Cutting planes from extended LP formulations

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Extended LP formulations

Consider

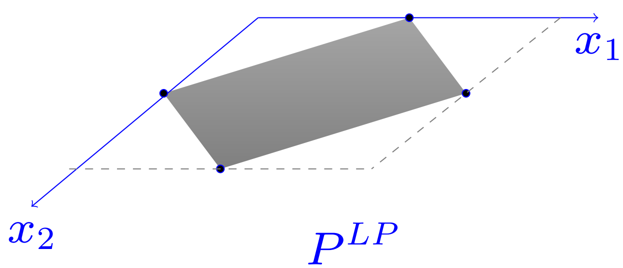
$$P^{IP} = \left\{x \in \mathcal{R}^n \ : \ Ax \leq b, \ x_i \in \mathcal{Z} \ \text{ for } i \in I\right\}$$

and

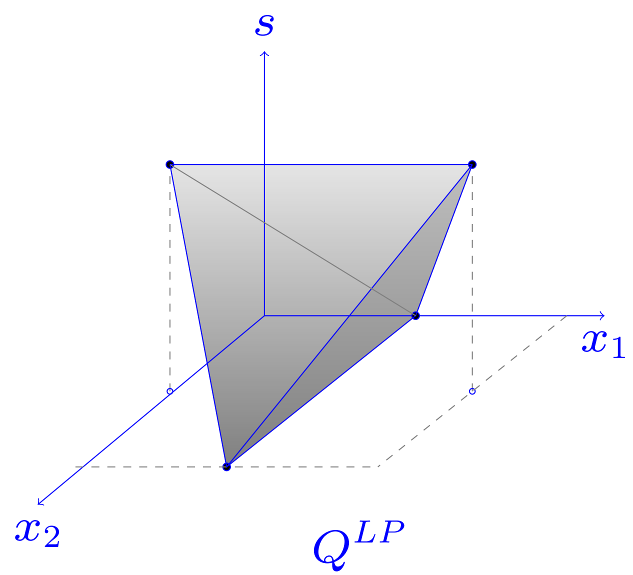
$$Q^{IP} = \left\{(x, s) \in \mathcal{R}^n \times \mathcal{R}^k \ : \ Cx + Gs \leq d, \ x_i \in \mathcal{Z} \ \text{ for } i \in I\right\}$$

such that

$$P^{LP} = \text{proj}_x \left(Q^{LP}\right)$$



P^{LP}



Q^{LP}

Split sets and split cuts

- *Consider*

$$P^{IP} = \{x \in \mathcal{R}^n \ : \ Ax \leq b, \ x_j \in \mathcal{Z} \ \text{ for } j \in J\}$$

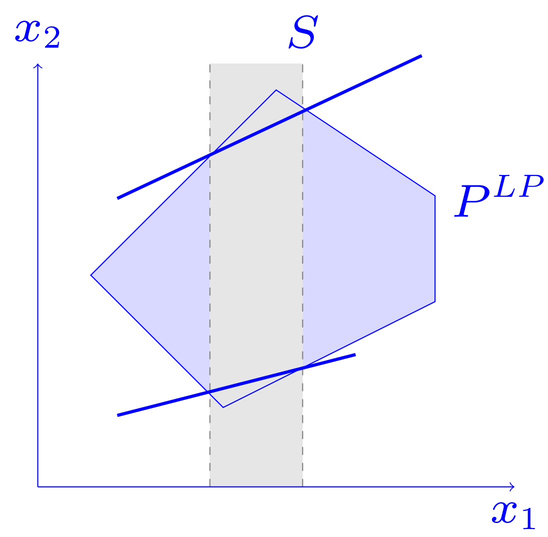
- *and the* **split set:**

$$S = \{x \in \mathcal{R}^n \ : \ \gamma + 1 > \pi x > \gamma\}$$

where $\pi \in \mathcal{Z}^n$, $\gamma \in \mathcal{Z}$, *and* $\pi_j \neq 0$ *only if* $j \in J$.

- *Clearly*

$$P^{LP} \supseteq \text{conv}(P^{LP} \setminus S) \supseteq P^{IP}.$$



Is there a benefit in applying split cuts to extended LP formulations?

- *Given:*
  - $P^{LP} \subseteq \mathcal{R}^n$ *and* $Q^{LP} \subseteq \mathcal{R}^{n+k}$ *such that* $P^{LP} = \mathrm{proj}_x\left(Q^{LP}\right)$
  - *Split sets* $S_i \subseteq \mathcal{R}^n$ *and* $S_i^+ = S_i \times \mathcal{R}^k$ *for* $i \in I = \{1, \ldots, m\}$
- *Compare:*

$$\bigcap_{i \in I} \mathrm{conv}(P^{LP} \setminus S_i) \quad \textit{vs.} \quad \mathrm{proj}_{\mathcal{R}^n}\left(\bigcap_{i \in I} \mathrm{conv}(Q^{LP} \setminus S_i^+)\right)$$

- *We can show that:*
  - *If* $|I| = 1 \Rightarrow$ *no gain.*
  - *If* $|I| > 1 \Rightarrow$ *splits on extended formulation can be strictly better.*

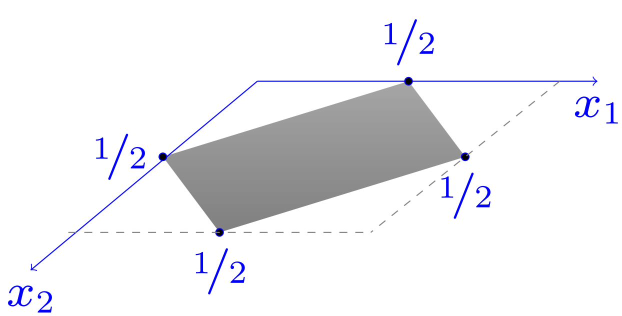
Proof by example

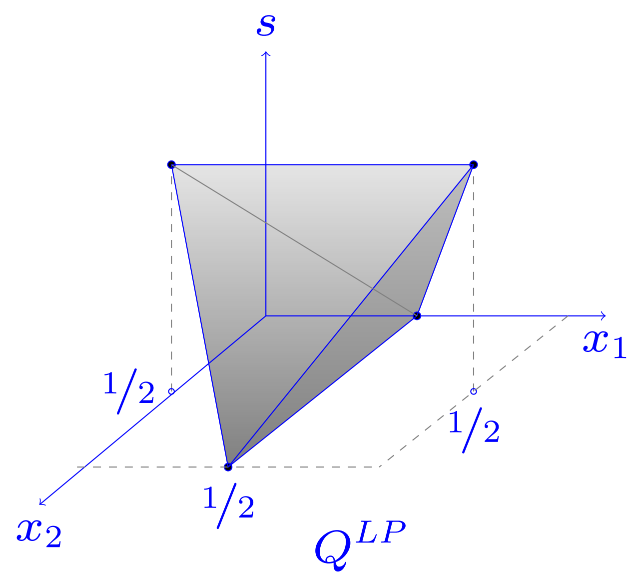
- *It is easy to argue that* $P^{LP} \setminus S_i = \text{proj}_{\mathcal{R}^n} \left(Q^{LP} \setminus S_i^+\right)$
- *Furthermore,*

$$\bigcap_{i \in I} \text{conv}(P^{LP} \setminus S_i) \supseteq \text{proj}_{\mathcal{R}^n} \left(\bigcap_{i \in I} \text{conv}(Q^{LP} \setminus S_i^+) \right)$$

- *The split closure of* P^{LP} *below is* $(1/2, 1/2)$ *whereas that of* Q^{LP} *is empty.*

P^{LP} : conv$((0, 1/2), (1, 1/2), (1/2, 0), (1/2, 1))$





[joint with Jim Luedtke]

Power of extended formulations

Theorem : *Every* $0-1$ *mixed integer set in* $\mathcal{R}^{n+k}$ *has a reformulation in* $\mathcal{R}^{2n+k}$, *such that the split closure of the extended formulation is integral.*

- *Let* $P^{IP} = P^{LP} \cap \{0,1\}^n \times \mathcal{R}^k$ *where*

$$P^{LP} = \text{conv}(x^1, \ldots, x^m) + \text{cone}(r^1, \ldots, r^\ell)$$

- *Consider* $X^{LP} = \text{conv}(\hat{x}^1, \ldots, \hat{x}^m) + \text{cone}(\hat{r}^1, \ldots, \hat{r}^\ell)$, *where*
  - $\hat{r}^t = (r^t, \mathbf{0})$
  - $\hat{x}^t = (x^t, z^t)$ *where*

$$z_i^t = \begin{cases} 1 & \textit{if } x_i^t \textit{ fractional} \\ 0 & \textit{o.w.} \end{cases}$$

  for $i = 1, \ldots, n.$

- *Elementary splits* $S_i = \{x \in \mathcal{R}^{n+k} \ : \ 1 > x_i > 0\}$ *give the integral hull.*

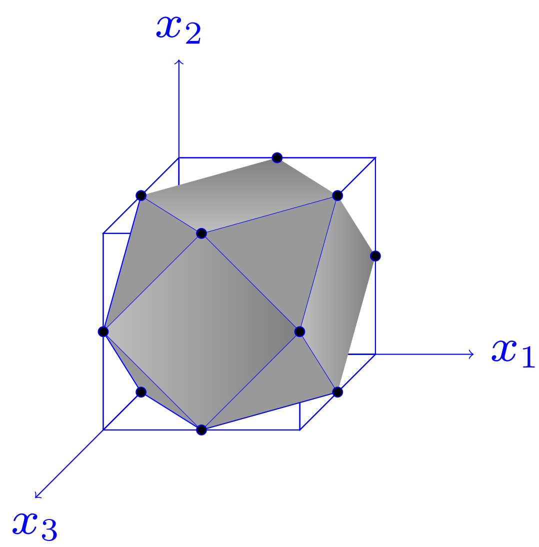
$(n-1)$

The cropped cube

- *Let* $N = \{1, \ldots, n\}$ *and consider*

$$P^{LP} = \Big\{ x \in \mathcal{R}^n \; : \quad \sum_{i \in I} x_i + \sum_{i \in N \setminus I} (1 - x_i) \geq 1/2, \quad \forall I \subseteq N$$

$$0 \leq x_i \leq 1, \quad \forall i \in N \quad \Big\}$$



- *All* $2^n + 2n$ *inequalities are facet defining*
- *All vertices are of the form* $x_i = 1/2$ *for one* $i \in N$ *and* $x_j \in \{0, 1\}$ *for the rest.*

Extended LP formulation for the cropped cube

- *Consider*

$$X^{LP} = \Big\{ (x,z) \in \mathcal{R}^n \times \mathcal{R}^n \; : \quad \begin{aligned} & z_i \leq 2x_i, \;\; \forall i \in N \\ & z_i + 2x_i \leq 2, \;\; \forall i \in N \\ & z_i \geq 0, \;\; \forall i \in N \\ & \sum_{i \in N} z_i = 1 \Big\} \end{aligned}$$

- *Extreme points of* X^{LP} *are of the form* $\hat{x}^t = (x^t, z^t)$ *where* x^t *is an extreme point of* P^{LP} *and*

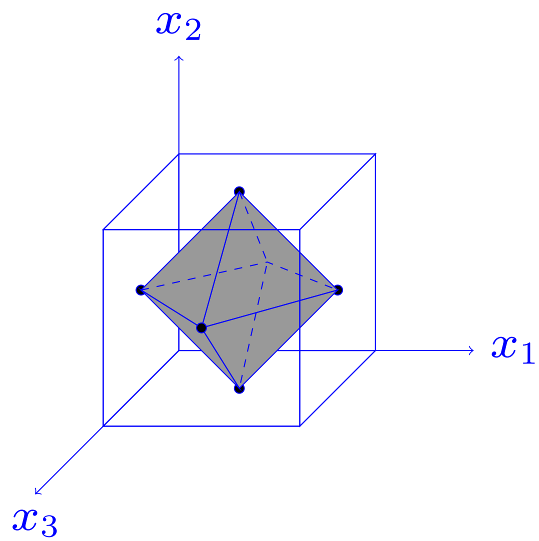
$$z_i^t = \begin{cases} 1 & \textit{if } x_i^t \textit{ fractional} \\ 0 & \textit{o.w.} \end{cases}$$

- $P^{LP} = \text{proj}_x \left(X^{LP} \right)$

- $SC(X^{LP}) = \emptyset$ *whereas* $SC^t(P^{LP}) \neq \emptyset$ *for all* $t < n$.

Generalized cropped cube

- *Extreme points of* P^{LP} *have exactly* k *fractional components with* $x_i = 1/2$ *.*



- *Exponentially many extreme points/facets as before.*

- *The (compact) extended formulation is:*

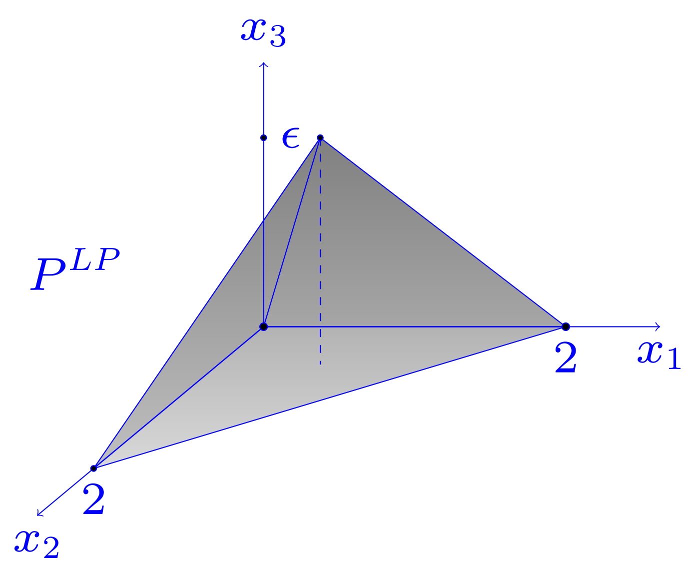
$$X^{LP} = \Big\{ (x, z) \in \mathcal{R}^n \times \mathcal{R}^n \ : \quad \begin{array}{l} z_i \leq 2x_i, \ \forall i \in N \\ z_i + 2x_i \leq 2, \ \forall i \in N \\ z_i \geq 0, \ \forall i \in N \\ \sum_{i \in N} z_i = k \end{array} \Big\}$$

General mixed integer case

The Cook, Kannan and Schrijver's example: $P^{IP} = P^{LP} \cap \mathcal{Z}^2 \times \mathcal{R}$ *where*

$$P^{LP} = \text{conv}((0,0,0),(2,0,0),(0,2,0),(1/2,1/2,\epsilon))$$

[P^{IP} *has* $x_3 = 0$ *but* $SC^t(P^{LP})$ *has* $x_3 > 0$ *for all* $t = 1, 2, \ldots$ *]*



Is there a good extended LP formulation for P^{LP} *?*

Properties of good extended formulations: minimality

- *Let* $S(P^{LP})$ *denote the split closure of* P^{LP} *w.r.t. a collection of split sets* S.
- *If* $Q_1^{LP} \subset Q_2^{LP}$ *in* $\mathcal{R}^{n+k}$ *are extended formulations of* $P^{LP} \subset \mathcal{R}^n$*, then*

$$S(Q_1^{LP}) \subseteq S(Q_2^{LP}) \;\Rightarrow\; \mathrm{proj}_{\mathcal{R}^n}\left(S(Q_1^{LP})\right) \subseteq \mathrm{proj}_{\mathcal{R}^n}\left(S(Q_2^{LP})\right)$$

$\Rightarrow$ *Smaller extended formulations are better.*

- *Each extreme point/ray of* P^{LP} *should have at least 1 pre-image in* Q^{LP}.
- *Ideally each extreme point/ray of* P^{LP} *should have* **exactly** *1 pre-image.*
  - *If less than one, not a valid extended formulation*
  - *If more than one, not minimal.*
- *Minimal extended formulations are not unique even for fixed* k.

Properties of good extended formulations: increasing dimension

- *Let* $S(P^{LP})$ *denote the split closure of* P^{LP} *w.r.t. a collection of split sets* S *.*

- *Let* $Q_1^{LP} \subset \mathcal{R}^{n+k}$ *be an extended formulation of* $P^{LP} \subset \mathcal{R}^n$ *.*

- *If* $\dim(Q_1^{LP}) \;=\; \dim(P^{LP})$ *, then* $\mathsf{proj}_{\mathcal{R}^n}\left(S(Q_1^{LP})\right) = S(P^{LP})$

- *More generally, if*

$$k > \dim(Q_1^{LP}) - \dim(P^{LP})$$

then there is an extended formulation $Q_2^{LP} \subset \mathcal{R}^{n+t}$ *such that*

$$\mathsf{proj}_{\mathcal{R}^n}\left(S(Q_1^{LP})\right) = \mathsf{proj}_{\mathcal{R}^n}\left(S(Q_2^{LP})\right)$$

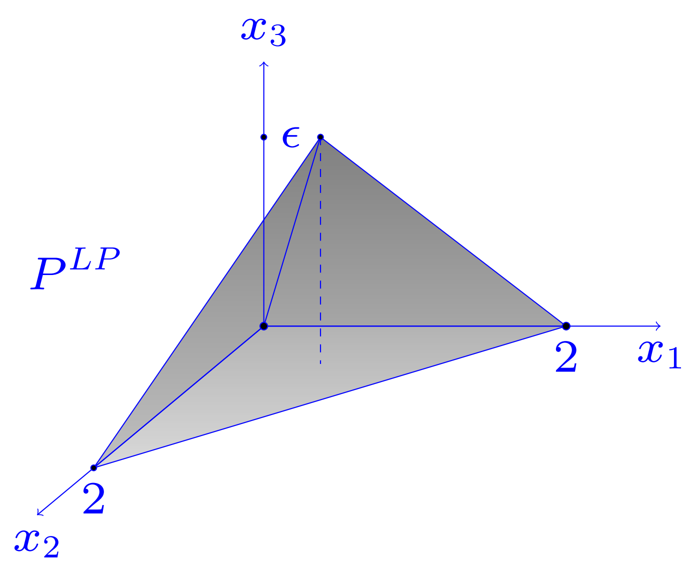
where $t = \dim(Q_1^{LP}) - \dim(P^{LP})$ *.*

$\Rightarrow$ *extended formulations are useless unless they increase dimension.*

Limitations of extended formulations

The Cook, Kannan and Schrijver's example: $P^{IP} = P^{LP} \cap \mathcal{Z}^2 \times \mathcal{R}$ *where*

$$P^{LP} = \text{conv}((0,0,0),(2,0,0),(0,2,0),(1/2,1/2,\epsilon))$$



1. P^{LP} *has* 4 *extreme points* $\Rightarrow$ Q^{LP} *should ideally have* 4 *extreme points.*

2. $\dim(Q^{LP}) \leq 3 = \dim(P^{LP}) \Rightarrow$ *no gain!*

A minimal extended formulation for mixed integer case

- *Let*

$$P^{LP} = \{x \in \mathcal{R}^n : x = \sum_{i=1}^{k} \alpha_i \hat{x}^i + \sum_{j=1}^{t} \nu_j \hat{r}^j \text{ s.t. } \sum_{i=1}^{k} \alpha_i = 1,\ \alpha \in \mathcal{R}_+^k,\ \nu \in \mathcal{R}_+^t\}$$

where $\hat{x}^i$ *are the extreme points and* $\hat{r}^j$ *are the extreme rays.*

- *Consider*

$$X^{LP} = \{q \in \mathcal{R}^{n+k+t} : q = \sum_{i=1}^{k} \alpha_i \hat{q}^i + \sum_{j=1}^{t} \nu_j \hat{w}^j \text{ s.t. } \sum_{i=1}^{k} \alpha_i = 1,\ \alpha \in \mathcal{R}_+^k,\ \nu \in \mathcal{R}_+^t\}$$

where $\hat{q}^i = (\hat{x}^i, e_i)$ *,* $\hat{w}^j = (\hat{r}^j, e_{k+j})$ *and* e_i *denotes the unit vector in* $\mathcal{R}^{k+t}$ *.*

- *Then*

$$\text{proj}_{\mathcal{R}^n}\left(SC(X^{LP})\right) \subseteq \text{proj}_{\mathcal{R}^n}\left(SC(Q^{LP})\right)$$

for any extended LP formulation Q^{LP} *of* P^{LP} *.*

The strength of the extended formulation

- *Consider split sets* $S^\ell = \{x \in \mathcal{R}^n : \pi_0^\ell + 1 > (\pi^\ell)^T x > \pi_0\}$ *for* $\ell \in L$ *.*

- *The split closure of* P^{LP} *with respect to* L *is:*

$$
\begin{aligned}
S^L(P^{LP}) = \Big\{ \; x \in \mathcal{R}^n : \quad & \quad x = \bar{x}^\ell + \bar{\bar{x}}^\ell, & \ell \in L,\\
\bar{x} = \sum_{i=1}^{k} \bar{\alpha}_i^\ell \hat{x}^i + \sum_{j=1}^{t} \bar{\nu}_j^\ell \hat{r}^j, \quad & \quad \bar{\bar{x}} = \sum_{i=1}^{k} \bar{\bar{\alpha}}_i^\ell \hat{x}^i + \sum_{j=1}^{t} \bar{\bar{\nu}}_j^\ell \hat{r}^j, & \ell \in L,\\
\sum_{i=1}^{k} \bar{\alpha}_i^\ell = \mu_\ell, \quad & \quad \sum_{i=1}^{k} \bar{\bar{\alpha}}_i^\ell = 1 - \mu_\ell, & \ell \in L,\\
(\pi^\ell)^T \bar{x}_\ell \le \mu_\ell \pi_0^\ell, \quad & \quad (\pi^\ell)^T \bar{\bar{x}}_\ell \ge (1-\mu_\ell)(\pi_0^\ell + 1), & \ell \in L,\\
\bar{\alpha}^\ell, \bar{\nu}^\ell \ge 0, \quad & \quad \bar{\bar{\alpha}}^\ell, \bar{\bar{\nu}}^\ell \ge 0, \qquad 0 \le \mu \le 1 \; \Big\}. &
\end{aligned}
$$

- $\text{proj}_{\mathcal{R}^n}\left(S^L(X^{LP})\right)$ *also imposes* $\alpha^* = \bar{\alpha}^\ell + \bar{\bar{\alpha}}^\ell$ *and* $\nu^* = \bar{\nu}^\ell + \bar{\bar{\nu}}^\ell$ *for* $\ell \in L$.

Computational experiments with the two row relaxation

- *Consider a two-row relaxation of a generic IP using the LP tableau:*

$$P^{IP} = \left\{ (x, s) \in \mathcal{Z}^2 \times \mathcal{R}^k_+ \ : \ x = f + \sum_{j=1}^{k} \hat{r}^j s_j \right\}$$

where f *and all* $\hat{r}^j$ *are in* $\mathcal{R}^2$ *.*

- $\dim(P^{LP}) =$ *number of extreme points and rays* $\Rightarrow$ *no gain.*

- *Compare 16 simple splits applied to* P^{LP} , $P^{LP}_+ = P^{LP} \cap \mathcal{R}^{2+k}_+$, *and* X^{LP}_+ *.*

- *Average gap closed by split cuts:*

$\lvert J\rvert$	$S(P^{LP})$	$S(P^{LP}_+) - S(P^{LP})$	$S(X^{LP}_+) - S(P^{LP}_+)$	$P^{IP}_+ - S(X^{LP}_+)$
20	*88.82 (42/100)*	*16.21 (23/58)*	*5.99 (10/35)*	*15.44 (25)*
40	*91.20 (47/100)*	*11.87 (17/53)*	*5.17 (6/36)*	*12.87 (30)*
60	*88.48 (36/100)*	*11.90 (28/64)*	*5.31 (9/36)*	*16.94 (27)*
80	*91.32 (44/100)*	*15.44 (27/56)*	*2.51 (11/29)*	*13.59 (18)*
100	*89.53 (43/100)*	*12.33 (25/57)*	*6.09 (6/32)*	*19.78 (26)*

Lovaśz-Schrijver extended formulation

- *Let* $P^{IP} = P^{LP} \cap \{0,1\}^n$ *and*

$$P^{LP} = \{x \in \mathcal{R}^n \ : \ Ax \geq b\}$$

where $1 \geq x \geq 0$ *is included in* $Ax \geq b$ *.*

- *The Lovaśz-Schrijver extended formulation* $Q(P^{LP})$ *:*

  1. *Generate the nonlinear system*

$$\begin{aligned} x_j(Ax - b) &\geq 0 \\ (1 - x_j)(Ax - b) &\geq 0 \qquad j = 1, \ldots, n. \end{aligned}$$

  2. *Linearize by substituting* y_{ij} *for* $x_i x_j$ *(and* $y_{ij} = y_{ji}$ *.)*

  3. *[But do not strengthen by substituting* x_i *for* y_{ii} *yet.]*

- *Note that* $P^{LP} = \text{proj}_x\left(Q(P^{LP})\right)$

- *Further,* $P^{LP} \supseteq S^{01}(P^{LP}) \supseteq N(P^{LP}) = \text{proj}_x\left(Q(P^{LP}) + \textit{strengthening step 3}\right)$

Split cuts for the Lovaśz-Schrijver extended formulation

- *The strengthening step (substituting* x_i *for* y_{ii} *) is a* $0/1$ *split cut for* $Q(P^{LP})$.
- *There are more split cuts for* $Q(P^{LP})$ *(even from* $0/1$ *splits).*
- *Let* $S^{01}(Q(P^{LP}))$ *be the split closure of* $Q(P^{LP})$ *w.r.t.* $0/1$ *splits.*

We can show that

$$\text{proj}_x\left(S^{01}(Q(P^{LP}))\right) \subseteq \underbrace{\text{proj}_x\left(Q(S^{01}(P^{LP})) + \textit{strengthening step 3}\right)}_{\textit{Lovaśz-Schrijver (w/ strengthening) applied to 0/1 split closure of } P^{LP}}$$

Which also implies $\text{proj}_x\left(S^{01}(Q(P^{LP}))\right) \subseteq S^{01}(S^{01}(P^{LP}))$ *and therefore:*

> *Applying this procedure* $n/2$ *times gives an integral polyhedron.*

(as $(S^{01})^n(P^{LP}) = P^{IP}$ *)*

Computations with the Lovaśz-Schrijver extended formulation

- *Random instances of the stable set problem with density* 0.25%
  (higher density instances do not have gap between N^2 *and* SA^2 *)*
- *For the the stable set problem,* $N(P^{LP}) = S^{01}(P^{LP}) =$ *odd cycle inequalities*
- *Consequently, for the stable set problem:*

$$P^{IP} \subseteq \underbrace{SA^2(P^{LP})}_{\text{2nd level Sherali-Adams}} \subseteq \underbrace{\tilde{N}(P^{LP})}_{\text{new}} \subseteq N^2(P^{LP}) \subseteq \underbrace{N(P^{LP})}_{\text{Lovasz-Schrijver}} \subseteq P^{LP}.$$

$\lvert V\rvert$	N	$N^2 - N$	$\tilde{N} - N^2$	$SA^2 - \tilde{N}$	% Gap left
20	100	0	0	0	0
25	99.53	0.46	0	0	0
30	97.50	2.49	0	0	0
35	90.29	9.52	0.0527	0	0.1236
40	89.45	10.37	0.0843	0.0003	0.0796
45	84.70	14.79	0.1214	0.0002	0.3727
50	80.55	18.33	0.0862	0.0001	1.0299

thank you...