5 Calibrated Submanifolds

The talks on calibrated submanifolds covered a range of ambient geometries: nearly Kähler 6-manifolds, hyperkähler 4-manifolds, and 7-manifolds equipped with $G_2$-structures with torsion.

#### 4.5.1 Pseudoholomorphic curves in nearly Kähler 6-manifolds

Holomorphic curves are an essential part of complex (and particularly Kähler) geometry and pseudoholomorphic curves play a crucial role in symplectic geometry. Nearly Kähler 6-manifolds are neither complex nor symplectic, but their pseudoholomorphic curves are nonetheless important, particularly since the cone on a pseudoholomorphic curve is associative in the $G_2$ cone over the nearly Kähler 6-manifold. That said, the general theory of pseudoholomorphic curves in nearly Kähler 6-manifolds has not previously been studied.

Benjamin Aslan described his study of this general theory, with a focus on twistor spaces, namely $\mathbb{C}P^3$ and the flag manifold $F_2$. The main result was a classification of $S^1$-invariant pseudoholomorphic curves in the nearly Kähler $\mathbb{C}P^3$.

#### 4.5.2 The minimal sphere in the Atiyah–Hitchin manifold

The (double cover of the) Atiyah–Hitchin manifold is a key example of a hyperkähler 4-manifold which is topologically $\mathbb{C}P^2 \setminus \mathbb{R}P^2$ and is closely related to monopoles on $\mathbb{R}^4$. In the Atiyah–Hitchin manifold there is a central 2-sphere that is well-known to be minimal, but not complex for any of the hyperkähler structures.

Chung-Jun Tsai described his work with Mu-Tao Wang in [11] which proves that the minimal 2-sphere is in fact area-minimizing by showing that it is a calibrated submanifold. Moreover, Tsai showed that the minimal 2-sphere is the unique compact minimal submanifold of the Atiyah–Hitchin manifold of dimension 2 or 3, and that the minimal 2-sphere is stable under mean curvature flow.

#### 4.5.3 Minimality and local non-existence of calibrated submanifolds

In the final talk of the meeting, Jesse Madnick described joint work with Gavin Ball in [1], which investigated associative 3-folds and coassociative 4-folds in general 7-manifolds with $G_2$-structures. Associative and coassociative submanifolds are minimal (in fact, volume-minimizing) and have good local existence theory in $G_2$-manifolds, so it is interesting to ask when these properties persist in other $G_2$-structures.

The main result was to give necessary and sufficient conditions on a $G_2$-structure for all associative, and respectively all coassociative, submanifolds to be minimal, by providing a formula for the mean curvature determined by the torsion of the $G_2$-structure. In addition, Madnick described an obstruction to even the local existence of coassociative 4-folds for certain $G_2$-structures.
5 Scientific Progress Made

We summarize the scientific progress made in each of the main themes highlighted in the previous section.

5.1 Instantons

There has clearly been a significant increase in our understanding of higher-dimensional gauge theory beyond the established settings of compact Calabi–Yau, $G_2$, and Spin(7)-manifolds. There have been extensions to non-integrable structures, such as almost complex 6-manifolds, nearly $G_2$-manifolds, and Sasaki–Einstein 7-manifolds, and in the study of the non-compact setting of asymptotically conical $G_2$-manifolds. In particular, we have seen classification and deformation theory results.

In the compact $G_2$-manifold setting, which holds the greatest interest in $G_2$ geometry, there has been exciting progress towards potentially constructing a large number of $G_2$-instantons on the new examples of $G_2$-manifolds due to Joyce–Karigiannis.

5.2 Symmetries

The progress made in the use of symmetries to understand exceptional holonomy and related geometries has yielded both positive and negative results. On the one hand, there is a better understanding of toric geometry in this context and there are new examples of complete metrics with exceptional holonomy. On the other hand, it is now clear that there are significant challenges to using symmetry techniques to understand compact manifolds with closed $G_2$ structures, now that the standard methods have been shown to not provide any non-trivial examples.

5.3 Special Structures

Special structures have received relatively little detailed attention and are generally quite poorly understood. The results presented in the meeting clearly show a marked improvement in our ability to study and understand balanced Spin(7)-structures and closed $G_2$-structures. In particular, the results provided new examples of and classification results for such structures.

5.4 Geometric Flows

There were some interesting results concerning the $G_2$-Laplacian flow with symmetries, building on the general theory developed in recent years. Specifically there were some impressive long-time existence results in the setting of trivial semi-flat coassociative torus fibrations, and some intriguing potential relations between $S^1$-invariant $G_2$-Laplacian flow and Kähler geometry. Both of these results certainly merit further examination and reveal exciting future research avenues for investigation.

The analytic foundations were developed for a flow of isometric $G_2$-structures which is a new research topic that has links to several research groups in $G_2$ geometry, and so will certainly continue to be studied.

5.5 Calibrated Submanifolds

There was notable progress made in the study of calibrated submanifolds outside of the setting of well-known areas of manifolds with special holonomy equipped with their usual calibrations. In particular, the techniques developed to study and classify pseudoholomorphic curves in nearly Kähler 6-manifolds, classify minimal submanifolds in the Atiyah–Hitchin manifold, and analyse the properties of calibrated submanifolds in $G_2$-structures with torsion, are certainly to yield further results in related areas.

6 Outcome of the Meeting

The key outcome of the meeting was the increase in communication and collaboration between researchers in $G_2$ geometry, which has and will continue to lead to exciting new research directions and results. It is particularly worth emphasizing the positive outcome of the meeting for early career researchers present,
mainly for PhD students but also some postdocs and other participants, who unanimously expressed how enjoyable and productive the meeting was for them. Senior researchers also remarked on how refreshing it was to have so many early career researchers interacting significantly with them and each other, which provided a unique opportunity to learn about and to offer input towards the research avenues pursued by the next generation of researchers in the field.

More specifically, the Open Problem session identified several interesting research problems that the participants considered worth pursuing, which we describe below.

### 6.1 Gauge theory

Benoit Charbonneau explained a result of Lewis that states that if a bundle $E$ over a compact Calabi–Yau 4-fold admits a Hermitian–Yang–Mills connection, then any Spin(7)-instanton on that bundle must be Hermitian–Yang–Mills. He therefore posed the problem:

- find a Spin(7)-instanton which is not Hermitian–Yang–Mills on some bundle over a compact Calabi–Yau 4-fold, or prove there are no such Spin(7)-instantons.

As a follow-up, he posed the related problem:

- find a $G_2$-instanton on a bundle over a product of a circle with a compact Calabi–Yau 3-fold which is not the pullback of a Hermitian–Yang–Mills connection, or prove that there are none.

Based on the known relationship between stable bundles and Hermitian–Yang–Mills connections, Spiro Karigiannis asked:

- is there potentially some analogue of the Donaldson–Uhlenbeck–Yau or Hitchin–Kobayashi correspondence for instantons on compact $G_2$/Spin(7)-manifolds?

Derek Harland remarked that while we now know much more about pseudoholomorphic curves on nearly Kähler 6-manifolds, we know much less about instantons on nearly Kähler 6-manifolds. He therefore suggested to try to:

- find more examples of instantons on nearly Kähler 6-manifolds;
- prove an analogue of Walpuski’s gluing result for $G_2$-instantons in the setting of nearly Kähler 6-manifolds;
- classify homogeneous instantons on nearly Kähler 6-manifolds, with all possible structure groups.

He also considered a 6-dimensional nearly Kähler twistor space, where one can take bundles $E$ whose restriction to each twistor fibre is trivial, and asked:

- are there instantons on $E$ which are not pulled back from the base of the twistor fibration?

Finally he asked:

- are there any smooth instantons on $S^6$ with structure group SU(2)?

### 6.2 $G_2$-manifolds

Considering the known examples of complete non-compact $G_2$-manifolds, which are asymptotically cylindrical, asymptotically conical, or asymptotically locally conical and thus respectively have $O(r)$, $O(r^0)$, or $O(r^7)$ volume growth for geodesic balls of radius $r$ as $r \to \infty$, Benoit Charbonneau asked:

- what are the possible volume growths for geodesic balls of radius $r$ as $r \to \infty$ for complete non-compact $G_2$-manifolds?

Spiro Karigiannis recalled the work of Madsen–Swann on toric $G_2$-manifolds which produced incomplete holonomy $G_2$ metrics. He therefore asked:
• can we construct compact holonomy $G_2$-manifolds by gluing building blocks that have incomplete holonomy $G_2$ metrics?

Jason Lotay responded to this by pointing out the fundamental work by Gross–Wilson, which produced hyperkähler metrics on K3 surfaces by gluing in the incomplete Ooguri–Vaga metric, and the recent work by Hein–Sun–Viaclovsky–Zhang, which also produced metrics on K3 surfaces by gluing in an incomplete hyperkähler metric, now on an interval times a 3-dimensional nilmanifold. He therefore suggested:

• find an incomplete holonomy $G_2$ metric on an interval times a 6-dimensional nilmanifold which could be used as a building block in a gluing construction for compact $G_2$-manifolds.

6.3 Special Structures

Henrique Sá Earp posed the following problem:

• formulate the right definition of “extremally Ricci-pinched” for co-closed $G_2$-structures.

Gavin Ball responded to this problem by suggesting the following analogy of his study of quadratic closed $G_2$-structures:

• study co-closed $G_2$-structures such that the exterior derivatives of their torsion forms are quadratic in the torsion.

Following on from this, Ball suggested the following problem:

• find a 1-parameter family of homogeneous co-closed $G_2$-structures containing a nearly parallel $G_2$-structure (that is, a $G_2$-structure whose exterior derivative is a constant multiple of its Hodge dual), but so that all other members of the 1-parameter family are not nearly parallel.

A solution to this problem would show that there is unlikely to be a satisfactory answer to Sá Earp’s question. Finally, Ball asked:

• can a co-closed $G_2$-structure which is purely of “type $\tau_3$” (that is, such that its exterior derivative has zero component in the direction of its Hodge dual), be Einstein?

Sá Earp explained that the condition to be a critical point for the flow of isometric $G_2$-structures is equivalent to a certain natural map associated to isometric $G_2$-structures be harmonic. He explain how this harmonic condition could be extended to other situations by analogy and in particular he suggested:

• study harmonic Spin(7)-structures.

6.4 Calibrated Submanifolds

Jesse Madnick asked the fundamental question:

• are there any topological obstructions for complete, embedded special Lagrangian 3-folds in $\mathbb{C}^3$, or can every topological type occur?

Jason Lotay recalled that Harvey–Lawson proved that, given any real analytic surface in $\mathbb{R}^7$, there is a (locally unique) associative containing that surface. He therefore posed the question:

• when is a real analytic surface in $\mathbb{R}^7$ the boundary of a compact associative 3-fold?

He also posed the related problem:

• given a map $u$ from the boundary of a domain in $\mathbb{R}^3$ to $\mathbb{R}^4$, when does there exist a map on the domain to $\mathbb{R}^4$, with $u$ as its boundary value, so that the graph of the map is associative?

He suggested that there may be some relation to work of Harvey–Lawson on pluripotential theory for calibrated manifolds, and pointed to the fact that Haskins–Pacini have proved some obstructions for the special Lagrangian boundary value problem.
References


