Orthogonal and Multiple Orthogonal Polynomials (15w5022)

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1 Overview of the Field

The theory of orthogonal and multiple orthogonal polynomials has greatly advanced in recent decades. Multiple orthogonal polynomials have been known to play an important role in the proofs of the irrationality of zeta(3). Also orthogonal and multiple orthogonal polynomials play a key role in the understanding of random matrices. These advances were inspired by Riemann-Hilbert and later semi-classical analysis of various quantities associated with systems of orthogonal and multiple orthogonal polynomials. The techniques have also shown how various nonlinear differential equations of the Painleve type arise depending on the random matrix model under consideration and the scaling limits taken. Complex analytic techniques and sophisticated potential theory continue to generate advances on Bergman polynomials, and polynomials orthogonal with respect to exponential weights, and multiple orthogonal polynomials. Furthermore these techniques are now being applied to extend the theory of multiple orthogonal polynomials into the complex plane. The computation of Lax pairs associated with these systems gives a connection to integrable systems. Also the connection between matrix orthogonal polynomials and the bispectral problem is an important development that is in the processes of being investigated. A recent important development has been the interplay between spectral theory, Jacobi matrices, and orthogonal polynomials on the unit circle. Also the spectral theory of Jacobi matrices has been enriched and the connection of Dunkl operators in orthogonal polynomials has been amplified. From other recent work it appears that the recurrence formulas associated with Nikishin systems maybe an appropriate generalization of three term recurrences to higher order difference equations, including the connection of conformal maps on Riemann surfaces.

The connection between special functions and orthogonal polynomials is an extremely important problem as this sheds light on both areas.

This conference brought together experts in the various areas mentioned above to foster collaborations and interactions between these experts.

2 Topics Covered by the Workshop

2.1 Multiple Orthogonal Polynomials

The meeting was started off by Arno Kuijlaars who talked about multiple orthogonal polynomials and their role in matrix models with external sources. In particular he discussed two matrix models related to the products of complex Ginibre matrices. Hiroshi Miki discussed a generalization of the Toda lattice for which both continuous and discrete multiple orthogonal polynomials are the eigenfunctions. Walter Van Assche discussed some new results (with Aptekarev and Yattselev) on multiple orthogonal polynomials whose orthogonality intervals intersect and Jorge Arvesu presented results on the nth root asymptotic behavior of multiple Meixner polynomials using a constrained equilibrium problem. Guillermo Lopez discussed the connection of solutions to higher order constant coefficient recurrence relations and Nikishin systems. He showed that in many cases modified Nikishin systems give rise to constant coefficient recurrence formulas and vise versa. This indicates that multiple orthogonal polynomials play a role similar to orthogonal polynomials in the solution of nth order difference equations. Results similar to this were presented by Abey Lopez for multiple orthogonal polynomials on star-like sets. In this case the higher order recurrences have all coefficients equal to zero except three and the coefficients are allowed to vary with the degree of the solution. Maxim Derevyagin discussed how the discrete Toda equation in higher dimensions are related to multiple orthogonal polynomials. Since one of the main themes of the conference was on multiple orthogonal polynomials a number of workers in this area were invited which has lead to some collaborations. For instance the talk of Miki was based on some work done by Aptekarev and VanAssche and this meeting provided a venue for these researcher to meet. His talk engendered a lively discussion and start a collaboration between those three researchers and Derevyagin. This has resulted in the preprint *Multidimensional Toda lattices: continuous and discrete time* (arXiv 1511.08098).

2.2 Special Functions and multivariate orthogonal Polynomials

A part of the meeting was devoted to special functions, multivariate and matrix orthogonal polynomials. Albert Grunbaum gave a historical review of the bispectral problem indicating the roots of the problem and ending with the new results on the matrix bispectral problem and its connection with matrix orthogonal polynomials. The talk by Pablo Román discussed the connection between quantum groups and q-matrix orthogonal polynomials. In particular for the cases he was studying he was able to find a q-difference equation. Jan van Diejen considered recurrence formulas for multivariate orthogonal polynomials and showed that these could be considered as eigenvalue equations for lattice quantum integrable systems. An interesting aspect of these systems is that there are bispectral dual particle models. Luc Lapointe considered generalizations of the Macdonald polynomials to the superspace setting. Interesting in this talk was how the partition function for the 6 vertex model arises from certain rules on these polynomials. The theme of multivariate orthogonal polynomials was also picked up by Plamen Iliev. Here the connections coefficients between to bases of Jacobi polynomials on the simplex were calculated. These connection coefficients can be written in terms of Racah polynomials. Special cases include the connection coefficients for multivariable Hahn and Krawtchouk polynomials. Luc Vinet considered the connection between the non-symmetric Wilson polynomials and a certain degenerate double affine Hecke algebra. He showed that these coincide with the Bannai-Ito polynomials and the algebra associated with these polynomials is isomorphic to the above Hecke algebra. Mourad Ismail presented a new set of polynomials orthogonal on the disk. James Henegan obtained results on the asymptotic behavior of polynomial that are orthogonal with respect to area measure over a bounded domain with certain components removed. In the same vein Nikos Stylianopoulos considered the orthogonal polynomials associated with bounded simply connected domain in the complex plane whose boundary is a Jordan curve. He discussed the asymptotics of the polynomials on the boundary of the domain and also the relation between the leading coefficients of these polynomials and the Grunsky coefficients of the normalized mapping from the exterior of the simply connected to the exterior of the unit circle. Laurent Baratchart considered the exterior asymptotics of polynomials orthogonal with respect to weights supported on the unit disk having mild regularity conditions. For instance the weight should have certain radial limits and satisfy a mild integrability condition. He then discussed extensions to analytic simply connected domains.

2.3 Univariate orthogonal Polynomials

Many new and interesting connections and results for univariate orthogonal polynomials were presented. Christian Berg presented an important generalization of a result of Schoenberg relating the Gegenbauer expansion of a function to it being positive definite. The extension is to functions continuous on $[-1, 1] \times G$ where G is a locally compact group. Maurice Duits discussed the multi-time fluctuations on certain noncolliding processes. The global fluctuations were analyzed via the recurrence coefficients in the three term recurrence formula satisfied by the orthogonal polynomials associated with this problem. He showed that the fluctuations are governed by the Gaussion Free Field. Don Wang discussed nonintersecting Browning motions starting from a common point and ending at a common point. He considered various boundary conditions at the end points and used Riemann-Hilbert analysis to solve the limit as the number of points go to infinity and the degree of the polynomials associated with the problem goes to infinity, the so called double scaling limit. Antonio Duran considered the regularity properties for the zeros of Wronskian determinants whose entries are orthogonal polynomials. If the orthogonal polynomials are chosen sequentially the problem was one that was considered by Karlin and Szegő. The new results pertained to when the sequence has gaps. Manuel Domínguez constructed a new family of polynomials from the Casorati determinant of the Hahn polynomials and another arbitrary set of polynomials. He showed in certain cases the new polynomials obtained were in fact orthogonal polynomials. Natig Atakishiyev obtained an explicit form of the difference analogue of the quantum number operator for the 5D discrete Fourier transform. The eigenvalues of this operator are given by distinct nonnegative numbers so that the operator can be used to classify the eigenvectors of the 5D discrete Fourier transform. Doron Lubinsky discussed how the scaling limit of certain sequences of polynomials give entire functions. The relation with orthogonal polynomials was discussed. Brian Simanek discussed the relationship between the Bergman Shift matrix and the asymptotic behavior of orthogonal polynomials. He showed that ratio asymptotics for the polynomials is related to the asymptotic behavior of the entries of the diagonal elements of the Bergman Shift matrix. Alexander Aptekarev discussed the Pade approximants associated with analytic functions having multiple branch points. He discussed the strong asymptotics for the denominators of the Pade approximants and as well as various applications. Katarzyna Kozlowska discussed the change in asympotitics in Toeplitz determinants as the symbol changed from being smooth to having Fisher-Hartwig singularities. The change in the asymptotics model the phase transitions found in statistical physics. Luis Garza discussed some recent results on characterizations of semiclassical orthogonal polynomials based on lower semi-infinite matrices containing the coefficients of the polynomials. Matrix relations involving the Jacobi matrices associated with these orthogonal polynomials were also discussed. Jeff Geronimo presented the asymptotices of polynomials orthogonal with respect finite uniform discrete measure supported on an arc of the unit circle in the limit as the number of points of increase tend to infinity. This problem is related to the Fourier continuation method and results show that in the sup norm there are cases where the polynomials increase exponentially fast.