Effectiveness of Sparse Cutting-planes for Integer Programs with Sparse Formulations

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Modern Techniques in Discrete Optimization: Mathematics, Algorithms and Applications, 2015

Outline

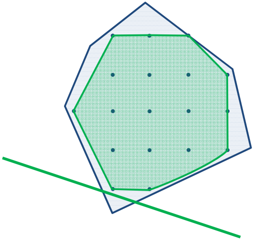
1

Introduction and Motivation

Cutting-planes: Introduction

Cutting Plane

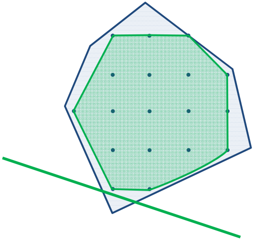
1. Cutting-planes in a linear inequality that is valid for all integer feasible points, but may not be valid for the linear programming relaxation.



Cutting-planes: Introduction

Cutting Plane

1. Cutting-planes in a linear inequality that is valid for all integer feasible points, but may not be valid for the linear programming relaxation.
2. Huge amount of research in Integer Programming on problem-specific and general purpose cutting-planes.
3. General purpose cutting-planes have been extremely useful in practice to solve IPs.



Cutting plane selection is non-trivial

Most commercial/successful IP solvers have very sophisticated methods of cutting-planes selection and use.

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1. "Dept of cut"
2. "Parallelism"
3. "Numerical stability"
4. "Cutting-plane sparsity"

'Cut pool management system', 'Cutting-plane filter system'

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Pros...

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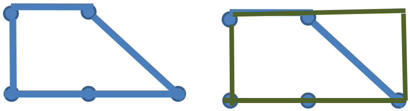
1. Sparse constraints may not approximate the integer hull (a polytope) well!



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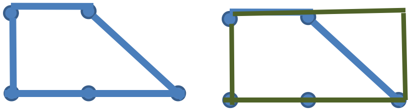
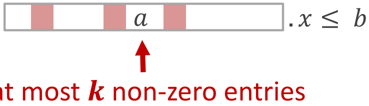


Main Goal: Theoretically analyze performance of sparse cutting-planes.

Prior results: Quality of sparse closure

SSD, Marco Molinaro, Qianyi Wang, "Approximating Polyhedra with Sparse Inequalities," Mathematical Programming, 2015.



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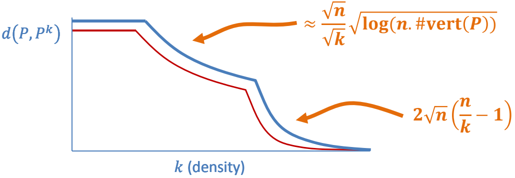
Theorem

Let $n \geq 2$. Let $P \subseteq [0,1]^n$ be the convex hull of points $\{p^1, \ldots, p^t\}$. Then

1. $d(P, P^k) \leq 4 \max \left\{ \frac{n^{1/4}}{\sqrt{k}} \sqrt{8 \max_{i \in [t]} \|p^i\|} \sqrt{\log 4tn}, \frac{8\sqrt{n}}{3k} \log 4tn \right\}$
2. $d(P, P^k) \leq 2\sqrt{n} \left(\frac{n}{k} - 1 \right)$.

□



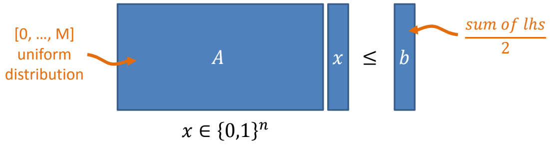
Consequences

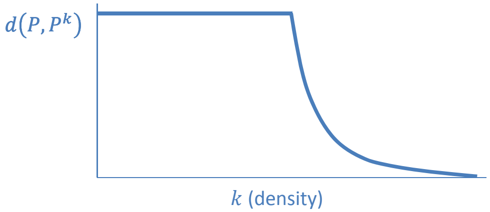
1. Polynomial number of vertices as a function of dimension with $\sim \frac{1}{2}$ sparsity, implies $d(P, P^k)$ is very small ($\approx \sqrt{\log n}$), i.e. sparse cutting planes are good.

Prior results: Sparse cuts are bad for dense packing IPs

Random packing instances: Sparse cutting-planes are bad.

Hard packing IPs





We want analysis for “real" IPs

- “Real" IPs are sparse: The average number (median) of non-zero entries in the constraint matrix of MIPLIB 2010 instances is 1.63% (0.17%).

How does sparsity of IPs effect the performance of sparse cutting-planes?

1.1
Some examples

Why sparse cuts may be useful for sparse IPs?

1. Consider the following IP set

$$\begin{array}{lll} \sum_{j=1}^{5} A_j x_j & & \leq b^1 \\ & \sum_{j=6}^{10} A_j x_j & \leq b^2 \\ x \in \mathbb{Z}^{10}. & & \end{array}$$

   1. Clearly the convex hull is given by inequalities in the support of the first five examples and separately on the last 5 inequalities.
   2. In practice many instances are "loosely decomposable".

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2. Classic computation paper: "Solving Large-Scale Zero-One Linear Programming Problems" by H. Crowder, E. L. Johnson, M. Padberg (1982). Some quotes:
   1. *"All problems are characterized by sparse constraint matrix with rational data."*
   2. *"We note that the support of an inequality obtained by lifting (2.7) or (2.9) is contained in the support of the inequality (2.5) ... Therefore, the inequalities that we generate preserve the sparsity of the constraint matrix."*

Another example of sparse IPs: Two-stage stochastic IPs

$$\begin{array}{llllllll}
\max & c^T y & +(d^1)^T z^1 & +(d^2)^T z^2 & +(d^3)^T z^3 & + \dots & +(d^k)^T z^k & \\
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Scenario-specific cuts: $\alpha^T y + \beta^j z^j \leq \gamma$

2

Main results

Overview of results

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1. Packing-type IPs:

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where $\underline{c, A, b}$ are non-negative.

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3. "Packing-type" problem with milder assumptions:

$$\begin{array}{rc} \max & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} & Ax \leq b \\ & x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \end{array} \qquad (P - Arbitrary\ A)$$

   where only c is non-negative.

Overview of results contd.

1. We present a method to "quantity" the sparsity of A.
2. We present a specific way to describe a hierarchy of sparse cutting-planes with different supports.
3. We present multiplicative bounds:
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   3. Packing-type arbitrary A problem (max objective):
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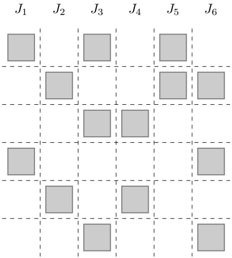
2.1
Main results: Packing-type problems

Describing sparsity of A



The matrix A with:

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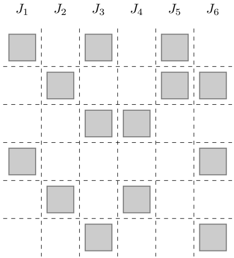

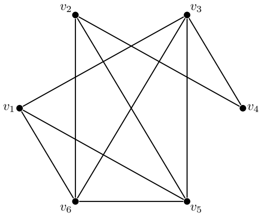

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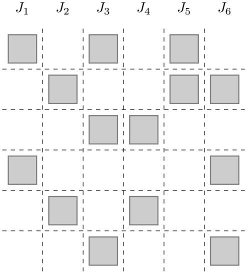
The corresponding graph $G^{\text{pack}}_{A,\mathcal{J}}$:

1. One node for every block of variables.
2. $(v_i, v_j) \in E$ if and only if there is a row in A with non-zero entries in both parts J_i and J_j.

Describing the sparsity of cutting-planes: Notation

Given the problem (P), let $\mathcal{J} := \{J_1, J_2, \ldots, J_q\}$ be a partition of the index set of columns of A (that is $[n]$).

1. For a set of nodes $S \subseteq V$, we say that inequality $\alpha x \leq \beta$ is a *sparse cut on S* if the support of α is on the variables corresponding to vertices in S, namely $\bigcup_{v_j \in S} J_j$.



Adding a cut of the form:

$$(\alpha^1)^T x^1 + (\alpha^4)^T x^4 \leq \beta$$

corresponds to:

$$S = \{v_1, v_4\}.$$

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3. *Support list* of sparse cuts: Given a collection $\mathcal{V} = \{S^1, S^2, S^3, ..., S^q\}$ of subsets of the vertices V, we use $P^{\mathcal{V},\text{pack}}$ to denote the closure obtained by adding all sparse cuts on the sets in $\mathcal{V}$'s, namely

$$P^{\mathcal{V},\text{pack}} := \bigcap_{S^i \in \mathcal{V}} P^{(S^i)}.$$

Example of notation

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1. $\mathcal{J} = \{y, z^1, \dots, z^k\}$, that is $V = \{v_0, \dots, v_k\}$
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2. $E = \{(v_0, v_1), (v_0, v_2), \dots, (v_0, v_k)\}$
3. Specific-scenario closure: closure using all valid inequalities of the form: $\alpha^T y + \beta^T z^i \leq \gamma$, i.e.,

$$P^{\mathcal{V},\text{pack}} = \bigcap_{i=1}^{k} P^{(\{v_0, v_i\})},$$

where $\mathcal{V} = \{\{v_0, v_1\}, \{v_0, v_2\}, \dots, \{v_0, v_k\}\}$.

Some graph-theoretic definition I: Mixed stable set

Definition (Mixed stable set)

Let $G = (V, E)$ be a simple graph. Let $\mathcal{V}$ be a collection of subsets of the vertices V. We call a collection of subsets of vertices $\mathcal{M} \subseteq 2^V$ a *mixed stable set subordinate to* $\mathcal{V}$ if the following hold:

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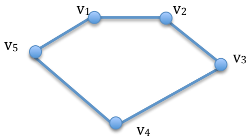
Example:

1. $\mathcal{V} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_1, v_5\}\}$
2. $\mathcal{M} = \{\{v_3\}, \{v_1, v_5\}\}$

Original Graph:



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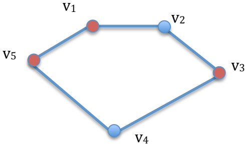

Some graph-theoretic definition II: Mixed chromatic number

Consider a simple graph $G = (V, E)$ and a collection $\mathcal{V}$ of subset of vertices.

- *Mixed chromatic number with respect to* $\mathcal{V}$ (Denoted as $\bar{\eta}^{\mathcal{V}}_{(G)}$): It is the smallest number of mixed stables sets $\mathcal{M}^1, \dots, \mathcal{M}^k$ subordinate to $\mathcal{V}$ that cover all vertices of the graph (that is, every vertex $v \in V$ belongs to a set in one of the $\mathcal{M}^i$'s).

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- *Fractional mixed chromatic number with respect to $\mathcal{V}$* (Denoted as $\eta^{\mathcal{V}}_{(G)}$): Given a mixed stable set $\mathcal{M}$ subordinate to $\mathcal{V}$, let $\chi_{\mathcal{M}} \in \{0,1\}^{|V|}$ denote its incidence vector (that is, for each vertex $v \in V$, $\chi_{\mathcal{M}}(v) = 1$ if v belongs to a set in $\mathcal{M}$, and $\chi_{\mathcal{M}}(v) = 0$ otherwise.) Then we define the *fractional mixed chromatic number*

$$\begin{aligned} \eta^{\mathcal{V}}_{(G)} = \ & \min \sum_{\mathcal{M}} y_{\mathcal{M}} \\ & \text{s.t.} \sum_{\mathcal{M}} y_{\mathcal{M}} \chi_{\mathcal{M}} \geq \mathbf{1} \\ & \quad y_{\mathcal{M}} \geq 0 \quad \forall \mathcal{M}, \end{aligned} \tag{1}$$

where the summations range over all mixed stable sets subordinate to $\mathcal{V}$.

Main result: Packing Problem

Theorem

Consider a packing integer program. Let $\mathcal{J}$ be a partition of the index set of columns of A and let $G^{pack}_{A,\mathcal{J}}(V,E)$ be the packing-type induced graph of A. Then for any sparse cut support list $\mathcal{V} \subseteq 2^V$ we have

$$z^{cut} \leq \eta^{\mathcal{V}}_{(G^{pack}_{A,\mathcal{J}})} \cdot z^I,$$

where $z^{cut} = \max\{c^T x \mid x \in P^{\mathcal{V}}\}$.

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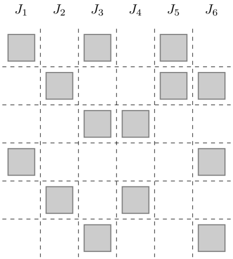
where $z^{cut} = \max\{c^T x \mid x \in P^{\mathcal{V}}\}$.

Comments:

1. The results depend only on the packing-type induced graph and sparse cut support list.
2. $\eta^{\mathcal{V}}_{(G^{\text{pack}}_{A,\mathcal{J}})}$ is upper bounded by the standard fractional chromatic number.

"Natural" Sparse Closure

Natural sparse closure Let $A_1, \dots, A_m$ be the rows of A. Let V^i be the set of (nodes corresponding to) block variables that have non-zero entries in A_i. Then for this sparse cut support list $\mathcal{V} = \{V^1, V^2, \dots, V^m\}$.



Natural sparse closure corresponds to support list:

$$\mathcal{V} = \{\{v_1, v_3, v_5\}, \{v_2, v_5, v_6\}, \{v_3, v_4\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_3, v_6\}\}$$

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Theorem

Consider a two-stage packing integer program with k scenarios.

$$z^{\text{specific-scenario closure}} \leq \left(\frac{2k-1}{k}\right) z^I.$$

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More general result:

Theorem

If $G^{pack}_{A,\mathcal{J}}$ is a tree max degree k, then $\eta^{\mathcal{V}}_{(G^{pack}_{A,\mathcal{J}})} = \left(\frac{2k-1}{k}\right)$ where $\mathcal{V}$ corresponds to natural sparse closure.

Some consequences for stochastic programs

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Some consequences for stochastic programs

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Theorem

For any $\epsilon > 0$, there exists a two-stage packing integer program with k scenarios such that

$$z^{\text{specific-scenario closure}} \geq \left(\frac{2k-1}{k} - \epsilon\right) z^I.$$

Natural sparsity for cycles

Theorem (Natural sparse closure of cycles)

Consider a packing integer program as defined in (P). Let $\mathcal{J} \subseteq 2^{[n]}$ be a partition of the index set of columns of A and let $G^{pack}_{A,\mathcal{J}}$ be the packing-type induced graph of A. If $G^{pack}_{A,\mathcal{J}}$ is a cycle of length K, then:

1. *If $K = 3k, k \in \mathbb{Z}_{++}$, then $z^{N.S.} \leq \frac{3}{2} z^I$.*
2. *If $K = 3k+1, k \in \mathbb{Z}_{++}$, then $z^{N.S.} \leq \frac{3k+1}{2k} z^I$.*
3. *If $K = 3k+2, k \in \mathbb{Z}_{++}$, then $z^{N.S.} \leq \frac{3k+2}{2k+1} z^I$.*

Moreover, for any $\epsilon > 0$, there exists a packing integer program with a suitable partition $\mathcal{V}$ of variables, where $G^{pack}_{A,\mathcal{J}}$ is a cycle of length K such that

1. *If $K = 3k, k \in \mathbb{Z}_{++}$, then $z^{N.S.} \geq \left(\frac{3}{2} - \epsilon\right) z^I$.*
2. *If $K = 3k+1, k \in \mathbb{Z}_{++}$, then $z^{N.S.} \geq \left(\frac{3k+1}{2k} - \epsilon\right) z^I$.*
3. *If $K = 3k+2, k \in \mathbb{Z}_{++}$, then $z^{N.S.} \geq \left(\frac{3k+2}{2k+1} - \epsilon\right) z^I$.*

2.2
Main results: Covering-type problems

Packing-type sparsity description does not work!

Example of packing instance

$$\begin{array}{rl} \max & (c^1)^T x^1 + (c^2)^T x^2 \\ \text{s.t.} & A^1 x^1 + A^2 x^2 \leq b \\ & x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \end{array} \qquad (P)$$

If we add cuts on the support of x^1 and x^2 variable blocks separately, then

$$z^{cut} \leq 2z^I.$$

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If we add cuts on the support of x^1 and x^2 variable blocks separately, then

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Example of covering instance

$$\begin{array}{rl} \min & (c^1)^T x^1 + (c^2)^T x^2 \\ \text{s.t.} & A^1 x^1 + A^2 x^2 \geq b \\ & x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \end{array} \qquad (C)$$

If we add cuts on the support of x^1 and x^2 variable blocks separately, then

$$z^{cut} \geq (?)z^I.$$

Packing-type sparsity description does not work!

Example of packing instance

$$\begin{array}{rl} \max & (c^1)^T x^1 + (c^2)^T x^2 \\ \text{s.t.} & A^1 x^1 + A^2 x^2 \leq b \\ & x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \end{array} \qquad (P)$$

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$$\begin{array}{rl} \min & (c^1)^T x^1 + (c^2)^T x^2 \\ \text{s.t.} & A^1 x^1 + A^2 x^2 \geq b \\ & x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \end{array} \qquad (C)$$

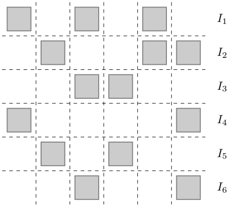
If we add cuts on the support of x^1 and x^2 variable blocks separately, then

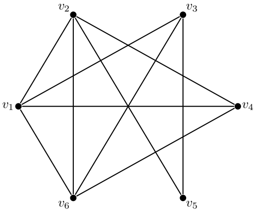
$$z^{cut} \geq (?) z^I.$$

It turns out, for any $\epsilon > 0$ there exists an instance such that:

$$z^{cut} \leq \epsilon z^I \qquad (\text{and } z^I > 0)$$

Describing sparsity of A





The matrix A with:

1. Row partition $\mathcal{I} := \{I_1, ..., I_6\}$.
2. Unshaded boxes correspond to zeros in A.
3. Shaded boxes have non-zero entries.

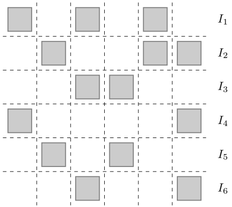
The corresponding graph $G^{\text{cover}}_{A,\mathcal{J}}$:

1. One node for every block of rows.
2. $(v_i, v_j) \in E$ if and only if there is a column in A with non-zero entries in row corresponding to I_i and I_j.

Describing sparsity of cutting-planes: Notation

Given the problem (C), let $\mathcal{I} = \{I_1, I_2, \ldots, I_p\}$ be a partition of index set of rows of A (that is $[m]$).

1. For a set of nodes $S \subset V$, we say that the inequality $\alpha \leq \beta$ is a sparse cut on S if the support of α is on the variables which have non-zero coefficients in the rows corresponding to vertices in S.



Adding a cut of the form:

$$(\alpha^2)^T x_2 + (\alpha^3)^T x_3 + (\alpha^4)^T x_4 + (\alpha^6)^T x_6 \geq \beta$$

corresponds to:

$$S = \{v_5, v_6\}.$$

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2. The closure of sparse cuts on S: $P^{(S)}$.
3. Support list of sparse cuts: Given a collection $\mathcal{V} = \{S^1, S^2, \ldots, S^q\}$ of subsets of vertices V, we use $P^{\mathcal{V},\text{pack}}$ to denote the closure obtained by adding all the sparse cuts in the sets in $\mathcal{V}$, namely

$$P^{\mathcal{V},\text{cover}} := \bigcap_{S^i \in \mathcal{V}} P^{S^i}.$$

Example of notation

$$
\begin{array}{llllllll}
\max & c^T y & +(d^1)^T z^1 & +(d^2)^T z^2 & + \dots & +(d^k)^T z^k & & \\
\text{s.t.} & A^1 y & +B^1 z^1 & & & & \leq b^1 & \} \to I_1 \\
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1. $\mathcal{I} = \{I_1, \dots, I_k\}$, that is $V = \{v_1, \dots, v_k\}$
2. Complete graph!

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1. $\mathcal{I} = \{I_1, \dots, I_k\}$, that is $V = \{v_1, \dots, v_k\}$
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$$P^{\mathcal{V},\text{pack}} = \bigcap_{i=1}^{k} P^{\{v_i\}},$$

where $\mathcal{V} = \{\{v_1\}, \{v_2\}, \dots, \{v_k\}\}$.

Main result

Theorem

Consider a covering integer program. Let $\mathcal{I}$ be a partition of the index set of columns of A and let $G^{cover}_{A,\mathcal{J}}(V,E)$ be the covering-type induced graph of A. Then for any sparse cut support list $\mathcal{V} \subseteq 2^V$ we have

$$z^{cut} \geq \frac{1}{\bar{\eta}^{\mathcal{V}}_{(G^{cover}_{A,\mathcal{I}})}} z^I,$$

where $z^{cut} = \min\{c^T x \mid x \in P^{\mathcal{V},cover}\}$.

1. The above Theorem holds even if upper bounds are present on some or all of the variables (in this case, we also need to assume that the instance is feasible).

Consequence and bounds is tight: Two stage stochastic problem

Corollary

Consider a covering-type two-stage stochastic problem for k scenario. Let z^* be the objective function obtained after adding all weakly specific-scenario cuts. Then:

$$z^I \leq k z^{\text{scenario-specific cuts}}.$$

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Consider a covering-type two-stage stochastic problem for k scenario. Let z^* be the objective function obtained after adding all weakly specific-scenario cuts. Then:

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Bound is tight:

Theorem

Let z^* be the objective function obtained after adding all weakly specific-scenario cuts for a covering type two-stage stochastic problem. Given any $\epsilon > 0$ there exists an instance of the covering-type two-stage stochastic problem with k scenarios such that:

$$z^I \geq (k - \epsilon)z^{\text{scenario-specific cuts}}.$$

2.3
Main results: “Packing-type problems" with arbitrary matrix A

Corrected density of sparse cutting-planes

- We use the same notation as the packing case.
- Specifically, we use the same kind of definition of sparsity of A and cuts as in the packing case.

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Definition (Corrected average cutting-plane density)

Let $\mathcal{V} = \{V^1, V^2, \ldots, V^t\}$ be the sparse cut support list. For any subset $\tilde{V} = \{V^{u_1}, V^{u_2}, ..., V^{u_k}\} \subseteq \mathcal{V}$ define its density as

$$D(\tilde{V}) = \frac{1}{k}\sum_{i=1}^{k} |V^{u_i}|.$$

We define the corrected average cutting-plane density of $\mathcal{V}$ (denoted as $D_{\mathcal{V}}$) as the value of $D(\mathbb{V})$ where:

1. $\mathbb{V}$ covers V, that is, $\bigcup_{\bar{V}\in\mathbb{V}} \bar{V} = V$.
2. Among all subsets of $\mathcal{V}$ that cover V, $\mathbb{V}$ is the subset with largest density.

Main results

Theorem

Let (P) be defined by an arbitrary $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{R}^n_+$ and $\mathcal{L} \subseteq [n]$. Let $\mathcal{J} := \{J_1, J_2, \dots, J_q\}$ be a partition of the index set of columns of A (that is $[n]$). If P^I is non-empty, then:

$$z^{\mathcal{V}} \leq (|V| + 1 - D_{\mathcal{V}})\, z^I.$$

Moreover these results are tight:

Corollary

Consider a two stage packing-type problem with arbitrary $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ and with k scenarios. Suppose that P^I is non-empty. Then:

$$z^{\textit{scenario-specific closure}} \leq (k) z^I.$$

Proposition

For every $k \in \mathbb{Z}_{++}$, there exists an instance of two stage packing-type problem with arbitrary $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ and k scenarios such that:

$$z^{\textit{scenario-specific closure}} = (k) z^I.$$

Conclusion

1. Introduced a natural framework to analyze strength of sparse cuts for sparse IPs.
2. The results obtained show that in many cases, sparse cuts provide good bounds for sparse IPs.
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1. Introduced a natural framework to analyze strength of sparse cuts for sparse IPs.
2. The results obtained show that in many cases, sparse cuts provide good bounds for sparse IPs.
3. The analysis is tight: all the bounds obtained are tight.
4. Can we design supports of cuts so that we get good bounds?

Thank You!