Elementary Polytopes, their Lift-and-Project Ranks and Integrality Gaps

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- $$C_{k+1} \neq C_k \text{ unless } C_k = \operatorname{conv}\left(C_k \cap \{0,1\}^d\right).$$

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are clear for every j, it makes sense to consider applying this operator iteratively, each time for a new index j.

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Therefore, the notation $BCC_{(J)}(\cdot)$ is justified.

A beautiful, fundamental property of these operators is:

Lemma

For every $J \subseteq \{1, 2, \ldots, d\}$, *we have*

$$BCC_{(J)}(P) = \operatorname{conv}\left(P \cap \{x : x_j \in \{0,1\}, \forall j \in J\}\right).$$

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Theorem

(Balas [1974]) Let P be as above. Then

$$BCC_{(\{1,2,\ldots,d\})}(P) = P_I.$$

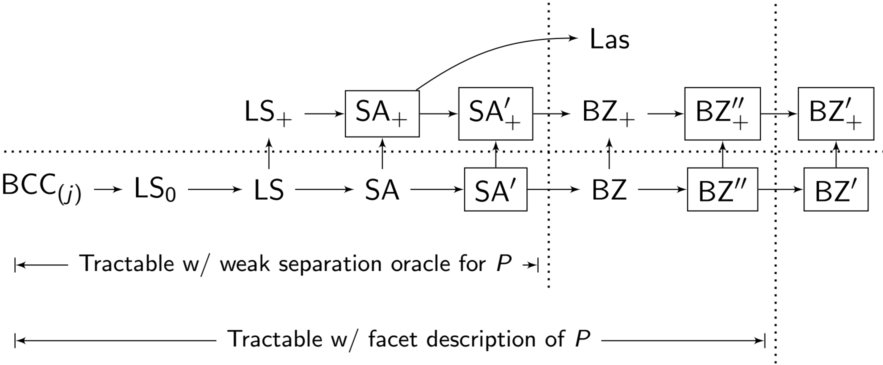

Figure: Various properties of lift-and-project operators (Au and T. [2011, 2013]).

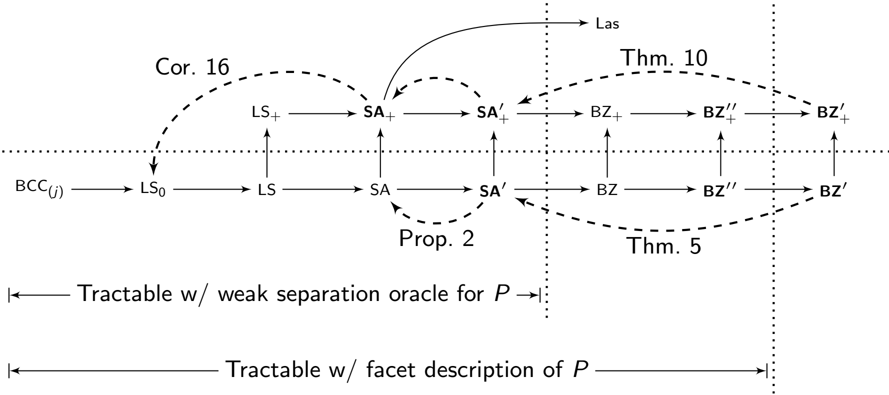

Figure: An illustration of several restricted reverse dominance results (dashed arrows) Au and T. [2013, 2015].

Lovász and Schrijver [1991] proposed:

$$M_0(K) := \Big\{ Y \in \mathbb{R}^{(d+1)\times(d+1)} : \quad Ye_0 = Y^T e_0 = \mathrm{diag}(Y), \\ Ye_i \in K, Y(e_0 - e_i) \in K, \\ \forall i \in \{1, 2, \ldots, d\} \Big\}$$

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$$\mathrm{LS}_0(K) := \{ Ye_0 : Y \in M_0(K) \}.$$

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Let K be as above. Then

$$K_I \subseteq \mathrm{LS}_+(K) \subseteq \mathrm{LS}(K) \subseteq \mathrm{LS}_0(K) \subseteq K.$$

Theorem

(Lovász and Schrijver [1991]) Let P be as above. Then

$$P \supseteq \mathrm{LS}_0(P) \supseteq \mathrm{LS}_0^2(P) \supseteq \cdots \supseteq LS_0^d(P) = P_I.$$

Similarly for LS as well as LS_+.

Moreover, the relaxations obtained after a few iterations are still tractable if the original relaxation P is.

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There is a wide spectrum of lift-and-project type operators: Balas [1974], Sherali and Adams [1990], Lovász and Schrijver [1991], Balas, Ceria and Cornuéjols [1993], Kojima and T. [2000], Lasserre [2001], de Klerk and Pasechnik [2002], Parrilo [2003], Bienstock and Zuckerberg [2004].

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Can it be really good on any problem?

Let $G = (V, E)$ be an undirected graph.
We define the *fractional stable set polytope* as

$$\mathrm{FRAC}(G) := \left\{ x \in [0,1]^V : x_i + x_j \leq 1 \text{ for all } \{i,j\} \in E \right\}.$$

This polytope is used as the initial approximation to the convex hull of incidence vectors of the *stable sets* of G, which is called the *stable set polytope*:

$$\mathrm{STAB}(G) := \mathrm{conv}\left(\mathrm{FRAC}(G) \cap \{0,1\}^V \right).$$

Let us define the class of *odd-cycle inequalities*. Let $\mathcal{H}$ be the node set of an odd-cycle in G then the inequality

$$\sum_{i \in \mathcal{H}} x_i \leq \frac{|\mathcal{H}| - 1}{2}$$

is valid for $\mathrm{STAB}(G)$. We define

$$\mathrm{OC}(G) := \{x \in \mathrm{FRAC}(G) : x \text{ satisfies all odd-cycle constraints for } G\}.$$

If $\mathcal{H}$ is an odd-anti-hole then the inequality

$$\sum_{i \in \mathcal{H}} x_i \leq 2$$

is valid for $\text{STAB}(G)$.

If we have an odd-wheel in G with *hub* node represented by x_{2k+2} and the *rim* nodes represented by $x_1, x_2, \ldots, x_{2k+1}$, then the *odd-wheel inequality*

$$kx_{2k+2} + \sum_{i=1}^{2k+1} x_i \leq k$$

is valid for $\mathrm{STAB}(G)$.

Based on these classes of inequalities we define the polytopes

$$\mathrm{OC}(G), \mathrm{ANTI\text{-}HOLE}(G), \mathrm{WHEEL}(G).$$

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Note that this theorem provides a compact lifted representations of the odd-cycle polytope of G (in the spaces $\mathbb{R}^{(\{0\}\cup V)\times(\{0\}\cup V)}$ and $\mathbb{S}^{\{0\}\cup V}$). This polytope can have exponentially many facets in the worst case.

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However, $M(G)$ is represented by

$$|V|(|V|+1)/2 \text{ variables and } O\left(|V|^3\right) \text{ linear inequalities.}$$

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Some partial results by Lipták [1999] and by Lipták and T. [2003].

A *clique* in G is a subset of nodes in G so that every pair of them are joined by an edge. The *clique polytope* of G is defined by

$$\mathrm{CLQ}(G) := \left\{ x \in \mathbb{R}_+^V : \sum_{j \in C} x_j \leq 1 \text{ for every clique } C \text{ in } G \right\}.$$

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Optimizing a linear function over FRAC(G) is easy!
Linear optimization over CLQ(G) (and STAB(G)) is $\mathcal{NP}$-hard!

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$u^{(1)}, u^{(2)}, \ldots, u^{(|V|)} \in \mathbb{R}^d$ such that

$$\langle u^{(i)}, u^{(j)} \rangle = 0, \text{ for all } i \neq j, \{i, j\} \notin E,$$

and

$$\langle u^{(i)}, u^{(i)} \rangle = 1, \quad \text{for all } i \in V.$$

$$\mathrm{TH}(G) := \left\{ x \in \mathbb{R}_+^{|V|} : \sum_{i \in V} \left(c^T u^{(i)} \right)^2 x_i \leq 1, \right.$$
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but infinitely many linear inequalities!

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(v) G *does not contain an odd-hole or odd anti-hole*

(vi) *the ideal generated by* $\left\{(x_v^2 - x_v), \forall v \in V; x_u x_v, \forall \{u, v\} \in E\right\}$ *is* $(1,1)$*-SoS.*

There is a strong connection between $\text{LS}_+(G)$ and $\text{TH}(G)$:

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Theorem

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$$\mathrm{TH}(G) = \left\{ x \in \mathbb{R}^V : \begin{pmatrix} 1 \\ x \end{pmatrix} = Ye_0;\ Y_{ij} = 0, \forall \{i, j\} \in E; \right.$$
$$\left. Ye_0 = \mathrm{diag}\,(Y);\ Y \succeq 0 \right\}.$$

Theorem

(Lovász and Schrijver [1991]) For every graph G,

$$LS_+(G) \subseteq \text{OC}(G) \cap \text{ANTI-HOLE}(G) \cap \text{WHEEL}(G) \cap \text{CLQ}(G) \cap \text{TH}(G).$$

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Open Problem: Give full, elegant, combinatorial characterizations for $\mathrm{LS}_+(G)$.

Is $\mathrm{LS}_+(G)$ polyhedral for every G?

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Let $G_{\alpha\beta}$ be the graph in the following figure:

. . .

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Let $G_{\alpha\beta}$ be the 8-node graph in the following figure, on the board:

A two dimensional cross-section of the compact convex relaxation $\text{LS}_+(G_{\alpha\beta})$ has a nonpolyhedral piece on its boundary.

We say that $z \in \mathbb{R}^8$ is an *$\alpha\beta$-point*, if α and β are both nonnegative and $z_i := \begin{cases} \alpha & \text{if} \quad i \in \{1,2,3,4\}, \\ \beta & \text{if} \quad i \in \{5,6,7,8\}. \end{cases}$

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Theorem

(Bianchi, Escalante, Nasini, T. [2014]) An $\alpha\beta$-point with $\frac{1}{4} \le \alpha \le \frac{1}{2}$ belongs to $\mathrm{LS}_+(G_{\alpha\beta})$ if and only if $\beta \le \frac{3-\sqrt{1+8(-1+4\alpha)^2}}{8}$.

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We call these LS_+*-perfect graphs*.

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A graph is called *near-bipartite* if after deleting the closed neighborhood of *any* node, the resulting graph is bipartite. Let us denote by NB the class of all near-bipartite graphs.

For every graph G,

- $\mathrm{LS}_+(G) \subseteq \mathrm{NB}(G)$ and

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Open Problem: Find a combinatorial characterization of LS_+-perfect graphs.

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Current best characterization (Bianchi, Escalante, Nasini, T. [2014])

$$\text{LS}_+(G) \subseteq \text{NB}(G) \cap \hat{\text{TH}}(G).$$

What is the smallest graph which is LS_+-imperfect?

In a related context, Knuth (1993) asked what is the smallest graph for which $\mathrm{STAB}(G) \neq \mathrm{LS}_+(G)$?

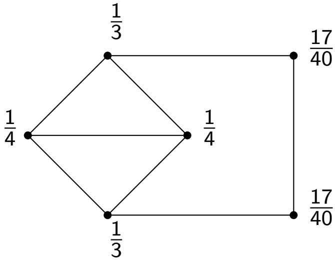

Figure: Little graph that could! G_2 with corresponding weights

Proposition

(Lipták, T., 2003) G_2 *is the smallest graph for which* $\mathrm{LS}_+(G) \neq \mathrm{STAB}(G)$.

The LS-*rank of P* is the smallest k for which $\mathrm{LS}^k(P) = P_I$.
Analogously, LS_0-*rank of P*, LS_+-*rank of P_I relative to P* ...
We denote these ranks by $r(G)$, $r_0(G)$, and $r_+(G)$, respectively.

Theorem

(Lipták, T. [2003]) For every graph $G = (V, E)$, $r_+(G) \leq \left\lfloor \frac{|V|}{3} \right\rfloor$.

$$n_+(k) := \min\{|V(G)| : r_+(G) = k\}.$$

Open Problem: What are the values of $n_+(k)$ for every $k \in \mathbb{Z}_+$? In particular,
Conjecture (Lipták, T. [2003]): Is it true that $n_+(k) = 3k$ for all $k \in \mathbb{Z}_+$?

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bipartite graphs, series-parallel graphs, perfect graphs and odd-star-subdivisions of graphs in $\mathcal{B}$ (which contains cliques and wheels, among many other graphs), antiholes and graphs that have N_0-rank ≤ 2. Also true for all 8-node graphs, and for 9-node graphs that contain a 7-hole or a 7-antihole as an induced subgraph Au [2008].

Other lower bound results: Stephen and T. [1999], Cook and Dash [2000], Goemans and T. [2001], Laurent [2002], Laurent [2003], Aguilera, Bianchi and Nasini [2004], Escalante, Montelar and Nasini [2006], Arora, Bollobás, Lovász and Tourlakis [2006], Cheung [2007], Georgiou, Magen, Pitassi, Tourlakis [2007], Schoeneback, Trevisan and Tulsiani [2007], Charikar, Makarychev and Makarychev [2009], Mathieu and Sinclair [2009], Raghavendra and Steurer [2009], Benabbas and Magen [2010], Karlin, Mathieu and Thach Nguyen [2010], Chan, Lee, Raghavendra and Steurer [2013]. Many of the lower bound proofs have been unified/generalized: Hong and T. [2008].

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Stronger "lower bound" results via study of extended complexity.

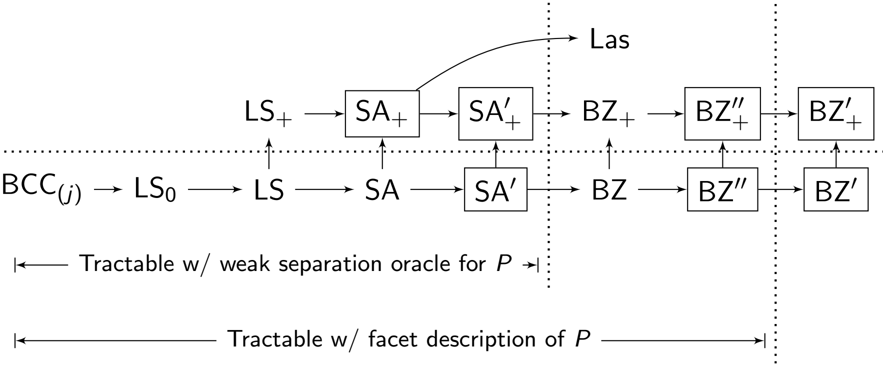

Figure: Various properties of lift-and-project operators (Au and T. [2011, 2013]).

Denote $\{0,1\}^d$ by $\mathcal{F}$. Define $\mathcal{A} := 2^{\mathcal{F}}$. For each $x \in \mathcal{F}$, we define the vector $x^{\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ such that

$$x_{\alpha}^{\mathcal{A}} = \begin{cases} 1, & \text{if } x \in \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

For any given $x \in \mathcal{F}$, if we define $Y_{\mathcal{A}}^{x} := x^{\mathcal{A}}(x^{\mathcal{A}})^{T}$, then, the following must hold:

- $Y_{\mathcal{A}}^{x} e_0 = (Y_{\mathcal{A}}^{x})^{T} e_0 = \operatorname{diag}(Y_{\mathcal{A}}^{x}) = x^{\mathcal{A}}$;
- $Y_{\mathcal{A}}^{x} e_\alpha \in \left\{0, x^{\mathcal{A}}\right\}, \ \forall \alpha \in \mathcal{A}$;
- $Y_{\mathcal{A}}^{x} \in \mathbb{S}_{+}^{\mathcal{A}}$;
- $Y_{\mathcal{A}}^{x}[\alpha, \beta] = 1 \iff x \in \alpha \cap \beta$;
- If $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, then $Y_{\mathcal{A}}^{x}[\alpha_1, \beta_1] = Y_{\mathcal{A}}^{x}[\alpha_2, \beta_2]$.

Given $S \subseteq [d]$ and $t \in \{0, 1\}$, we define

$$S|_t := \{x \in \mathcal{F} : x_i = t, \ \forall i \in S\}.$$

For any integer $i \in [0, d]$, define

$$\mathcal{A}_i := \{S|_1 \cap T|_0 : S, T \subseteq [n], S \cap T = \emptyset, |S| + |T| \leq i\}$$

and

$$\mathcal{A}_i^+ := \{S|_1 : S \subseteq [d], |S| \leq i\}.$$

1. Let $\tilde{\text{SA}}^k(P)$ denote the set of matrices $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k}$ that satisfy all of the following conditions:

(SA 1) $Y[\mathcal{F}, \mathcal{F}] = 1$;

(SA 2) $\hat{x}(Ye_\alpha) \in K(P)$ for every $\alpha \in \mathcal{A}_k$;

(SA 3) For each $S|_1 \cap T|_0 \in \mathcal{A}_{k-1}$, impose

$$Ye_{S|_1 \cap T|_0} = Ye_{S|_1 \cap T|_0 \cap j|_1} + Ye_{S|_1 \cap T|_0 \cap j|_0}, \ \forall j \in [n] \setminus (S \cup T).$$

(SA 4) For each $\alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta = \emptyset$, impose $Y[\alpha, \beta] = 0$;

(SA 5) For every $\alpha_1, \alpha_2 \in \mathcal{A}_1^+, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, impose $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Let $\tilde{\text{SA}}_+^k(P)$ denote the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_k}$ that satisfies all of the following conditions:

(SA_+ 1) (SA 1), (SA 2) and (SA 3);

(SA_+ 2) For each $\alpha, \beta \in \mathcal{A}_k$ such that $\text{conv}(\alpha) \cap \text{conv}(\beta) \cap P = \emptyset$, impose $Y[\alpha, \beta] = 0$;

(SA_+ 3) For any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, impose $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

3. Define

$$\text{SA}^k(P) := \left\{ x \in \mathbb{R}^d : \exists Y \in \tilde{\text{SA}}^k(P) : Ye_{\mathcal{F}} = \hat{x} \right\}$$

Given $P := \left\{x \in [0,1]^d : Ax \leq b\right\}$, and an integer $k \in [d]$,

1. Let $\tilde{\text{Las}}^k(P)$ denote the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_{k+1}^+}$ that satisfy all of the following conditions:

(Las 1) $Y[\mathcal{F}, \mathcal{F}] = 1$;

(Las 2) For each $j \in [m]$, let A^j be the j^{th} row of A. Define the matrix $Y^j \in \mathbb{S}^{\mathcal{A}_k^+}$ such that

$$Y^j[S|_1, S'|_1] := b_j Y[S|_1, S'|_1] - \sum_{i=1}^{n} A_i^j Y[(S \cup \{i\})|_1, (S' \cup \{i\})|_1]$$

and impose $Y^j \succeq 0$.

(Las 3) For every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k^+$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, impose $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Define

$$\text{Las}^k(P) := \left\{x \in \mathbb{R}^d : \exists Y \in \tilde{\text{Las}}^k(P) : \hat{x}(Ye_{\mathcal{F}}) = \hat{x}\right\}.$$

In our setting, the Las-rank of a polytope P (the smallest k such that $\text{Las}^k(P) = P_I$) is equal to the Theta-rank, defined by Gouveia, Parrilo, Thomas [2010].

Consider the set

$$P_{n,\alpha} := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \le n - \alpha \right\}.$$

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Theorem

(Au and T. [2015]) Suppose an integer $n \geq 5$ *is not a perfect square. Then there exists* $\alpha \in (\lfloor \sqrt{n} \rfloor, \lceil \sqrt{n} \rceil)$ *such that the* BZ'_+*-rank of* $P_{n,\alpha}$ *is at least* $\left\lfloor \frac{\sqrt{n}+1}{2} \right\rfloor$.

Theorem

(Au and T. [2015]) For every $n \geq 2$, the SA_+-rank of $P_{n,\alpha}$ is n for all $\alpha \in (0, 1)$.

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1. *For every even integer $n \geq 4$, the Las-rank of $P_{n,\alpha}$ is at most $n-1$ for all $\alpha \geq \frac{1}{n}$;*
2. *For every integer $n \geq 2$, there exists $\alpha \in \left(0, \frac{1}{n}\right)$ such that the Las-rank of $P_{n,\alpha}$ is n.*

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Theorem

(Au and T. [2015]) Suppose $n \geq 2$*, and*

$$0 < \alpha \leq n \left(\frac{3 - \sqrt{5}}{4n-4} \right)^n .$$

Then $P_{n,\alpha}$ *has* Las*-rank* n*.*

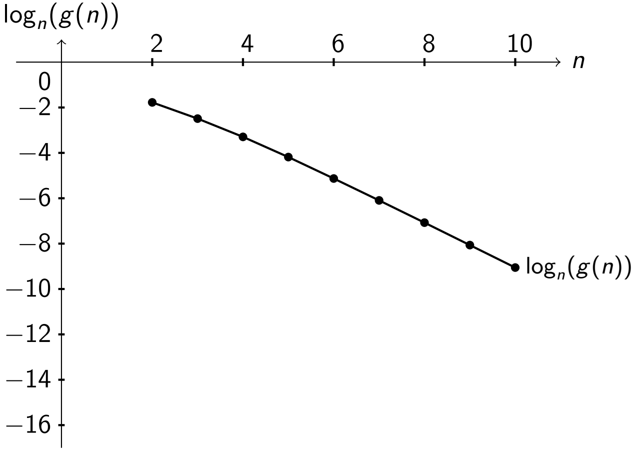

Figure: Computational results and upper bounds for $g(n) := \max\left\{\alpha : \text{Las}^{n-1}(P_{n,\alpha}) \neq (P_{n,\alpha})_I\right\}$ (Au and T. [2015]).

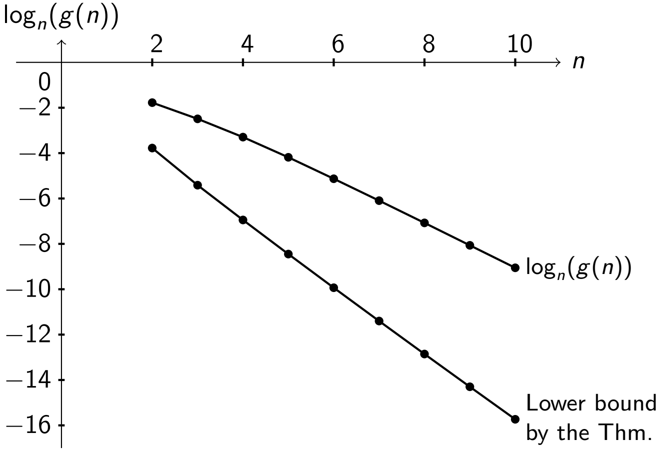

Figure: Computational results and upper bounds for $g(n) := \max \left\{ \alpha : \mathrm{Las}^{n-1}(P_{n,\alpha}) \neq (P_{n,\alpha})_I \right\}$ (Au and T. [2015]).

Given $\alpha > 0$, we define the set

$$Q_{n,\alpha} := \left\{ x \in [0,1]^n : \sum_{i \in S} (1 - x_i) + \sum_{i \notin S} x_i \geq \alpha, \ \forall S \subseteq [n] \right\}.$$

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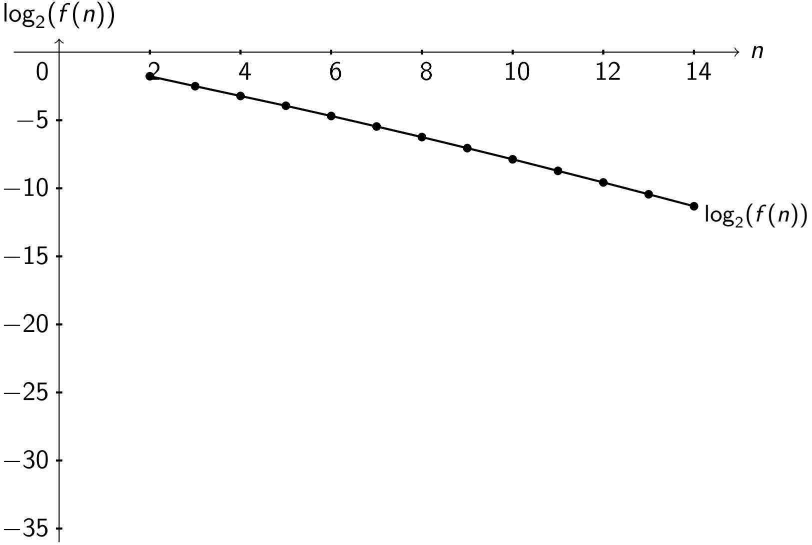

Figure: Computational results and possible ranges for $f(n) := \max\{\alpha : \mathrm{Las}^{n-1}(Q_{n,\alpha}) \neq \emptyset\}$ (Au and T. [2015]).

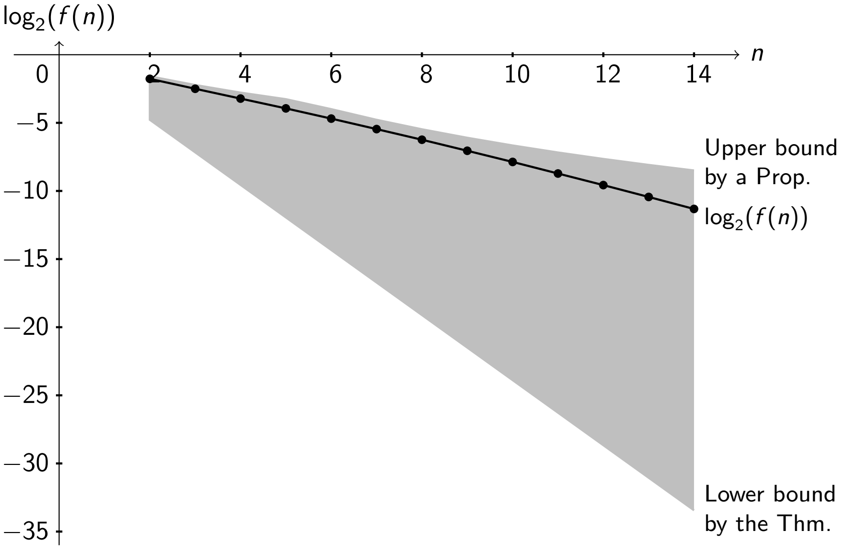

Figure: Computational results and possible ranges for $f(n) := \max\{\alpha : \text{Las}^{n-1}(Q_{n,\alpha}) \neq \emptyset\}$ (Au and T. [2015]).

For the complete graph $G := K_n$, $\text{FRAC}(G)$ has rank 1 with respect to LS_+, SA_+ and Las operators. However, the rank is known to be $\Theta(n)$ for all other operators that yield only polyhedral relaxations, such as SA and Lovász and Schrijver's LS operator.

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Theorem

(Au and T. [2013]) Suppose G is the complete graph on $n \geq 3$ vertices. Then the BZ'*-rank (and the* BZ*-rank) of* $\text{FRAC}(G)$ *is between* $\left\lceil \frac{n}{2} \right\rceil - 2$ *or* $\left\lceil \frac{n+1}{2} \right\rceil$.

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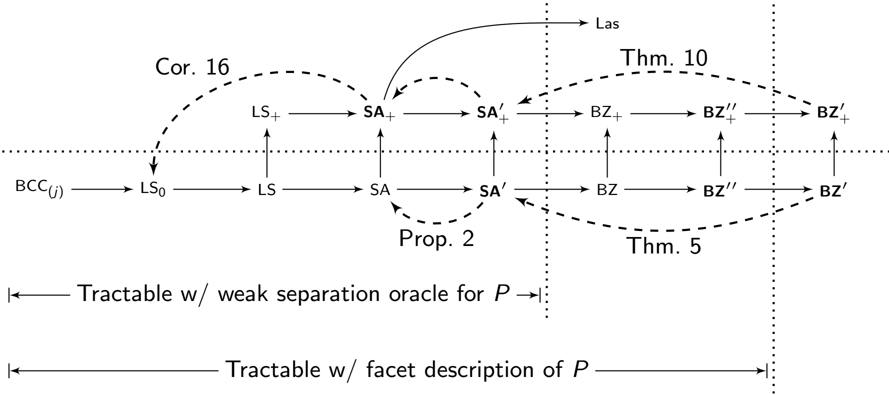

Figure: An illustration of several restricted reverse dominance results (dashed arrows) Au and T. [2013, 2015].