New results on k-sum and ordered median combinatorial optimization problems

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Summer Graduate School 2016

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Description

This school is oriented to the presentation of theory, algorithms and applications for the solution of **mixed integer nonlinear problems (MINLP).** This type of problems appears in numerous application areas where the modelization of nonlinear phenomena with logical constraints is important; we must remember here the memorable phrase "***the world is nonlinear***". Nowadays the theoretical aspects of this area are spread in a number of recent papers which makes it difficult, for non-specialist, to have a solid background of the existing results and new advances in the field. This school aims to organize and present this material in an organized way. Moreover, it also pursues to link theory with actual applications. In particular, remarkable applications can be found in air traffic control agencies, the air companies, the electric power generation companies, the chemical complex units, the analysis of financial products usually associated with risk dealing and in the algorithms in the statistical field and artificial intelligence as for instance artificial neural networks, or supporting vector machines, among many others.

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Description

The Theory of Mixed Integer Non-linear Programming

Robert Weismantel

- MINLP for convex and concave functions
- The Theory of MINLP for Polynomial functions.

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Modelling, algorithms and applications of MINLP
Jeff Linderoth

- Modeling with integer variables and applications of MINLP. The importance of convexity.
- Algorithms for MINLP and their theoretical properties. Relaxations, Branch and Bound, Linearization.
- Beyond MINLP
- Heuristics and software for MINLP

The raw problem

Let E be a finite set of elements, where each $e \in E$ is associated with a pair of real weights (c_e, d_e), where $d_e \geq 0$. Let S be a collection of subsets of E.

- The MINSUM problem is to find a subset $X \in S$ of minimum total weight, $c(X) + d(X) = \sum_{e \in X}(c_e + d_e)$.
- The MINMAX problem with respect to the d weights is to find a subset $X \in S$ minimizing the sum of $c(X)$ and the maximum element in $\{d_e : e \in X\}$.
- The k-SUM problem with respect to the d weights is to find a subset $X \in S$ minimizing the sum of $c(X)$ and the sum of the k-largest elements in the set $\{d_e : e \in X\}$.

Examples

assignment, shortest paths, matching, spanning trees, matroid, ...

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Background

- The name: Gupta and Punnen 1990.
- Trace back till k-centrum problem (Slater 1978).
- Kalcsics, Nickel, P. and Tamir (2002)
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- Bottleneck problems (Tamir DAM 1982, Burkard & Rendl, ORL 1991)
- Minimum deviation problems (Gupta and Punnen ORL 1988)
- Partial sum problems (Gupta and Punnen ORL 1990)
- Lexicographical (De la Croce et al. ORL 1999)
- Balance or range criterion (max-min) (Martello et al. ORL 1984)
- Multifacility location (Tamir, DAM 2001; Tamir, P., Perez, DAM 2002; Kalcsics, Nickel, P., Networks 2003)
- Robust optimization (Bertsimas and Sim, Math. Prog. 2003)
- Discrete ordered median location problems (Nickel and P., Networks 2005)
- Ordered path and spanning tree location in graphs (P. and Tamir, Math. Prog. 2005)
- The k-Centrum Shortest Path Problem, (Garfinkel, Fernandez, Lowe TOP, 2006)
- Universal Shortest Paths. (Turner and Hamacher.Report in Wirtschaftsmathematik 128, Universitt Kaiserslautern, 2010.)
- OWA Spanning trees (Galand and Spanjaard CORS 2012)
- Discrete optimization with ordering (Fernández, P., Rodríguez Annals OR 2012)
- OWA Combinatorial Optimization (Fernández, Pozo, P., DAM 2014)
- On the generality of the greedy algorithm for solving matroid base problems (Turner et al., DAM 2015)
- Shifted combinatorial optimization (Kaibel, Onn, Sarrabezolles, ORL 2015) . . .

Our program started in 2010 ...

General Complexity Results

- Garfinkel, Fernández and Lowe (2006) show that the class of k-centrum shortest $s-t$-paths problem among the paths with cardinality at least k is NP-hard (Reduction for $k = n - 1$ from Hamiltonian path).
- Our claim is that in a slightly modified setting solving the minimum k-centrum problem on the respective combinatorial model can be done by solving $O(t)$ linear optimization problems where t is the number of different cost coefficients of the elements (e.g., edges of a graph, nodes of a graph, etc.).

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Achievements in this paper...

Problem	*Best known complexity*	*Our complexity*
k-*centrum minimum cost network flow problem*	*Approximate alg., Bertsimas & Sim 2003*	*Strongly polynomial*
k-*centrum path problem on trees*	*Unknown*	$O(n^2 \log n)$
Continuous tactical k-*centrum subree problem on trees*	$O(n^3 + n^{2.5} l)$, *P.& Tamir 2005*	$O(n \log n)$
Continuous tactical k-*centrum path problem on trees*	*Unkonwn*	$O(n(n\alpha(n) \log n)^2)$
Continuous strategic k-*centrum subtree problem on trees*	$O(kn^7)$, *P.& Tamir 2005*	$O(n \log n)$
Single facility k-*centrum problem:*		
Undirected general networks	$O(nm \log n)$, *Kalcsics et al. 2002*	$O(mn \log n)$
Continuous ℓ_1-*norm*	$O(n)$, *Tamir 2003*	$O(n \log n)$
k-*centrum Chinese Postman Problem*	*Unknown*	*Strongly polyn.*
The k-*centrum* p-*facility problem on trees*	$O(pk^2n^2)$, *Kalcsics 2011*	$O(pn^4)$
The k-*centrum* p-*facility problem on paths*	*Unknown*	$O(pn^3)$
The discrete tactical k-*centrum path problem on trees*	*Unknown*	$O(n^3 \log n)$
The discrete strategic k-*centrum subtree problem on trees*	$O(kn^3)$, *P.& Tamir 2005*	$O(n^3)$
The k-*centrum shortest path problem*	$O(n^2m^2)$, *Garfinkel et al. 2006*	$O(m^2 + mn \log n)$
The continuous multifacility OMP $\lambda = (a, \overset{s}{\ldots}, a, b, \overset{n-s}{\ldots}, b)$	$O(pn^9s^2)$, *Kalcsics et al 2003*	$O(pn^8 \log^4 n)$
The convex continuous OMP	*Unknown*	*Polynomial*

OM-Combinatorial optimization, Fernandez, Pozo, P. (2014)

Let $\mathcal{P}$ be a problem with feasible region $\mathbf{Q}$ and $f_i(x) = d_i x$, $i = 1, \ldots, p$:

$$\mathcal{P} : \min\{\sum_{i=1}^{p} c^i x + \sum_{i=1}^{p} w_i d^{\sigma_i} x : x \in \mathbf{Q} \subset \mathcal{S}\}$$

where $d^{\sigma_1} x \geq d^{\sigma_2} x \geq \ldots \geq d^{\sigma_p} x$.

NP-hard for $p = 2$ and $\mathbf{Q}$ being shortest paths, matchings, spanning trees...

- $w = (1, 0, \ldots, 0, 0)$: minimize the maximum of the weights,
- $w = (1, \overset{(k)}{\ldots}, 1, 0, \ldots, 0)$: minimize the sum of the k-largest weights (k-centrum)
- $w = (0, \overset{(k_1)}{\ldots}, 0, 1, \ldots, 1, 0, \overset{(k_2)}{\ldots}, 0)$: minimization of the (k_1, k_2)-trimmed mean of m weights,...
- $w = (1, \alpha, \ldots, \alpha)$: minimizing the convex combination of the sum and the maximum of the weights (w-centdian).
- $w = (1, 0, \ldots, 0, -1)$: minimize the range of a set of weights.

$$\min_{x \in X}(cx + \max\{\sum_{j \in S_k} d_j x_j : S_k \subseteq \{1, ..., n\}, |S_k| = k\}),$$

where $X = \{(x_e)_{e \in E}\}$ characteristic vectors of subsets of E.

The inner maximization for a fixed $x \in X$ is ($d \geq 0$):

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} d_j x_j v_j \\ s.t. \quad & \sum_{j=1}^{n} v_j \leq k \\ & v_j \in \{0,1\}, \quad \forall j = 1, \ldots, n. \end{aligned}$$

The problem above is:

$$Z^* = \min_{r \geq 0} Z(r), \qquad (1)$$

$$Z(r) = kr + \min_{(x,p)}(cx + \sum_{j=1}^{n} p_j),$$

subject to $p_j \geq d_j x_j - r, j = 1, ..., n,$

$$p_j \geq 0, j = 1, ..., n,$$

$$x \in X.$$

+ constraint on the support!

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X *a polytope in* $\mathbb{R}^n$

Let $X_L := \{x : Ax = b, x \geq 0\}$ be the region X for this particular case

Theorem

1. $Z_{X_L}(r)$ *is a piecewise linear convex function.*
2. *Suppose that there is a combinatorial algorithm of* $O(T(n,m))$ *complexity to compute* $Z_{X_L}(r)$ *for any given* r. *Then,* $Z^*_{X_L}$ *can be computed in* $O((T(n,m))^2)$ *time. Moreover, if* $T(n,m) = O(n)$ *then* $Z^*_{X_L}$ *can be computed in* $O(n \log n)$ *time.*

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   Use duality from the previous reformulation!
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   Use Megiddo's parametric approach on $Z_{X_L}(r)$*.!*

1. Robust minimum cost network flow problem in Bertismas and Sim (2003). (Only approximately solved!)

$$\min_{x\in X}(cx + \max\{\sum_{j\in S_k} d_j x_j : S_k \subseteq \{1,....,n\}, |S_k| = k\}),$$

Our approach gives an exact algorithm with strongly polynomial complexity.
Indeed, the evaluation of $Z_{X_L}(r)$ can be done solving a flow problem with piecewise linear costs: $T(n,m) = O((m\log n)(m + n\log n))$.

2. The k-centrum path problem on trees.
Solved in $O(n^2 \log n)$ time. Uses the reformulation

$$\begin{aligned} \min \quad & \sum_{k=1}^{n-1} w_k \sum_{j:e_j\in P[v_k,v_0)} \ell_j(1-x_j) \\ s.t. \quad & \sum_{k\in ES(e_i)} x_k \le x_i, \quad \forall i = 1,\dots,n-1 \\ & 0 \le x_j \le 1, \quad \forall j = 1,\dots,n-1. \end{aligned}$$

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Solves also discrete version: property of k-centrum path

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The continuous tactical k-centrum subtree/path problem on trees

Consists of:

$$\min_{Y \subseteq A(T)} \quad \sum_{i=1}^{n} w_i d(v_i, Y)$$
$$s.t. \qquad L(Y) \leq L.$$

Best complexity bound P. and Tamir (2005): $O(n^3 + n^{2.5} I))$ where I is the total number of bits needed to represent the input.

Theorem

1. *The continuous tactical k-centrum subtree problem on trees can be solved in $O(n \log n)$ time.*
2. *The continuous tactical k-centrum path problem on trees can be solved in $O(n(n\alpha(n) \log n)^2)$ time, where $\alpha(n)$ is the inverse of the Ackermann function.)*

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The continuous strategic k-centrum subtree problem on trees

Consists of:

$$\min_{Y\subseteq A(T)} \sum_{i=1}^{n} w_i d(v_i, Y) + \delta L(Y), \text{ with } \delta \in \mathbb{R}.$$

Best complexity bound is $O(kn^7)$.

Theorem

The continuous strategic k-centrum subtree problem on trees is solvable in $O(n \log n)$ time.

The single facility k-centrum problem

Theorem

The following complexity bounds can be obtained for the single facility k-centrum problem.

1. *On undirected general networks the k-centrum is solvable in $O(mn \log n)$ time.*
2. *On a continuous d-dimension (d fixed) ℓ_1-norm space the k-centrum problem is solvable in $O(n \log n)$ time.*

Complexity bounds similar to those in Kalcsics, Nickel, P. and Tamir (2002) with the general methodology!

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k-centrum integer optimization

Let $X_I = \{x \in R^n : Ax = b, x_j \in \{0, 1, 2, ...\}, j = 1, ..., n\}$ be the region X for this case

Some negative results

Unlike the linear case, even for the binary case, the function $Z_{X_I}(r)$ is not generally convex when $k = 1$, and is not generally unimodal when $k = 3$.

Positive results

If all the integer variables are bounded by $M = M(n, m)$, where $M(n, m)$ is a polynomial in m, n, the integer model is polynomially solvable.

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$$Z_{X_I}(r) = kr + \min_{x \in X_I} \left(cx + \sum_{j=1}^{n} \max\{d_j x_j - r, 0\} \right).$$

Decompose $[0, M \max_{j=1,...,n} \{d_j\}]$ into consecutive intervals induced by the set of points $\{pd_j\}$, $p = 0, 1, ..., M$, and $j = 1, ..., n$.

Let $\mathcal{I} = [pd_s, qd_t]$ with $p, q \in \{0, \ldots, M\}$ and $s, t (s \le t) \in \{1, \ldots, n\}$.
For each $j = 1, ..., n$, let $h_j \in \mathbb{Z}^+$ such $\mathcal{I} \subseteq [h_j d_j, (h_j + 1) d_j]$.
Then, over the nonnegative integers for each $r \in \mathcal{I}$,

$$\max\{d_j x_j - r, 0\} = \begin{cases} 0 & \text{if } x_j \le h_j \\ d_j x_j - r & \text{if } x_j \ge h_j + 1 \end{cases}$$

The function $Z_{X_I}(r) = kr + \min_{x \in X_I} \left(cx + \sum_{\substack{j=1 \\ x_j > h_j}}^{n} (d_j x_j - r) \right)$

is concave for $r \in \mathcal{I}$.

Hence, we may conclude that without loss of generality $r^* \in \{pd_s, qd_t\}$.

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Theorem

Consider the k-sum integer optimization problem $Z^*_{X_I}$, and assume that the matrix A is totally unimodular. Suppose further that all integer variables are bounded by some polynomial $M(n,m)$. Then, $Z^*_{X_I}$ can be computed in strongly polynomial time.

Proof. $Z^*_{X_I}$ can be computed by evaluating $Z_{X_I}(r)$ for $O(nM(n,m))$ values of the parameter r. Specifically, for a fixed value of r, we need to solve the following problem:

$$\begin{aligned} \min \quad & cx + \sum_{j=1}^{n} \max\{d_j x_j - r, 0\}, \\ s.t. \quad & x \in X_I. \end{aligned}$$

The above can be solved in strongly polynomial time by substituting $x_j = u_j + v_j + z_j$, $j = 1, \ldots, n$, and solving the respective integer program, defined by a totally unimodular system,

$$\begin{aligned} \min \quad & c(u+v+z) + \sum_{j=1}^{n} (d_j(\lceil r/d_j \rceil - r/d_j)v_j + d_j z_j), \\ s.t. \quad & A(u+v+z) = b, \\ & u_j \in \{0,1,...,\lfloor r/d_j \rfloor\}, \quad j = 1,...,n, \\ & v_j \in \{0,1\}, \quad j = 1,...,n, \\ & z_j \in \{0,1,2,...\}, \quad j = 1,...,n. \end{aligned}$$

Since A is totally unimodular this problem is an LP with $\{0, \pm 1\}$-matrix and therefore, by Tardos (1985), it is solvable by a strongly polynomial algorithm.

Applications

The k-sum Chinese Postman Problem defined on undirected connected graphs and on strongly connected directed graphs is solvable in strongly polynomial time.

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In this section $\mathcal{X} = \{0, 1\}^n$.

Therefore, given a finite set of elements E, where each $e \in E$ is associated with a pair of real weights (c_e, d_e) and X_C be a collection of subsets of E; MINSUM problem is to find a subset $x \in X_C$ of minimum total weight, $c(x) + d(x) = \sum_{e \in x}(c_e + d_e)$.

***k*-sum optimization problem with respect to the d weights**

Find a subset $S \in \mathcal{S}$ minimizing the sum of $c(S)$ and the sum of the k-largest elements in the set $\{d_e : e \in S\}$.

Theorem

Punnen & Aneja (1996) Suppose that for each real r the MINSUM problem with respect to the weights $(c_e, \max(0, d_e - r))$, $e \in E$, is solvable in $T(m)$ time, where $m = |E|$. Then, the k-centrum problem with respect to the d weights can be solved in $O(m' T(m))$ time, where m' is the number of distinct elements in the set $\{d_e : e \in E\}$.

Remark

The supposition that $d_e \geq 0$, for each $e \in E$, which is made in the papers by Punnen & Aneja is used extensively in the proofs. Based on this nonnegativity supposition, they can relax the formulation and introduce the constraint that at most k elements are selected, i.e., $\sum_{e \in E} u_e \leq k$.

From the proof of the above result we note that it actually holds also for some specific linear functions as stated in the next theorem.

Consider the case of arbitrary $\{d_e\}$. For the general case we need to impose the constraint $\sum_{e \in E} u_e = k$. We will then obtain that the parameter θ is unrestricted in sign and we will get the following result for general $\{d_e\}$:

Theorem

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The k-centrum p-median problem on trees and paths

Let us denote by $X_{med(p)}$ the lattice points defined by p-median polytope. The sum version of above problem is solvable in polynomial time provided that c_{ij} are distances induced by the metric of shortest paths on a tree Hassin and Tamir (2002). (It is NP-hard for a general linear objective function.)

k-sum: requires to solve $O(G)$ problems of the form:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \max\{c_{ij} - c_{(\ell)}, 0\} x_{ij} \\ s.t. \quad & x \in X_{med(p)} \end{aligned}$$

The algorithm in Tamir (1996) also applies to the above problem. Therefore, by Theorem, the k-centrum p-facility on trees is solvable in $O(pn^4)$. This improves upon the $O(\min(k,p)kpn^5)$ bound in Tamir (2000) and equals the complexity reported in Kalcsics (2011), although in this case using ad hoc arguments.

1. The discrete tactical k-centrum path problem on trees
   The case of locating a discrete median path is solvable in $O(n \log n)$ time, see (Alstrup et al 1997). Following our approach, the k-centrum version of this model can be solve in $O(n^3 \log n)$ time.
2. The best complexity for the k-centrum version of locating a subtree using the strategic model is $O(kn^3)$ (P. & Tamir 2005). Using Theorem above we improved upon the complexity above to $O(n^3)$ time.
3. The k-centrum shortest path problem can be solved in $O(n^2 m^2)$ time provided that any simple $s-t$-path there are at least k arcs, otherwise this problem is NP-hard, see Garfinkel, Fernández, Lowe (2006). We improve the bound to $O(m^2 + mn \log n)$ time.
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A natural question

Can Theorem above be extended to the convex ordered median optimization problem?

$$\min_{x\in X}\left\{cx+\max_{\sigma\in\mathcal{P}(1,\dots,n)}\left\{\sum_{j=1}^{n}\lambda_j d_{\sigma_j}x_{\sigma_j}\ :\ d_{\sigma_1}x_{\sigma_1}\geq\ldots\geq d_{\sigma_n}x_{\sigma_n}\right\}\right\}.$$

Some partial answers

- Bottleneck problems (Tamir 1982, Burkard & Rendl, ORL 1991)
- Lexicographical (De la Croce et al. ORL 1999)
- Balance or range criterion (max-min) (Martello et al. ORL 1984)
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The formulation of the problem is:

$$\min_{x\in X} \left\{ cx + \max_{\sigma\in\mathcal{P}(1,\dots,n)} \left\{ \sum_{j=1}^{n} \lambda_j d_{\sigma_j} x_{\sigma_j} \ : \ d_{\sigma_1} x_{\sigma_1} \geq \ldots \geq d_{\sigma_n} x_{\sigma_n} \right\} \right\}.$$

Or equivalently, using $\lambda_{n+1} := 0$,

$$\begin{aligned} \min \quad & cx + \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1})(kt_k + \sum_{j=1}^{n} p_{jk}) \\ s.t. \quad & p_{jk} \geq d_j x_j - r_k, \quad j,k = 1,\ldots,n \\ & p_{jk} \geq 0, \quad j,k = 1,\ldots,n \\ & x \in X. \end{aligned}$$

Again, this problem can be reformulated as:

$$\min_{x\in X,(r_1,\dots,r_k)\in\mathbb{R}^k} \quad cx + \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1})(kt_k + \sum_{j=1}^{n} \max\{0, d_j x_j - r_k\})$$

Theorem

If the number of different values of the vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ is constant, let say k_0, we have that

1. *The discrete convex ordered median problem can be solved in $O(n^{k_0} T_d(n,m))$ time, where $T_d(n,m)$ is the combinatorial complexity of solving the sum problem on the discrete set X. (Solving n^{k_0} sum problems on X.)*
2. *The continuous convex ordered median problem can be solved in $O(k_0^3 T_c(n,m) \log^{2k_0} n)$ time, where $T_c(n,m)$ is the combinatorial complexity of solving the sum problem on the polytope X. (Using the multiparametric approach Cohen and Megiddo (1993).)*

Applications

1. Multifacility Ordered Median Problem on Trees
2. The centdian subtree on tree networks

Non constant number of λ values

Theorem

The continuous convex ordered median problem with monotone λ on the polytope X can be solved in polynomial time.

Proof.
We observe that

$$\max_{\substack{\sigma \in Perm(1,\dots,n) \\ d_{\sigma_1} x_{\sigma_1} \geq \dots \geq d_{\sigma_n} x_{\sigma_n}}} \sum_{j=1}^{n} \lambda_j d_{\sigma_j} x_{\sigma_j} = \max\{\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_j d_i x_i p_{ij} : \sum_{i=1}^{n} p_{ij} = 1,\ \forall j;\ \sum_{j=1}^{n} p_{ij} = 1,\ \forall i\}.$$

Next, dualizing the second problem one has the Problem is equivalent to:

$$\begin{array}{lll} \min & c'x + \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j & \\ \text{s.t.} & u_i + v_j \geq \lambda_j d_i x_i & \forall\, i,j \\ & x \in X. & \end{array}$$

The above is a linear programming problem that can be solved in polynomial time and thus the result follows.

Minimizing the middle range problem $(k3 > k1)$

Suppose that $d_1 \geq d_2 \geq \ldots \geq d_m$ are the values of the coefficients of the ground set E.

Our optimization problem is defined by:

$$\begin{array}{ll} \min & S^x_{k_3} - S^x_{k_1} \\ s.t. & x \in X \\ & (1, \overset{k_1}{\ldots}, 1, \overset{k_3-k_1}{\ldots}, 1, 0, \ldots, 0) \\ & (1, \overset{k_1}{\ldots}, 1, 0, \ldots, 0) \end{array}$$

where S^x_k is the sum of the largest k and is given by

$$\begin{array}{ll} \max & \sum_{j=1}^m v_j d_j \\ s.t. & \sum_{j=1}^m v_j = k, \\ & v_j \leq x_j, \quad \forall j \\ & 0 \leq v_j \leq 1, \quad \forall j \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & kt + \sum_{i=1}^m y_j \\ s.t. & y_j + t \geq d_j x_j, \quad \forall j \\ & y_j \geq 0. \end{array}$$

Joining both:

$$\min_{x\in X} \min_{y_j+t\geq d_j x_j; y_j, x_j\geq 0 \forall j} k_3 t + \sum_{i=1}^{n} y_j - \max_{\substack{\sum_{j=1}^{n} v_j = k_1,\ 0\leq v_j\leq x_j\leq 1 \\ v_j\in\{0,1\}}} \sum_{j=1}^{n} v_j d_j$$

It can be rewritten as:

$$\begin{aligned}
\min \quad & k_3 t + \sum_{i=1}^{n} y_j - \sum_{j=1}^{n} v_j d_j \\
s.t. \quad & x \in X \\
& y_j \geq d_j x_j - t, \quad \forall j = 1, \ldots, n \\
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\end{aligned}$$

Now, for any $t \in [d_\ell, d_{\ell-1}]$ we have an equivalent formulation of the above problem:

$$\begin{aligned} \min \quad & k_3 d_\ell + \sum_{j=1}^{\ell-1}(d_j - d_\ell)x_j - \sum_{j=1}^{n} v_j d_j \\ s.t. \quad & x \in X \\ & \sum_{j=1}^{n} v_j = k_1 \\ & 0 \le v_j \le x_j,\ v_j \in \{0,1\} \quad j = 1, \dots, n \end{aligned} \tag{2}$$

Next, for each d_ℓ consider the $\binom{\text{position of } d_\ell}{k_1}$ different forms of fixing the v-variables and for each one of them we solve the resulting linear problem (2) with those variables already fixed. Therefore the overall complexity seems to be $O(G^{k_1})$ where G is the number of distinct values for d_j. Clearly, this approach is in general non polynomial. If the number k_1 of trimmed components is fixed then is polynomial.

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Matroids and non-negative lambda weights

For a matroidal system any ordered median function with non negative λ-weights is optimized by the base that optimizes the minisum problem.

We can solve any ordered median problem on matroidal systems with a constant number of coefficient values using separators and matroid intersection algorithms.

Minimizing the mid-range problem

We solve the problem (2) for each d_ℓ as follow:

For each $e \in E$, we associate two costs with e, $d_e - d_\ell$ and $-d_\ell$. Sort in nondecreasing order the list $\{d_e - d_\ell,\ e \in E\} \cup \{-d_\ell :\ e \in E\}$. For solving the problem above, we start choosing edges associated with the costs from the beginning of this list following these rules:

1. The chosen element together with the previous ones is an independent set.
2. For a given element e, it can be chosen either $d_e - d_\ell$ or $-d_\ell$.
3. After choosing k_1 elements with associated cost $-d_\ell$, delete from the list the remaining costs of the type $-d_\ell$.

Minimizing the difference between the largest k and the smallest t elements.

Using separators

We solve $O(m^2)$ subproblems for each pair $i < j$. Consider the three subsets:
$\mathbb{E}_1 = \{e_1, \ldots, e_i\}$, $\mathbb{E}_2 = \{e_{i+1}, \ldots, e_{i+j}\}$ and $\mathbb{E}_3 = \{e_{i+j+1}, \ldots, e_m\}$.

With each $e_k \in E_1$ associate a coefficient d_k, with each $e_k \in E_2$ associate a coefficient 0, and with each $e_k \in E_3$ associate a coefficient $-d_k$.

Using matroid intersection find an optimal base of cardinality $\bar{n}$ w.r.t. these weights which contains at most k element from E_1, at most t elements from E_3, and at most $\bar{n} - k - t$ from E_2.

Thanks for your attention!