On zeros of the characteristic polynomial of representable matroids of bounded tree-width.

Carolyn Chun [§] Rhiannon Hall [§] Criel Merino[‡] Steven D. Noble[§]

[§]Departament of Mathematical Sciences
Brunel University

[‡]Instituto de Matemáticas UNAM,
sede Oaxaca

Modern Techniques in Discrete Optimization: Mathematics, Algorithms and Applications,
BIRS Oaxaca, 1-6 November 2015

Chromatic polynomial

The chromatic polynomial (for planar graphs) was first defined by Birkhoff in 1912 in an attempt to find an algebraic proof of the then open four-colour problem. The chromatic polynomial $\chi(G;\lambda)$ of a graph G counts, for every positive integer λ, the number of proper colouring of the vertices of G with λ colours. One can easily check that, if e is an edge of G, then

$$P(G,q) = P(G \setminus e, q) - P(G/e, q).$$

Chromatic polynomial

The following theorem is due to G. D. Birkhoff (and independently by H. Whitney1932)

Theorem

If $G = (V, E)$ is a graph, then

$$\chi(G;\lambda) = \lambda^{\kappa(E)} \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(E)-r(A)}. \tag{1}$$

Chromatic polynomial

Since the chromatic polynomial can be evaluated at real and complex values, Birkhoff hoped to use analytic methods to prove the 4-colour conjecture. In fact, the four-colour Theorem is equivalent to stating that $\chi(G;4) > 0$ for all planar graphs G. Such an analytic proof was never found, but Birkhoff and Lewis in 1946 did show that if G is planar then $\chi(G;\lambda) > 0$ for all real $\lambda \geq 5$. They conjectured that 5 could be changed to 4, and almost 70 years later their conjecture is still open.

Conjecture (Birkhoff-Lewis, 1946)

If G is a planar graph, then $\chi(G;\lambda) > 0$ for all real $\lambda \geq 4$.

Chromatic polynomial

What about largest real roots? A family of graphs $\mathcal{G}$ has an upper root-free interval if there is a real number a such that no graph in $\mathcal{G}$ has a real root larger than a. The class of all graphs does not have an upper root-free interval (since you can get arbitrarily large chromatic roots from cliques).

Chromatic polynomial

This is different for a proper minor-closed class of graphs.

Theorem (Mader '67)

For every $k \in \mathbb{N}$ there exists an integer $f(k)$ such that any graph with minimum degree at least $f(k)$ has a K_k-minor.

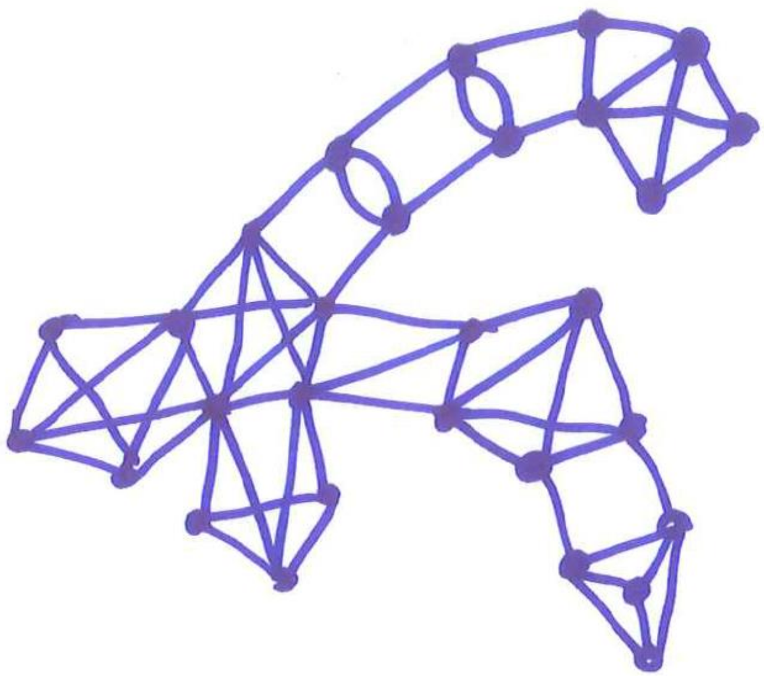
It follows that, for every proper minor-closed family of graphs $\mathcal{G}$, there exists a smallest integer $d = d(\mathcal{G})$ such that every graph in $\mathcal{G}$ has a vertex of degree at most d. Woodall and Thomassen proved the next result independently.

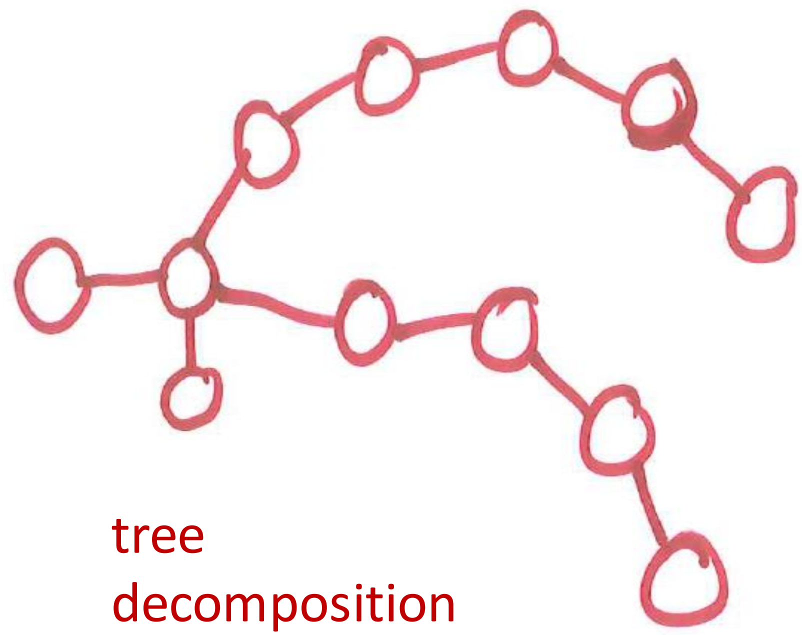
Theorem (Thomassen '97, Woodall '97)

If $\mathcal{G}$ is a proper minor-closed family of graphs, then $(d(\mathcal{G}), \infty)$ is an upper root-free interval for $\mathcal{G}$.

The idea of tree-width in graphs:

To say how “tree-like” the structure of a graph is.

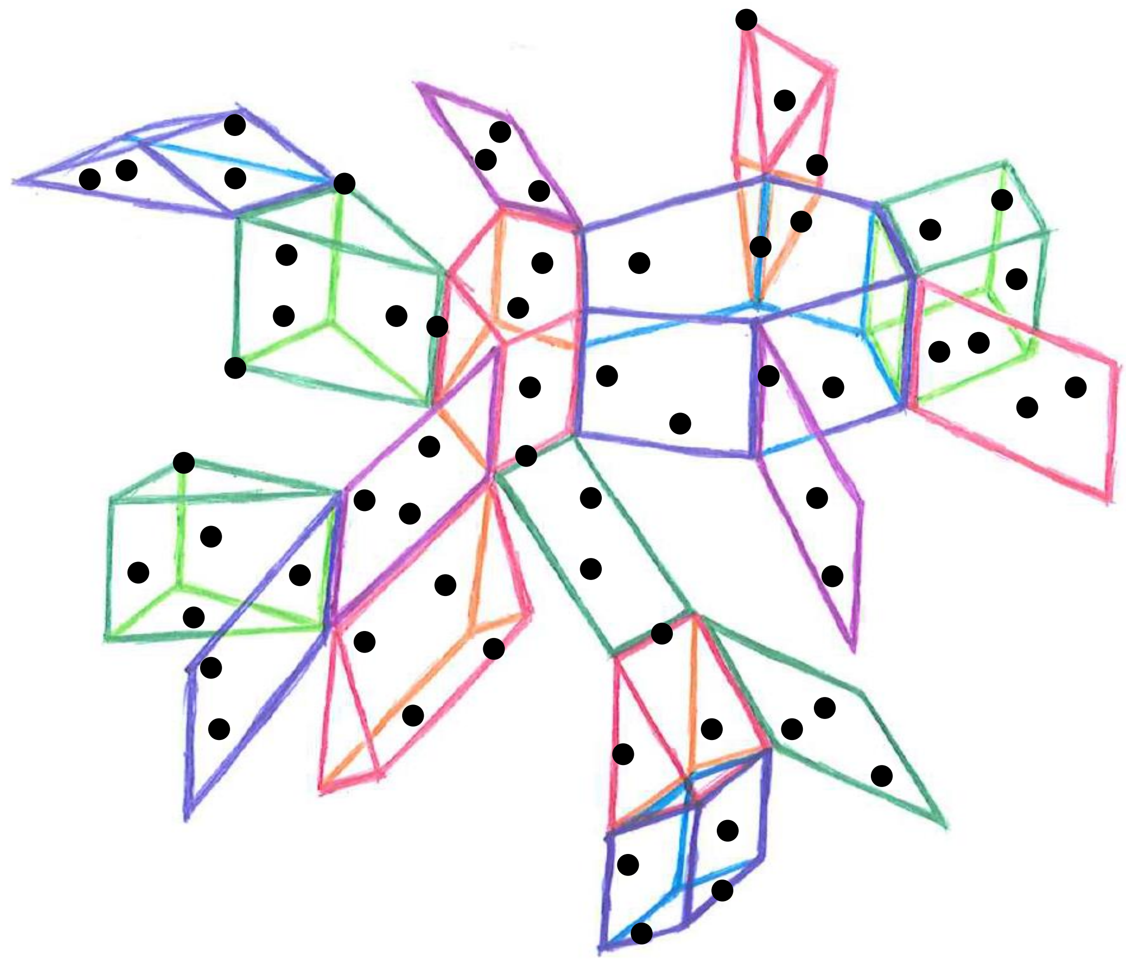




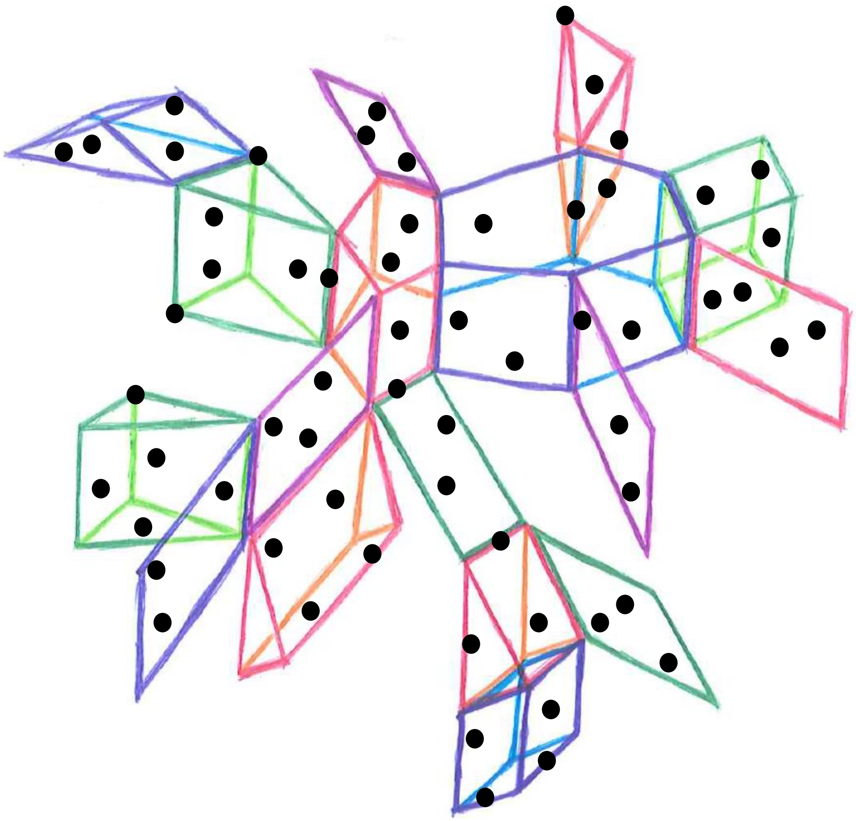
tree
decomposition

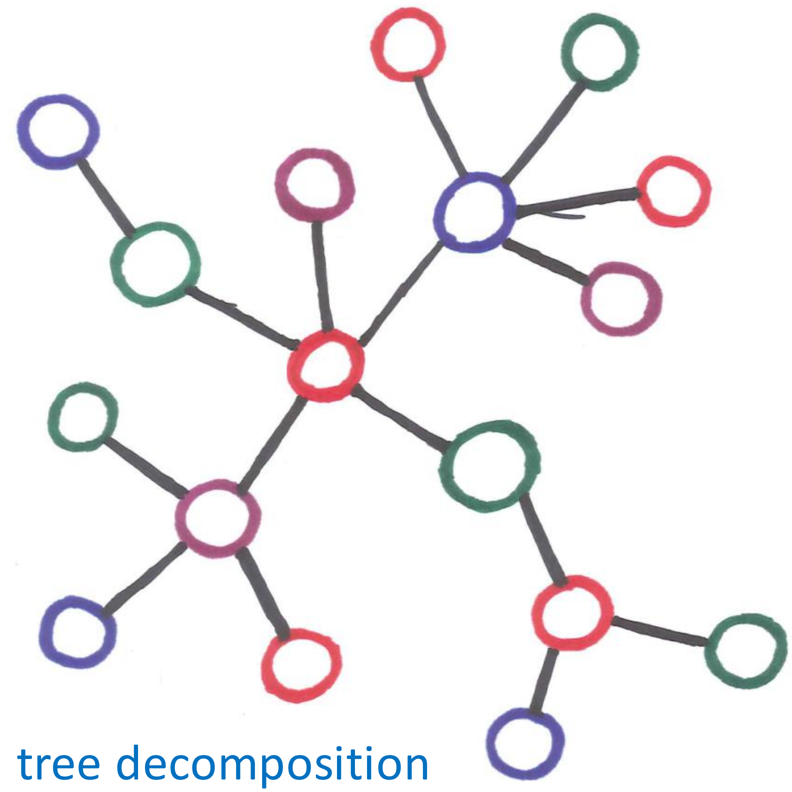
The width of a tree decomposition is the highest number of graph vertices contained in a node of the tree, minus one.

The idea was extended to matroids by Hlineny and Whittle



The idea was extended to matroids by Hlineny and Whittle





Tree decomposition:

- The nodes of T are “buckets”.
- Each element of M goes in one bucket (partitioning $E(M)$). Buckets can be empty or contain many elements.

Tree-Width for Matroids

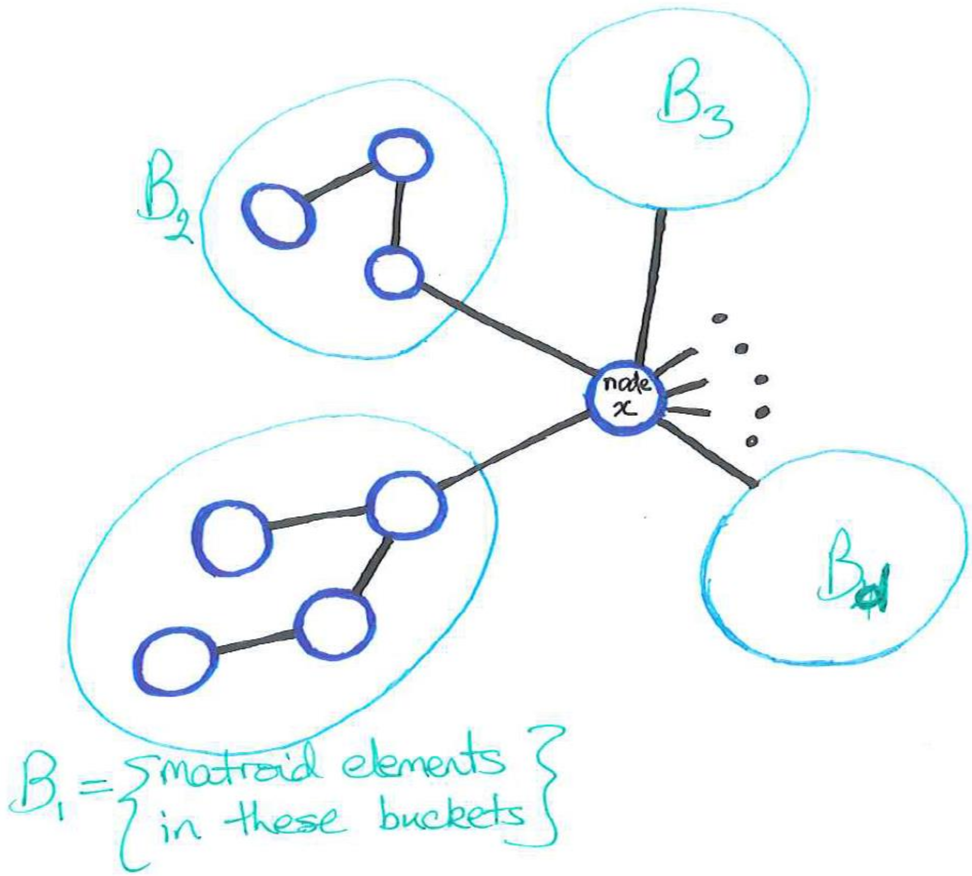
- Given a tree decomposition, each node gets a weight called its *node-width*.
- The *width* of a tree decomposition is the highest value of its node-widths.
- The *tree-width* of a matroid is the minimum width of all its tree decompositions.

Tree-Width for Matroids

- Given a tree decomposition, each node gets a weight called its *node-width*.
- The *width* of a tree decomposition is the highest value of its node-widths.
- The *tree-width* of a matroid is the minimum width of all its tree decompositions.
- How do we measure node-width?

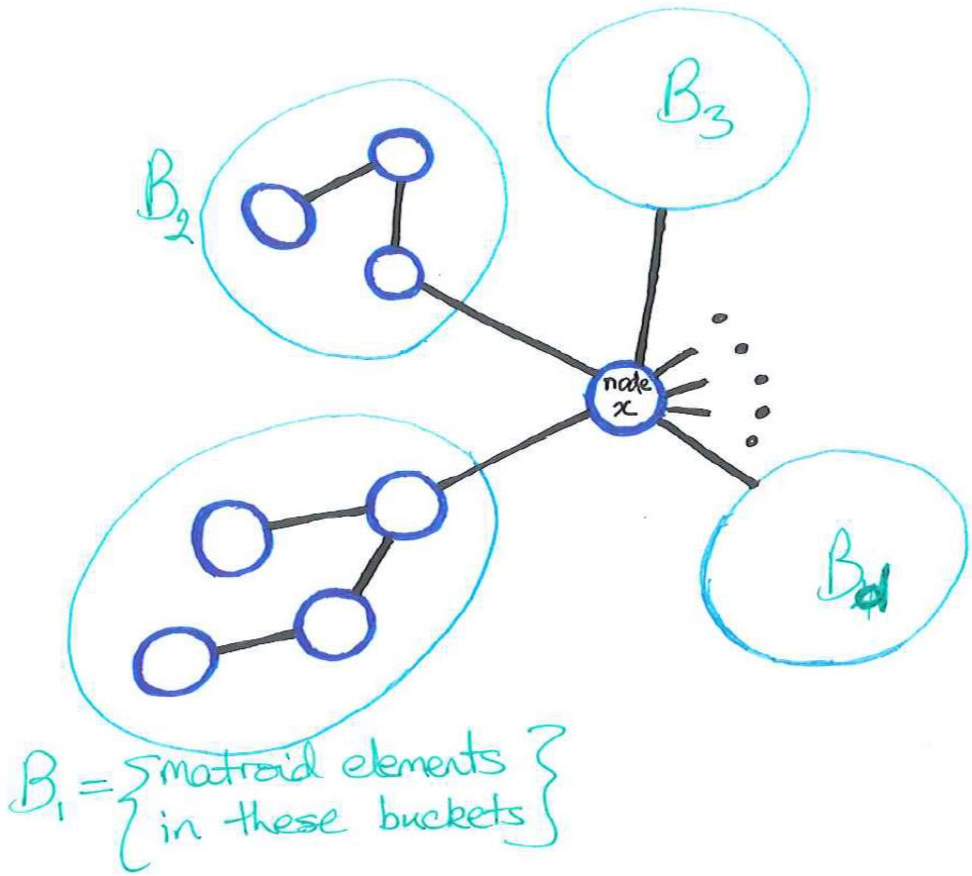
The *width* of a node x of T is

$$W(x) = r(M) - \sum[r(M) - r(E(M) - B_i)]$$
$$= \sum r(E(M) - B_i) - (d-1)r(M).$$



The *width* of a node x of T is

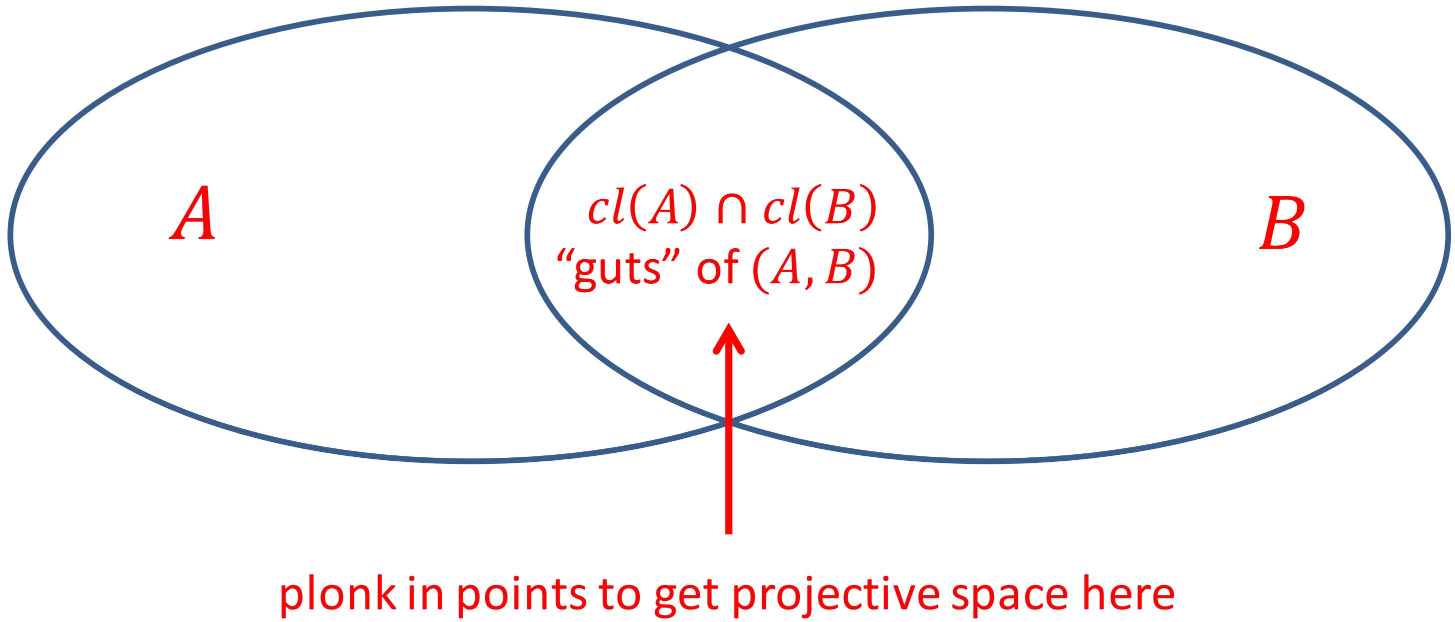
$$W(x) = r(M) - \sum[r(M) - r(E(M) - B_i)]$$
$$= \sum r(E(M) - B_i) - (d-1)r(M).$$



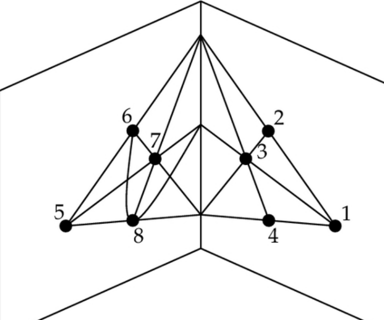
What does this value mean geometrically?

Brief Digression: A note on k-separations in representable matroids.

Let M be a matroid representable over $GF(q)$. Then we can embed M in the projective space $PG(r(M)-1,q)$. Let (A,B) be a k-separation of M.

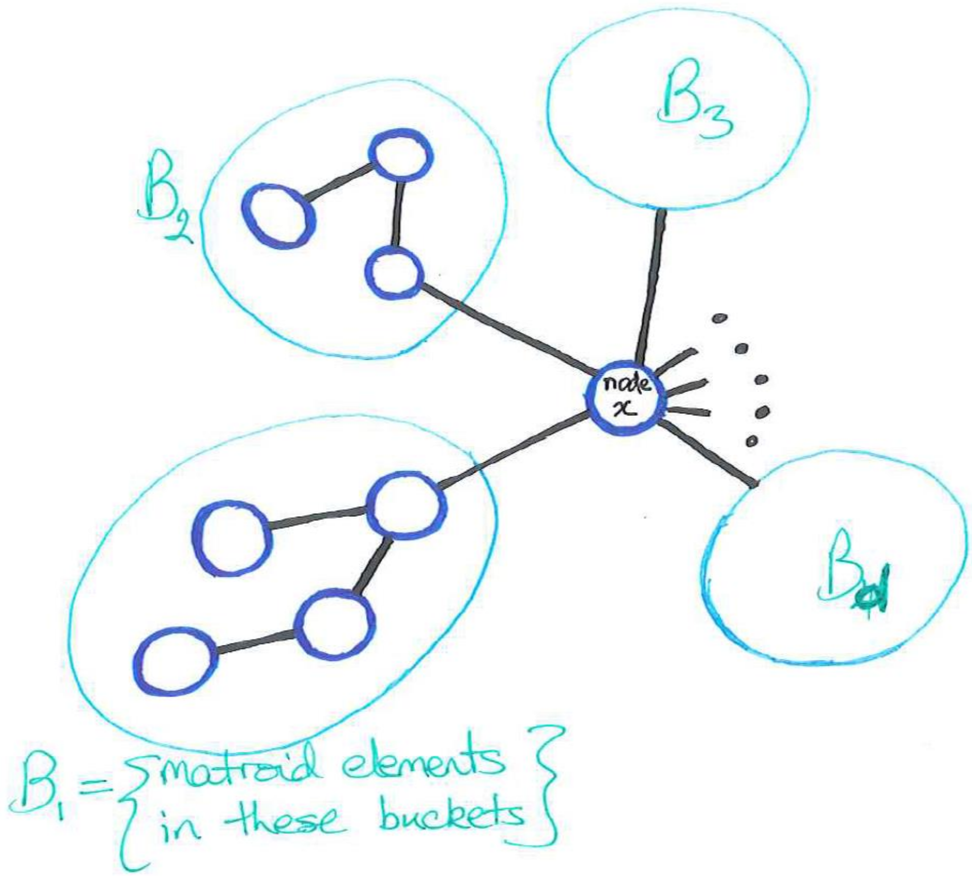


Then we can add points of $PG(r(M)-1,q)$ to M into the guts of (A,B) to obtain matroid M' and k-separation (A',B') such that the guts of (A',B') is the projective space $PG(k-2,q)$.


6
7
2
3
5
8
1
4

The *width* of a node x of T is

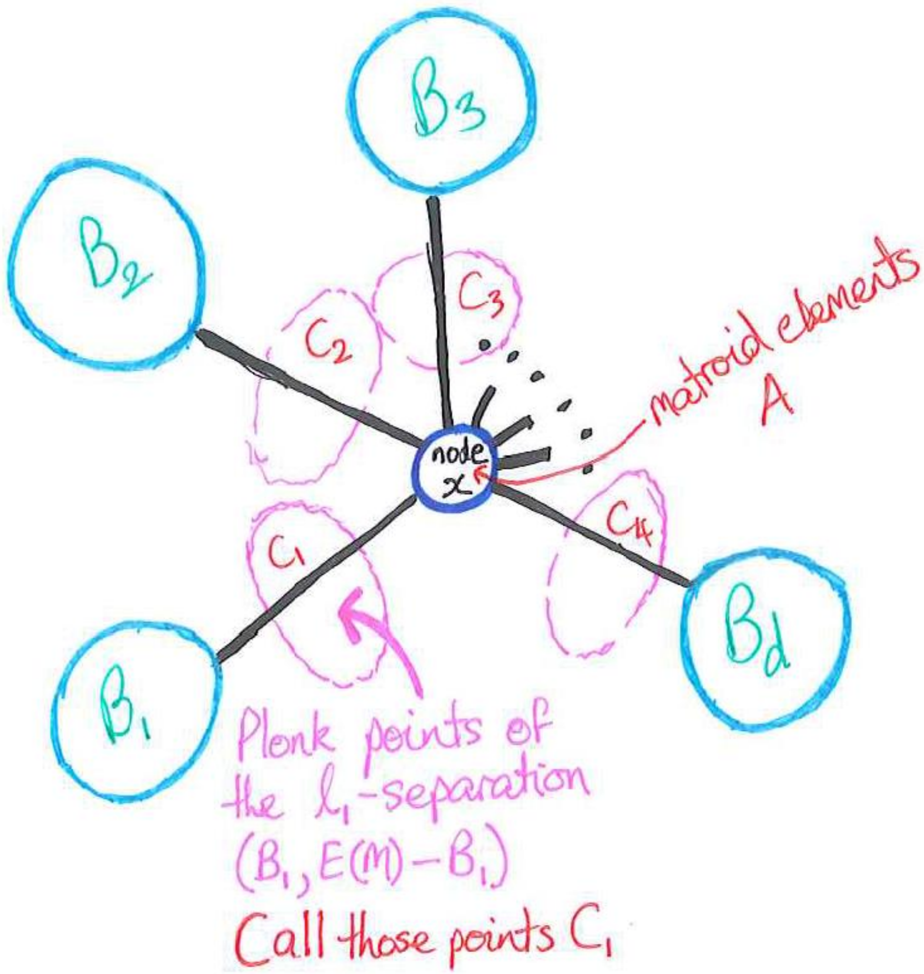
$$W(x) = r(M) - \sum[r(M) - r(E(M) - B_i)]$$
$$= \sum r(E(M) - B_i) - (d-1)r(M).$$



What does this value mean geometrically?

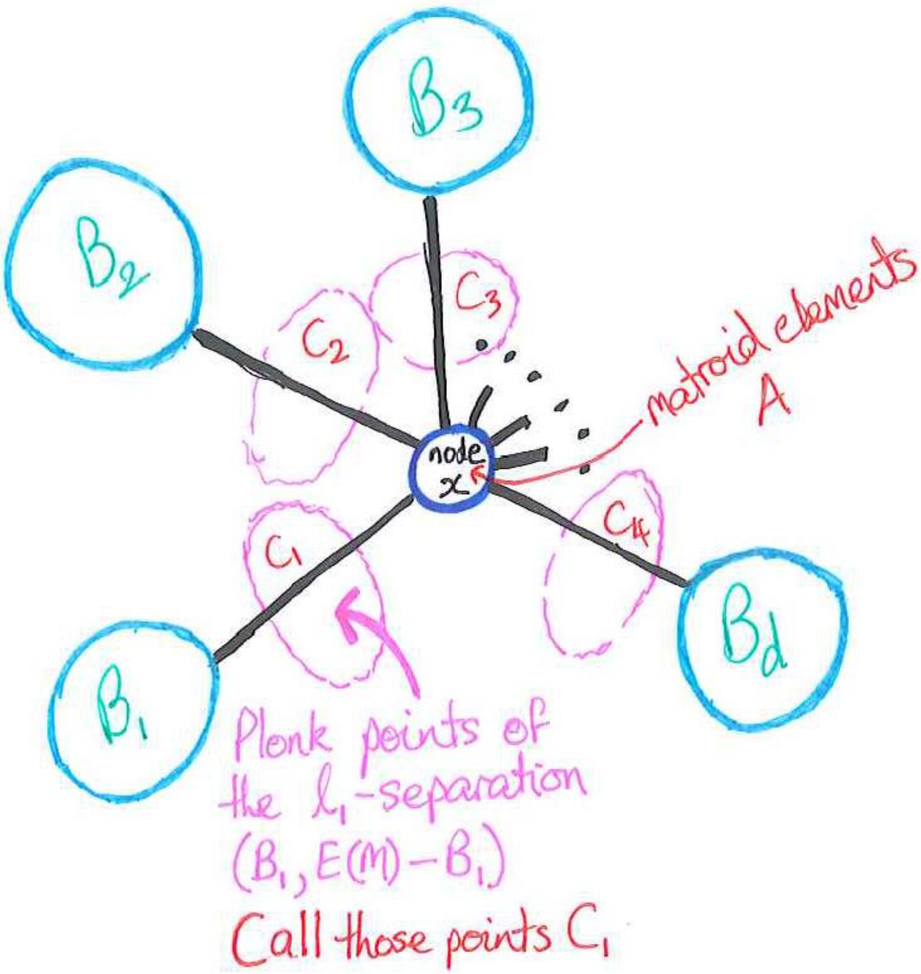
If M is representable, then the width of node x is

$$W(x) = r_{M'}(A \cup C_1 \cup C_2 \cup \cdots \cup C_d).$$



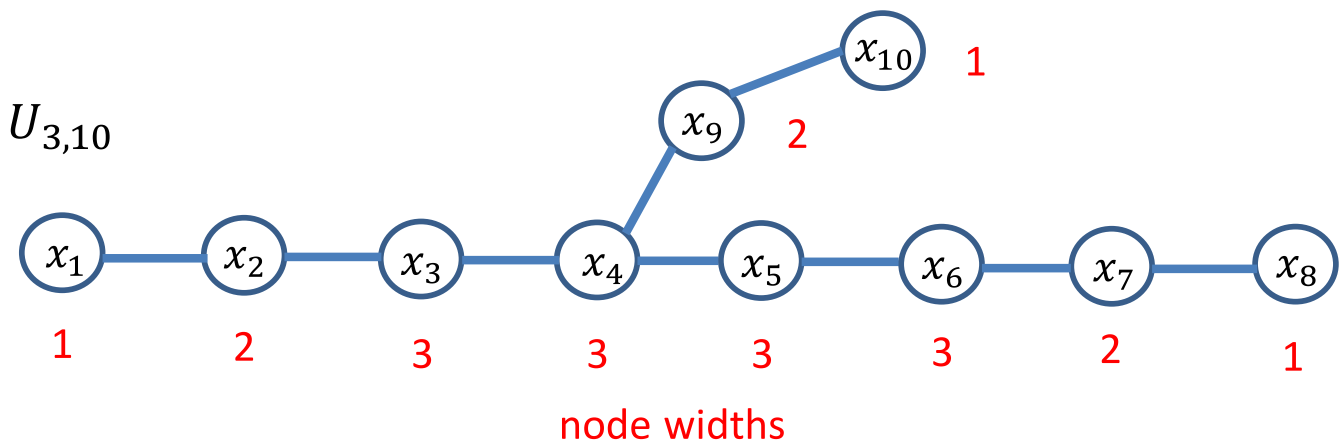
If M is representable, then the width of node x is

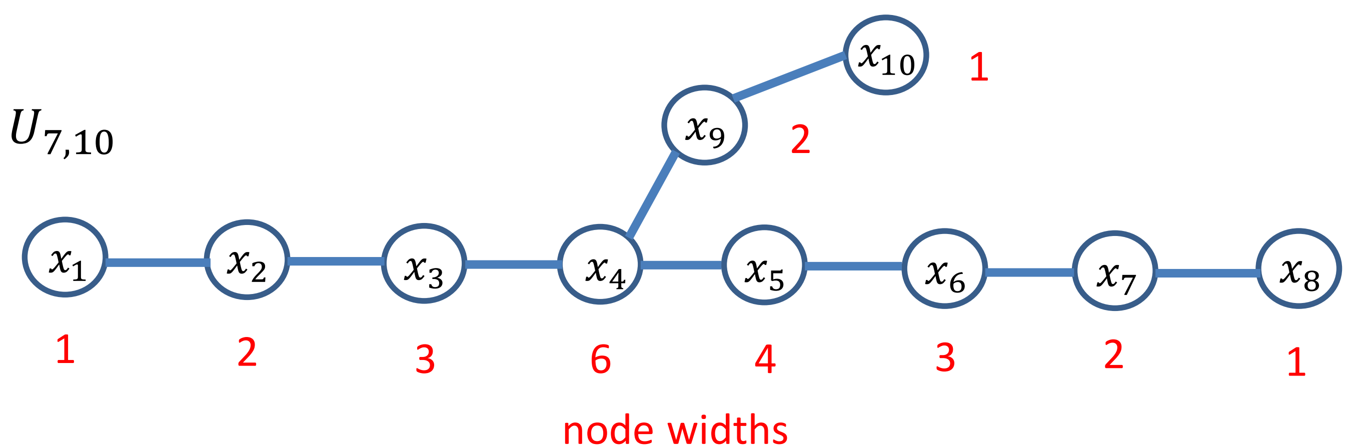
$$W(x) = r_{M'}(A \cup C_1 \cup C_2 \cup \cdots \cup C_d).$$



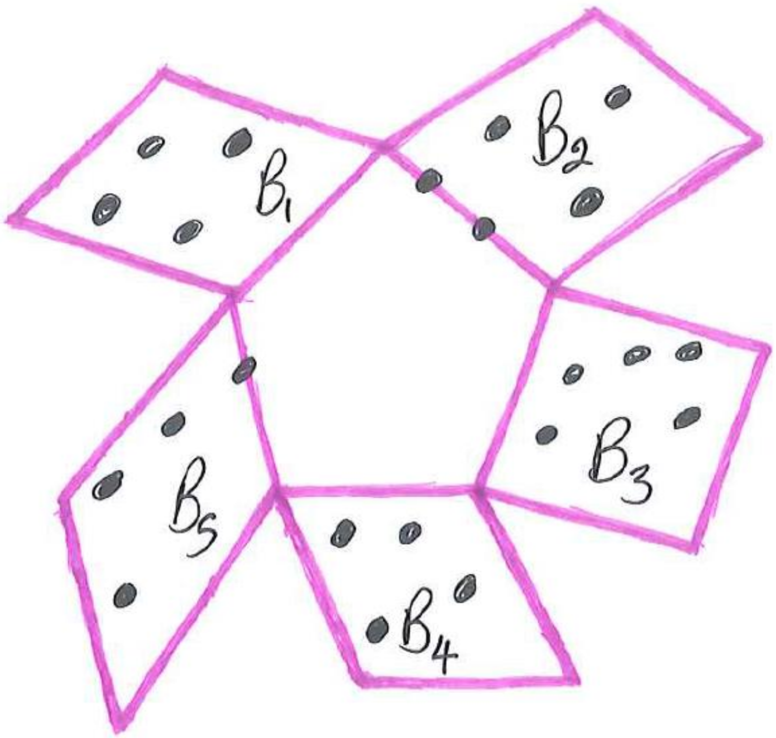
Note: If x is a leaf node of T, and A is the set of matroid elements in the x-bucket, then $W(x) = r_M(A)$.

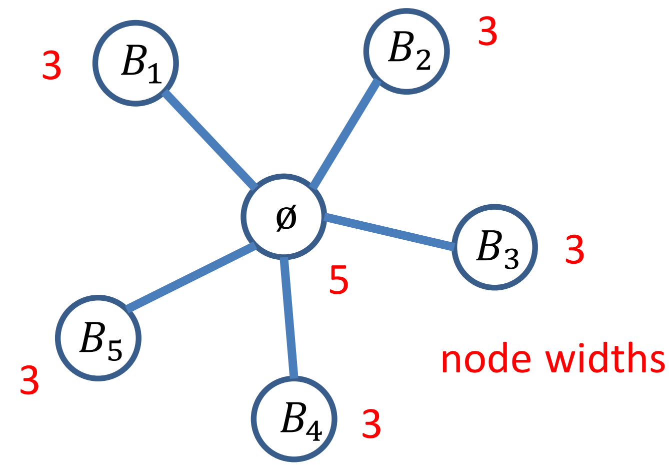
Example: Tree decompositions for $U_{3,10}$ and $U_{7,10}$



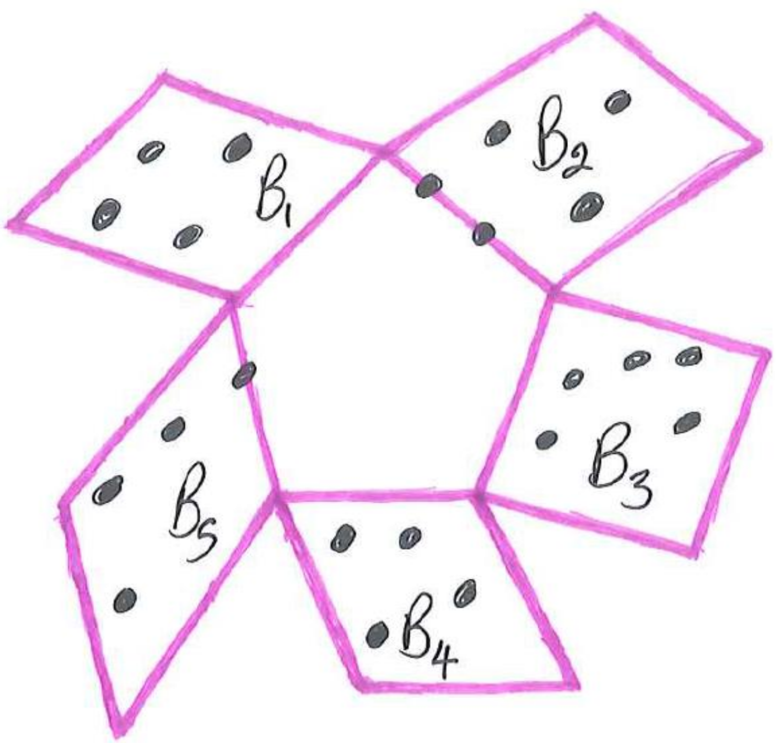


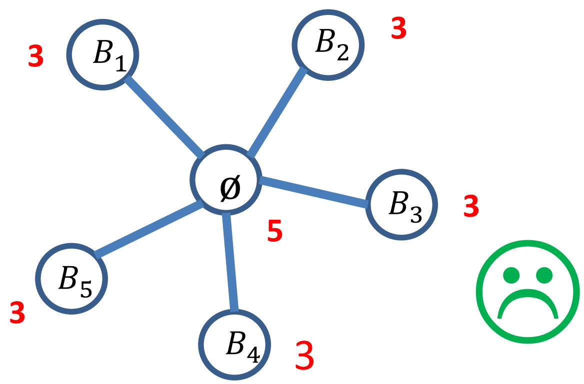
Example: Tree decompositions for a daisy

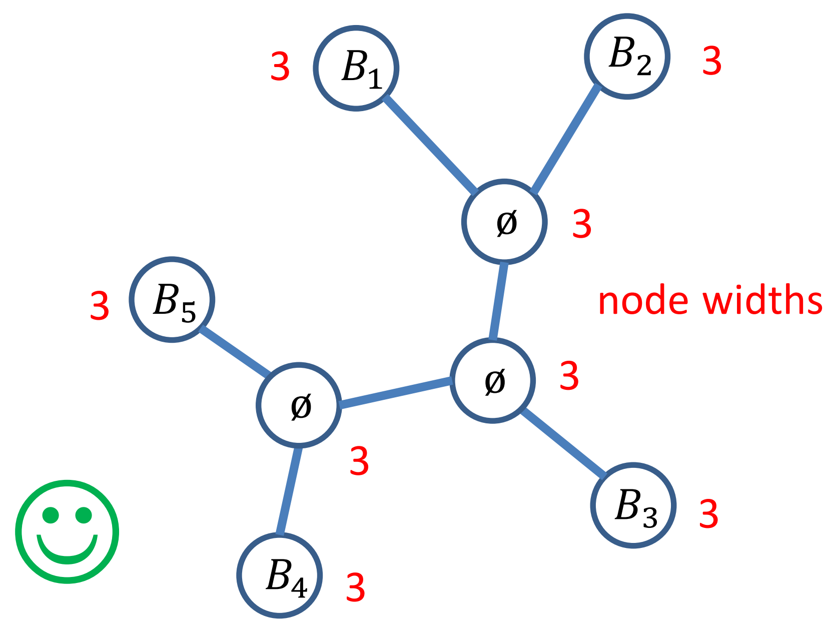




Example: Tree decompositions for a daisy







The Chromatic Polynomial for Matroids

For a matroid M,

$$\chi(M,\lambda) = \sum_{X \subseteq E(M)} (-1)^{|X|} \lambda^{r(M)-r(X)}.$$

The familiar deletion-contraction rules apply:

- If x is not a loop or coloop then $\chi(M,\lambda) = \chi(M\backslash x,\lambda) - \chi(M/x,\lambda)$,
- If x is a loop then $\chi(M,\lambda) = 0$,
- If x is a coloop then $\chi(M,\lambda) = (\lambda - 1)\chi(M\backslash x,\lambda)$.

The Characteristic polynomial

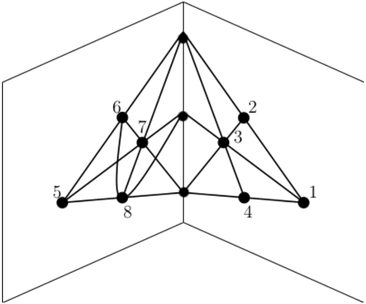
For example,the characteristic polynomial of $U_{m,n}$, $0 < m \leq n$, is $\chi_{U_{m,n}}(\lambda) = \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} (\lambda^{m-k} - 1)$.

For $PG(r-1, q)$, whose lattice of flats is isomorphic to the lattice of subspaces of the r-dimensional vector space over $GF(q)$, has characteristic polynomial
$\chi_{PG(r-1,q)}(\lambda) = (\lambda - 1)(\lambda - q)(\lambda - q^2) \cdots (\lambda - q^{r-1})$.

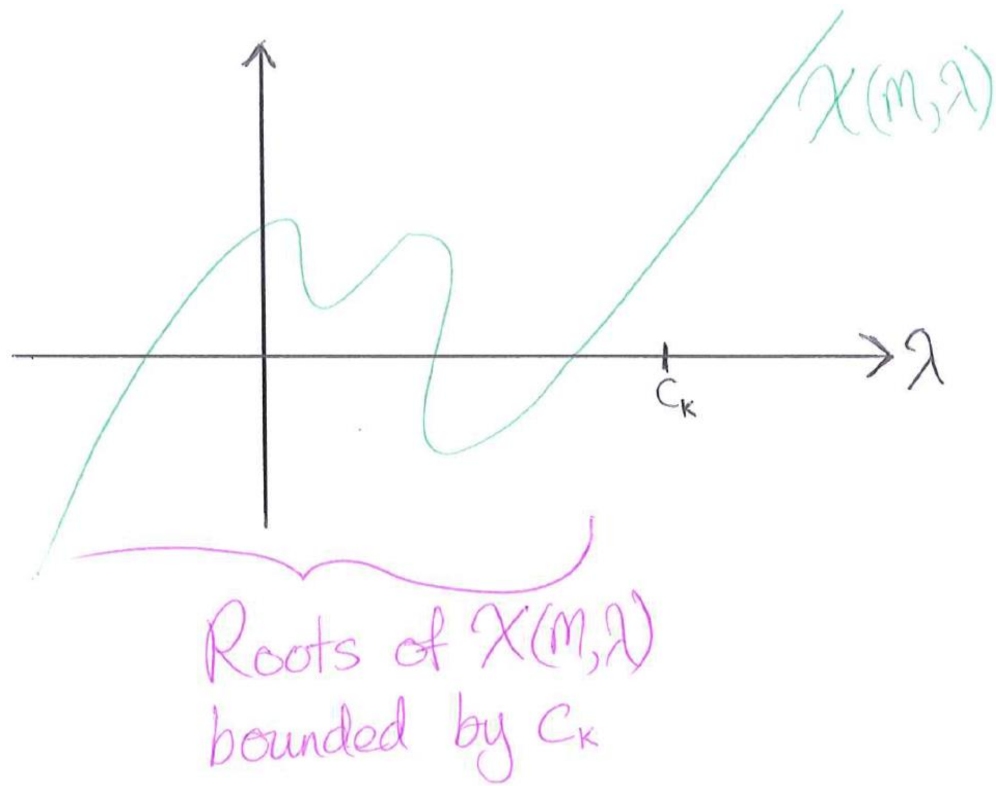
generalized parallel connection

Given two matroids $M_1 = (E_1, r_1)$ y $M_2 = (E_2, r_2)$ we define the *generalized parallel connection*, $P_N(M_1, M_2)$, as the matroid over $E_1 \cup E_2$ whose flats are the sets X de $E_1 \cup E_2$ such that $X \cap E_1$ is a flat in M_1 and $X \cap E_2$ is a flat in M_2 and where $N \cong M_1|T \cong M_2|T$ and $T = E_1 \cap E_2$. Then,

$$\chi_{P_N(M_1,M_2)}(\lambda) = \frac{\chi_{M_1}(\lambda)\chi_{M_2}(\lambda)}{\chi_N(\lambda)}. \qquad (3)$$



Theorem (Chun, H, Merino, Noble): Let M be a matroid representable over $GF(q)$. If M has tree-width at most k, then there exists $c_k \in \mathbb{R}$ such that $\chi(M,\lambda) > 0$ for all $\lambda > c_k$.

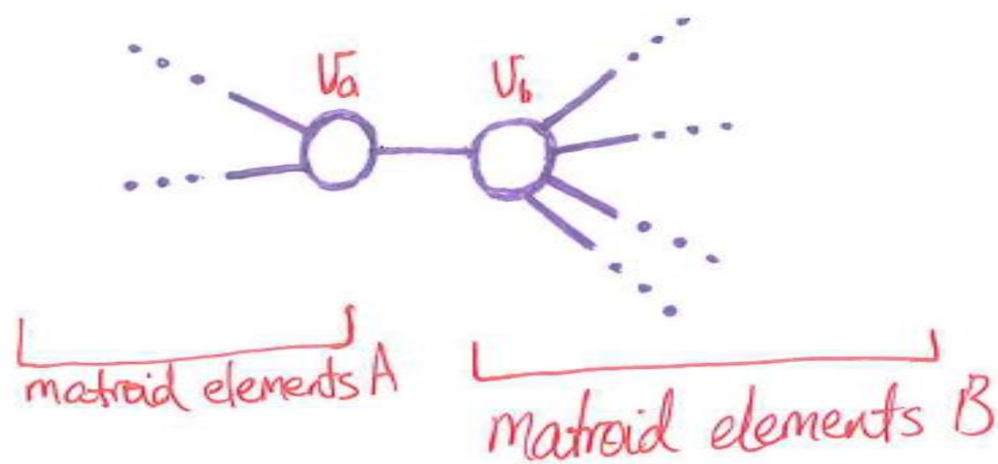


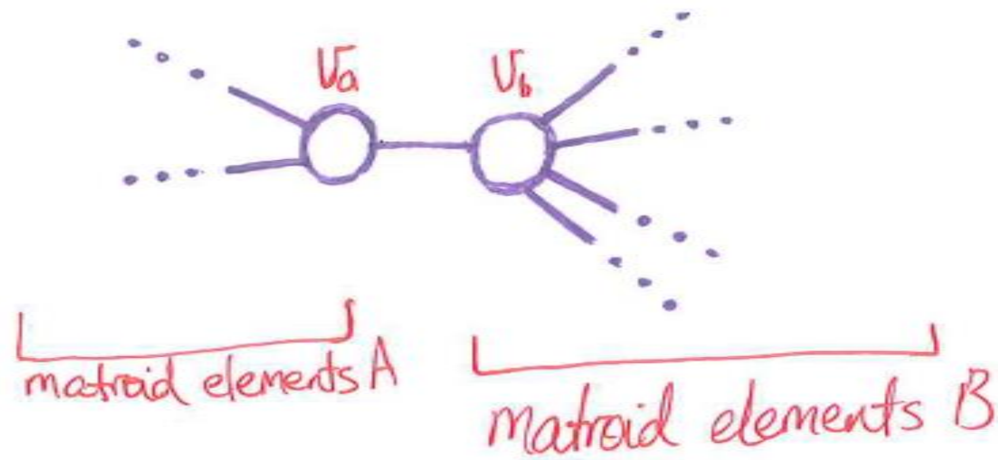
Proof Outline:

For induction, lexicographically order all $GF(q)$-representable matroids by (rank, $|E(M)|$, something).

We make sure c_k is large enough that $PG(0,q), PG(1,q), \ldots, PG(k-1,q)$ satisfy the theorem.

Induction: Let M be $GF(q)$-representable with tree-width $\leq k$. Assume all $GF(q)$-representable matroids of tree-width $\leq k$ that occur before M in the LEX ordering satisfy the theorem.
Let T be an optimal tree decomposition of M and let (A, B) be a k'-separation of M displayed by T (note $k' \leq k$):





Let $\{e_1, e_2, \ldots, e_t\}$ be the elements of $PG(r(M)-1, q)$ in the guts of (A, B).

Let M^{e_i} denote the matroid M extended by e_i.

<u>Lemma:</u> $M^{e_1, e_2, \ldots, e_t}$ has the same tree-width as M (construct the same tree decomposition with $e_1, e_2, \ldots, e_t$ in bucket v_a or v_b).

<u>Lemma:</u> For any matroid M and any element $e \in E(M)$, $TW(M/\mathrm{e}) \leq TW(M)$.

Induction:

$$\chi(M,\lambda) = \chi(M^{e_1},\lambda) + \chi(M^{e_1}/e_1,\lambda)$$

$$= \chi(M^{e_1 e_2},\lambda) + \chi(M^{e_1 e_2}/e_2,\lambda) + \chi(M^{e_1}/e_1,\lambda)$$

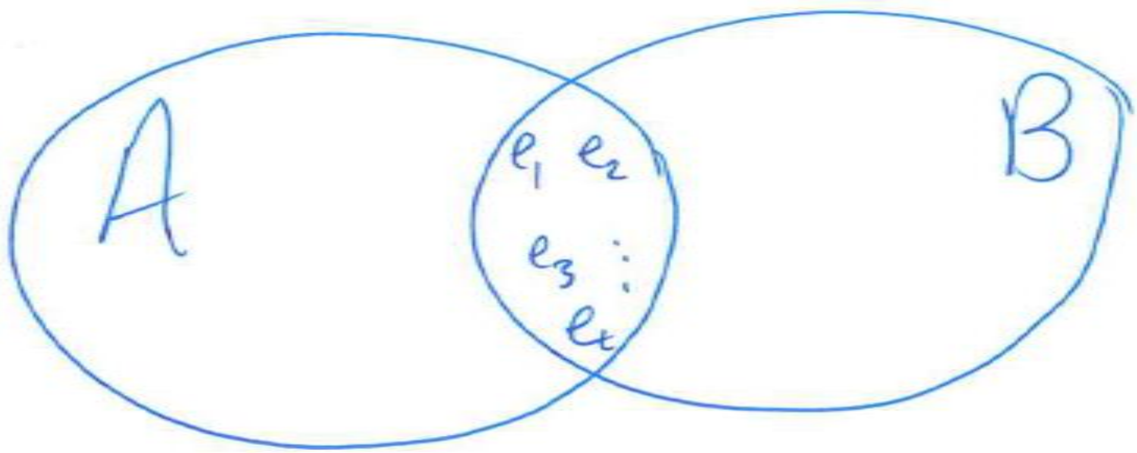
$$= \chi(M^{e_1 e_2 \cdots e_t},\lambda) + \chi(M^{e_1 e_2 \cdots e_t}/e_t,\lambda) + \cdots + \chi(M^{e_1}/e_1,\lambda)$$

Matroids of rank lower than M, so satisfy theorem by induction, ie. for $\lambda > C_K$, these are all positive.

We need to know that $\chi(M^{e_1,e_2,\ldots,e_t},\lambda)$ is positive for all $\lambda \geq c_k$.

Lemma: If the guts is a modular flat then

$$\chi(M^{e_1 e_2 \dots e_t}, \lambda) = \frac{\chi((M|A)^{e_1 e_2 \dots e_t}, \lambda)\chi((M|B)^{e_1 e_2 \dots e_t}, \lambda)}{\chi(\{e_1, e_2, \dots, e_t\}, \lambda)}$$



$$\chi(M^{e_1 e_2 \ldots e_t}, \lambda) = \frac{\chi((M|A)^{e_1 e_2 \ldots e_t}, \lambda)\chi((M|B)^{e_1 e_2 \ldots e_t}, \lambda)}{\chi(\{e_1, e_2, \ldots, e_t\}, \lambda)}$$

Rank lower than M so the induction works …

… except in the case where M has no k'-separations for any $k' < r(M)$. In which case, extend M to $PG(r(M) - 1, q)$ as on the previous slide.