Sandpile Groups

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1 Overview of the Field

The physicists Bak, Tang, and Wiesenfeld [5] created an idealized version of a sandpile in which sand is stacked on the vertices of a graph and is subjected to certain avalanching rules. They used the model as an example of what they called *self-organized criticality*. The abelian sandpile model is a variation, due to the physicist Deepak Dhar in 1990 [24], in which the avalanching obeys a useful commutativity rule. He realized that the model provided an expression of the dynamics inherent in the discrete Laplacian of a graph.

The long-term behavior of the abelian sandpile model on a graph is encoded by the *critical configurations*. These critical configurations have connections to parking functions [45], to the Tutte polynomial [46], and to the lattices of integral flows and cuts of a graph [53]. Among other properties, the critical configurations of the sandpile model have the structure of a group, and this group is our main object of study. It has been discovered in several different contexts and received many names: the *sandpile group* for graphs [24] and digraphs [49], the *critical group* [8], the *group of bicycles* [7], the *group of components* [51], and the *jacobian* of the graph [54].

The abelian sandpile model and its close relative the chip-firing game [44] have become a crossroads of a wide range of mathematics, physics, and computer science. Of particular interest in this workshop were connections with: commutative algebra, algebraic and tropical geometry, pattern formation, models of computation, generalizations of chip-firing, matroid theory, graph orientations, and tree-bijections, and random graphs.

2 Recent Developments and Presentation Highlights

Commutative algebra. To each graph, one may associate various ideals encoding chip-firing. At this workshop, Sam Hopkins presented joint work-in-progress with Spencer Backman [40] on various *power ideals* which have relations to areas as disparate as Schubert Calculus and approximation theory. Many open problems related to these ideals are presented in the Open Problems section, below.

Carlos Valencia and Carlos Alfaro presented recent work on *critical ideals*, [20]. These are determinantal ideals of a generalized Laplacian matrix for a graph.

The *toppling ideal* of a graph is the lattice ideal associated to the Laplacian matrix of a graph. Considerable work has gone into the study of the free resolution of this ideal, e.g., [63], [59]. Anton Dochtermann and Raman Sanyal have shown that the free resolution of the corresponding monomial ideal is supported on the graphical hyperplane arrangement. At this workshop, Dochtermann presented his joint work-in-progress with Sanyal generalizing this result to tropical hyperplane arrangements.

Tropical geometry and pattern formation. One of the themes of our subject is to think of a graph as a discrete version of a Riemann surface. In between graphs and Riemann surfaces lies a slightly less discrete version of a graph called a *metric graph*. These are graphs for which each edge has an assigned length. Metric graphs serve as models of tropical curves. During the workshop, Melody Chan presented an introduction to tropical geometry for sandpile theorists. Her talk was followed by those of Nikita Kalinin and Mikhail Shkolnikov in which they presented extraordinary new results accounting for the appearance of tropical curves in the scaling limits of sandpile configurations, [41].

Models of computation. It has been known for some time that the sandpile model can mimic a Turing machine, [35]. A generalization of the sandpile model called an *abelian network* ([10], [25]) is a system of communicating finite automata satisfying a certain local commutativity condition, and sandpiles are examples of such networks. Computationally, abelian networks model algorithms on graphs that can be implemented in a completely asynchronous distributed manner. The current ubiquity of data sets on large graphs (arising from biological networks, the brain, and the internet) has produced a growing demand for such algorithms. At this workship Lionel Levine presented his work on abelian networks, highlighting three open questions:

1. Energy: Following Mohammadi and Shokrieh (2014) and Guzman and Klivans (2015), define a notion of superstable for abelian networks. What energy do superstable configurations minimize?

2. Taxonomy: Given a list of gates, what is the class of functions computable by an abelian network built out of those gates?

3. Coefficients: How does the computational power of an abelian network with coefficients in a monoid C vary with C?

Swee Hong Chan presented attempts to extend to abelian networks Merino's theorem relating the Tutte polynomial of a graph to the generating function for its critical configurations.

M-matrices. The columns of the discrete Laplacian matrix of a graph encode the chip-firing rules for the sandpile model. A. Gabrielov has shown that if one substitutes an arbitrary M-matrix for the discrete Laplacian and uses its columns to define chip-firing rules, it is still the case that every configuration stabilizes. In Caroline Klivans' workshop talk, she showed how many aspects of graphical chip-firing (criticality, energy minimization and superstability) extend to the M-matrix setting. Vic Reiner then presented joint with Klivans, applying this more general theory to chip-firing on Dynkin diagrams and McKay quivers. Hugo Corrales discussed the relation between M-matrices and arithmetical graphs.

Matroid theory and partial orientations. One way of describing the critical group of a graph is as the quotient of the integral edge group of the graph by the sum of graph's integral cycle and cut spaces. Hence, some of the matroidal structure of a graph is encoded in its critical group. C. Merino exploited this fact in his proof of Stanley's h-vector conjecture for cographical matroids, [47].

In our workshop, Farbod Shokrieh gave an overview of matroid theory and discussed how to extend some of the notions of chip-firing to the class of regular matroids. For instance, one can define Jacobian groups whose cardinality/volume is related to the ”complexity” of the matroid.

Spencer Backman showed how partial orientations of the edges of a graph are related to graphical Riemann-Roch theory.

Tree bijections. The critical group of a graph may be viewed as the integral cokernel of the graph's discrete Laplacian operator. Hence, via the matrix-tree theorem, the cardinality of the critical group is the number of spanning trees of the graph. Much work has been done on combinatorial bijections between critical configurations and spanning trees. At this workshop, Chi Ho Yuen presented the state of the art for extensions to the case of tropical curves.

3 Ongoing projects stemming from the workshop

There are several ongoing collaborations facilitated or initiated at the workshop.

- Some of the foundational problems in our subject were introduced by Lorenzini, [51], who was concerned with arithmetical structures on graphs coming from the study of degenerating algebraic curves in algebraic geometry. A group of about 12 people at the workshop became interested in work involving the combinatorics of arithmetical structures on graphs. These include researchers from both the US and Mexico, and most of these are new collaborators. They already have partial results, and are continuing to work on a combined project.

  Specifically, they are studying exact enumeration of the arithmetical graphs for fixed graphs or families of graphs, e.g., paths, cycles, trees of bounded degree, etc. In addition, they are studying refined enumeration based on statistics on those arithmetical structures. For example, there are Catalan many arithmetical structures on cycle graphs, and when you refine by the number of 1's that appear in a fixed arithmetical structure (the “r-vector”) one sees that this corresponds to partitioning the arithmetical structures into groups with sizes given by Ballot Numbers. Further, they are studying sandpile groups, and have proven a relationship between the size of the cyclic sandpile group for a given arithmetical structure and the number of 1's in the associated r-vector for that arithmetical structure.

  H. Corrales and C. E. Valencia are preparing three manuscripts on arithmetical graphs, [21], [22], [22], and are collaborating with the above group on further work.

  For more information, contact Benjamin Braun.

- Georgia Benkart, Carly Klivans, and Vic Reiner have just posted ”Chip-firing on Dynkin diagrams and McKay quivers” to the arXiv. Conversations they had with Sam Payne at the conference led to one of the propositions in this paper.

- Avi Levy connected with Lionel Levine and Swee Hong Chan at the workshop and have developed a collaboration on the subject of abelian networks.

- Lionel Levine posed a problem during the workshop, about the probability that a random directed graph is coEulerian. Sam Payne has taken up that problem with his student Shaked Koplewitz.

It seems to be an interesting new direction to explore, and it is likely to end up being part of Koplewitz's PhD thesis.

- Laura Florescu, Lionel Levine, and Wilfried Huss developed a collaboration concerning phase transitions in the cover time of rotor walks on random graphs which they hope will lead to a paper in the future.
- Laura Florescu has had conversations with Jeremy Martin and Art Duval about some high-dimensional generalizations for the sandpile process which she spoke about in her workshop talk.
- David Perkinson has a student whose undergraduate thesis is motivated by Caroline Klivan's talk at the workshop.

4 Open problems

Each day at the workshop, time was alloted for posing open problems. We include here a list of these problems. (This compilation is due to Sam Hopkins.)

First, we briefly review the basic setup in the simplest case of an undirected, simple graph $G = (V, E)$, essentially following the presentation in [57]. From now on "graph" will mean "undirected, simple graph" unless it comes with other adjectives. We will always assume that G is connected.[1] A *divisor* of G is an element of $\mathbb{Z}V$, i.e., a formal linear combination of the vertices of G. The *degree* $\deg(D)$ of a divisor D is the sum of its coefficients. Two divisors are *linearly equivalent* if their difference belongs to the image of the graph Laplacian Δ of G. Note that linear equivalence preserves degree. The *Picard group* $\mathrm{Pic}(G)$ is the group of divisors modulo linear equivalence. It is graded by degree: $\mathrm{Pic}(G) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Pic}^d(G)$. Of special note are the parts $\mathrm{Pic}^g(G)$, where $g := \#E - \#V + 1$ is the cyclomatic number (i.e., first Betti number, and also sometimes called "genus") of G, and $\mathrm{Pic}^0(G)$. The group of divisors of degree zero modulo linear equivalence, $\mathrm{Pic}^0(G)$, is also called the *Jacobian* of the graph, denoted $\mathrm{Jac}(G)$. The Jacobian is also often called the *sandpile group* of G. (Yet another name for the sandpile group is the *critical group* of the graph; but we will never use this term from now on.) The sandpile group is isomorphic to $\mathrm{coker}(\widetilde{\Delta})$, the cokernel of the reduced Laplacian of G, and so has order equal to $\det(\widetilde{\Delta})$, which by Kirchoff's Matrix-Tree Theorem is equal to the number of spanning trees of G.

There are various ways to choose representatives for $\mathrm{coker}(\widetilde{\Delta})$. Most of these involve fixing the choice of a *sink vertex* $q \in V$ and can be described via *chip-firing* on the graph. Let $V^q := V \setminus \{q\}$ denote the nonsink vertices of G. A *configuration* on G (w.r.t. q) is an element of $\mathbb{N}V^q$, i.e., an assignment of a nonnegative number of chips to the nonsink vertices of G. If $c = \sum_{v\in V^q} c_v v$ is a configuration, then $v \in V^q$ is *unstable* if $c_v \geq \deg(v)$. When v is unstable, we can "topple" or "fire" v by having v send one chip to each of its neighbors (including potentially q.) We ignore all chips that accumulate at q. By repeatedly toppling unstable vertices in c, we arrive at a *stable configuration* $\widetilde{c}$, i.e., a configuration where no vertices are unstable. The "confluence" property of the Abelian sandpile model says that the map $c \mapsto \widetilde{c}$ is well-defined: it does not matter the order in which we stabilize the vertices; we always arrive at the same stable configuration. One choice of representatives for $\mathrm{coker}(\widetilde{\Delta})$, coming from the study of the longterm dynamics of chip-firing, are the *recurrent configurations*: these are the stable configurations which arise infinitely often in the dynamical process where we randomly

[1] For generalizations of sandpile groups and chip-firing to other settings, "connected" can mean different things: for chip-firing on directed graphs, we should assume that the graph is strongly connected; for chip-firing on matrices, we may want to assume the matrix is irreducible; et cetera.

add chips to the nonsink vertices and stabilize. Specifically, c is recurrent if for every configuration a there is some configuration b such that $\widetilde{a+b} = c$. The set of recurrent configruations, with the binary operation of vertex-wise addition and stabilizaiton, is isomorphic to $\mathrm{Jac}(G)$. Another choice of representatives is defined in terms of *set-toppling*. In a configuration c, a set $U \subseteq V^q$ can topple if every vertex $v \in U$ can simultaneously send one chip to each of its neighbors, and no one ends up with a negative number of chips. We say c is *superstable* if no nonempty set $U \subseteq V^q$ can topple. The set of superstable configurations, with the binary operation of vertex-wise addition and superstabilizaiton, is isomorphic to $\mathrm{Jac}(G)$. The superstables are essentially the same as the *q-reduced divisors*. And, at least in this case where G is undirected, they are the same as the *G-parking functions*. There is a straightforward (non-algebraic!) bijection between the recurrents and superstables: $c \mapsto c_{\max} - c$, where $c_{\max} := \sum_{v\in V^q}(\deg(v)-1)v$ is the *maximal stable configuration*.

Let us also briefly describe the important notion, introduced by Baker and Norine [6], of the *rank of a divisor*. We say a divisor is *effective* if all of its coefficients are nonnegative. The rank $r(D)$ of divisor is some number in $\{-1, 0, 1, \ldots\}$ and $r(D) = -1$ if and only if there is no effective divisor D' linearly equivalent to D. In general $r(D)$ is negative one plus the number of chips an adversary needs to remove from D so that it is not equivalent to any effective divisor. The rank of a graph divisor is supposed to be analogous to the algebro-geometric concept of the rank of a divisor on a curve. In particular, an analog of the Riemann-Roch theorem [6, Theorem 1.12] holds:

$$r(D) - r(K - D) = \deg(D) + 1 - g.$$

Here $K := \sum_{v\in V}(\deg(v) - 2)v$ is the *canonical divisor* of G.

4.1 The problems

4.1.1 David Perkinson: "Total Weierstrass weight of graphs"

G is a graph and $v \in V$ is some vertex. Choose a divisor class $[D] \in \mathrm{Pic}^{2g-1}(G)$. Consider the sequence a_i of integers $a_i := r(D - iV)$:

$\cdots$	a_{-2}	a_{-1}	a_0	a_1	a_2	$\cdots$
$\cdots$	$r(D+2v)$	$r(D+v)$	$r(D)$	$r(D-v)$	$r(D-2v)$	$\cdots$

and the sequence b_i of integers $b_i := g - 1 - i$ if $i < g$ and $b_i := -1$ if $i \geq g$:

$\cdots$	b_{-2}	b_{-1}	b_0	b_1	b_2	$\cdots$	b_{g+2}	b_{g+3}	$\cdots$
$\cdots$	$g+1$	g	$g-1$	$g-2$	$g-3$	$\cdots$	-1	-1	$\cdots$

Set $w_i := a_i - b_i$:

$\cdots$	w_{-2}	w_{-1}	w_0	w_1	w_2	$\cdots$	w_{2g}	w_{2g+1}	$\cdots$
$\cdots$	0	0	0	???	???	$\cdots$	0	0	$\cdots$

Note that w_i is zero for $|i| \gg 0$ thanks to the Riemann-Roch theorem. So we can define $\mathrm{weight}_v(D) := \sum_{i\in\mathbb{Z}} w_i$. And we can also define $t(v) := \sum_{[D]\in\mathrm{Pic}^{2g-1}(G)} \mathrm{weight}_v(D)$, the *total Weierstrass weight* of v. Note that $t(v)$ is independent of the choice of v; so it makes sense to define $t(G) := t(v)$ for any $v \in V$ to be the *total Weierstrass weight* of the graph G.

The problem is to explore $t(G)$ for various graphs G. How does it depend on G? A specific conjecture of Dave and his students is that for $G = K_n$ the complete graph,

$$t(G) = n^{n-3} \cdot \binom{n+1}{4}.$$

4.1.2 Farbod Shokrieh: "Generic submodularity of rank for graphs"

G is a graph or metric graph. Take D a divisor of G and P, Q points on G. Is it true that

$$r(D+P)+r(D+Q) \leq r(D)+r(D+P+Q),$$

if D, P, and Q are "generic"? Farbod has counterexamples if they are not generic. Certainly by the Riemann-Roch theorem the above inequality becomes an equality for D of sufficiently high degree. So the notion of "generic" is left open in this question. A related question is whether the Baker-Norine rank $r(D)$ is the rank of a matroid $M(D)$ in some natural way.

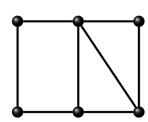
The motivation for this question is that many 19th century results about divisors of algebraic curves can be proved using only the submodularity of the rank function.

4.1.3 Chi Ho Yuen: "Admissible data for family of bijections from spanning trees to $\mathrm{Pic}^g(G)$"

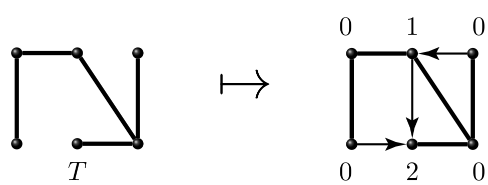
G is a graph. Fix a choice of orientation for every simple cycle (= matroid circuit) of G. Use this data to define a map

$$\begin{aligned} \{\text{spanning trees of } G\} &\to \mathrm{Pic}^g(G) \\ T &\mapsto \begin{array}{l} \text{class of divisor } D \text{ having one chip at head of each edge } e \notin T \\ \text{where } e \text{ is oriented in agreement with the way the unique cycle} \\ \text{of } T \cup \{e\} \text{ is oriented in our data} \end{array} \end{aligned}$$

For example, if G is the following planar graph



and we take our data to always orient simple cycles counterclockwise, then an example application of this map is



The question is if one can give concise necessary and sufficient conditions on the choice of data to make this map a bijection.

Chi Ho [65] has a nontrivial sufficient condition for the map to be bijective: namely, that $n_1C_1 + \cdots + n_tC_t = 0$ has no nonnegative non-zero solution where C_i's are the oriented cycles in our data set viewed as a formal sum of oriented edges. That is, one you pick a cycle with the chosen orientation of the data, you cannot add more cycles to get back to zero. Farbod Shokrieh conjectured that Chi Ho's condition is also necessary for the map to bijective for all graphs G.

4.1.4 Sam Hopkins: "Choices in Dhar's burning algorithm"

Dhar's burning algorithm [24] [25] can be defined to give a bijection

$$\{q\text{-reduced divisors}\} \to \{\text{spanning trees}\}$$

For example, see the bijections of Cori-Le Borgne [18] and Perkinson-Yang-Yu [58]. The specific bijection depends on a choice of "tiebreak" rule for the burning procedure. Can one classify all tiebreak rules? Each rule relates the degree of parking functions to some statistic of tree. For example, in the Cori-Le Borgne bijection the tree statistic is external activity and in the Perkinson-Yang-Yu bijection the tree statistic is Gessel's κ-inversion number [31].

4.1.5 Spencer Backman: "Superstables-spanning trees burning bijection for directed graphs"

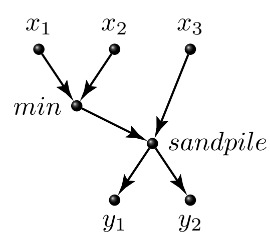
G is a digraph and $q \in V$ is a choice of sink. Can we find a "burning-style" bijective proof that

$$\#\left\{\begin{array}{c}\text{superstables of } G \text{ with}\\ \text{respect to } q\end{array}\right\} = \#\left\{\begin{array}{c} q\text{-rooted spanning trees of}\\ G\end{array}\right\} = \det(\Delta_q).$$

Note that such a bijection between spanning trees and G-parking functions is known (see [15]). For general digraphs, G-parking functions and superstables are not the same thing. They are the same when G is Eulerian. So this question is open only in the case where G is not Eulerian.

4.1.6 Lionel Levine: "Sandpile circuits"

This is a question about the computational power of a certain class of abelian networks [10] [11] [12] allowing only a small number of kinds of processors, as in [38]. Fix a digraph G with arcs divided into three classes: input edges (of which there are k), output edges (of which there are l), and interior edges. The nodes of G are "abelian processors" of the following four kinds: sandpile, min, max, and product. An example of this network is



Here $k = 3$ and $l = 2$. The circuit takes an input $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{N}^k$ and computes an output $\vec{y} = (y_1, \ldots, y_l) \in \mathbb{N}^l$. (When G is a directed acyclic graph then it is clear that there always is a well-defined output. When G has cycles it may run forever; but sometimes even when G has cycles it halts on all inputs and thus computes a function. A condition for halting on all inputs is given in [11].) For example, the above example computes the function

$$(x_1, x_2, x_3) \mapsto \left(\text{floor}\left(\frac{\min(x_1, x_2) + x_3}{2}\right), \text{floor}\left(\frac{\min(x_1, x_2) + x_3}{2}\right)\right).$$

The question is, choosing either "directed" or "directed acyclic", together with some subset $S \subseteq \{\min, \max, \text{product}\}$, and always allowing sandpile nodes, what class of functions can we compute with networks of this form? As an example, when $S = \emptyset$ (that is, allowing only sandpile nodes) the function can be expressed as a sum of a linear and a periodic function. Moreover, any function $F(\vec{x}) = P(\vec{x}) + L(\vec{x})$ with $P(\vec{x}) \in \mathbb{Q}^l$ periodic and $L(\vec{x}) \in \mathbb{Q}^l$ linear can be computed by these sandpile networks so long as $P(\vec{x}) + L(\vec{x}) \in \mathbb{N}^l$ and $P(\vec{x})$ and $L(\vec{x})$ are coordinatewise increasing.

Note that there is no distinction between directed or directed acyclic for this case of $S = \emptyset$. When we allow $\min$ or $\max$, we can now get functions which are just piecewise linear. Similarly, if we through in $\mathrm{product}$ we get functions that are polynomials. But the problem is to classify exactly which functions can be computed.

4.1.7 Lilla Tóthmérész: "Complexity of halting problem for sandpiles on Eulerian multidigraphs"

G is now a digraph. Let $c \in \mathbb{N}^V$ be a chip configuration on G. The general question is: what is the complexity of deciding whether the chip-firing stabilization process with halt? A theorem of Björner and Lovasz [9] says that halting is polynomial time decidable for simple (i.e. no multiple edges directed the same way) Eulerian digraphs G. On the other hand, a theorem of Farrell and Levine [28] shows that the halting problem for chip-firing is $\mathbf{NP}$-complete for general digraphs.

For Eulerian digraphs (with possibly multiple edges) the halting problem is in $\mathbf{NP}$ and $\mathbf{co}$-$\mathbf{NP}$. Lilla conjectures that it is actually in $\mathbf{P}$ in this case. In fact, we have a 2×2 chart of digraph properties for which the chip-firing halting problem is only understood for the upper-left and lower-right squares:

	Eulerian	General
Simple	$\mathbf{P}$	???
Multiple edges	???	$\mathbf{NP}$-complete

It would be interesting to fill in all the squares of this chart.

4.1.8 Dustin Cartwright and Farbod Shokrieh: "Realizing sandpile groups"

Can every finite abelian group A be $\mathrm{Jac}(G)$ for some 2-connected graph G? Note that it is easy to achieve if we do not require G to be 2-connected: if $A \simeq \bigoplus_{i=1}^{n} \mathbb{Z}/a_i\mathbb{Z}$ just let G be a wedge of n cycles of sizes $a_1, a_2, \ldots, a_n$. Here we need to allow multiple edges (i.e., 2-cycles) to achieve summands of $\mathbb{Z}/2\mathbb{Z}$.

Can every pair $(A, \langle,\rangle)$ where A is a finite abelian group and $\langle,\rangle$ is a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form on Γ be realized as $\mathrm{Jac}(G)$ together with its canonical pairing for some 2-connected graph G? Gaudet et al. [30] show that, conditional on the Generalized Riemann Hypothesis, every $(A, \langle,\rangle)$ arises in this way, but again without the requirement that G be 2-connected.

4.1.9 Vic Reiner: "Isomorphism between a group and the Jacobian of its Cayley graph"

Let A be a finite abelian group, and $S = \{a_1, \ldots, a_s\}$ a multiset of nonzero elements of A satisfying $\sum_{i=1}^{s} a_i = 0$. (This condition roughly corresponds to the a_i defining a mapping into $\mathrm{SL}_n(\mathbb{C})$.) Let G be the Cayley digraph of (A, S). Then a fact is that there exists a surjection $\mathrm{Jac}(G) \twoheadrightarrow A$.

Do we have $\mathrm{Jac}(G) \simeq A$ if and only if $A = \mathbb{Z}/m\mathbb{Z}$ for some m and $S = \{a, -a\}$ for some generator a of A? One direction is known: if A and S are of this form, then certainly $\mathrm{Jac}(G) \simeq A$.

4.1.10 Sam Hopkins: "Monomizations of power ideals"

A detailed write-up of this problem is available at [40]. Here is a brief summary. G is a graph, and $q \in V$ a choice of sink. Let $R := \mathbf{k}[x_v \colon v \in V^q]$ be a polynomial ring with generators indexed by nonsink vertices. For $r \geq 1$, define the power ideal

$$J^r := \left\langle \left(\sum_{u \in U} x_u \right)^{\deg(U)+r} : \emptyset \neq U \subseteq V^q \right\rangle$$

where $\deg_U(u) := \#\{e = \{u,v\} : v \in V - U\}$ and $\deg(U) := \sum_{u \in U} \deg_U(u)$. The Macaulay inverse systems to the ideals J^{-1}, J^0 and J^{+1} are the *internal*, *central*, and *external zonotopal algebras* associated to G. It follows from Ardila-Postnikov [1] and Holtz-Ron [39] that

$$\mathrm{Hilb}(R/J^{+1}; y) = y^g \cdot T_G\left(1+y, \frac{1}{y}\right);$$
$$\mathrm{Hilb}(R/J^{0}; y) = y^g \cdot T_G\left(1, \frac{1}{y}\right);$$
$$\mathrm{Hilb}(R/J^{-1}; y) = y^g \cdot T_G\left(0, \frac{1}{y}\right),$$

where $T_G(x,y)$ is the Tutte polynomial of G. We say a monomial ideal $\mathcal{I}$ of R is a *monomization* of any ideal $\mathcal{J}$ of R if the standard monomials of $\mathcal{I}$ are a linear basis of the quotient $R/\mathcal{J}$. Let $<$ be any order on V^q and define monomial ideals

$$I^0 := \left\langle \prod_{u \in U} x_u^{\deg_U(u)} : \emptyset \neq U \subseteq V^q \right\rangle;$$
$$I^{+1} := \left\langle x_{\min_<(U)} \cdot \prod_{u \in U} x_u^{\deg_U(u)} : \emptyset \neq U \subseteq V^q \right\rangle.$$

Observe that the standard monomials of I^0 are precisely the G-parking functions. Postnikov and Shapiro [60] showed that I^0 is a monomization of J^0. Desjardins [19] in his PhD thesis showed that I^{+1} is a monomization of J^{+1}. (Note that the I are not initial ideals of the J with respect to any term order and these results do not appeal to Gröbner basis theory.) Can we find an analogous monomization I^{-1} of the internal power ideal J^{-1} for all graphs G? Sam suggested an approach via partial graph orientations, which goes back to Gessel-Sagan [32], but which also uses a new class of partial orientations ("acyclic, cut internal") defined recently by Backman-Hopkins [4].

4.1.11 Art Duval and Caroline Klivans: "Chip-firing on invertible integer matrices"

Guzmán-Klivans [36] have defined a notion of chip-firing for M-matrices, and more recently a notion of chip-firing for general invertible integer matrices L [37]. Concepts such as recurrent and superstable configurations carry over to this setting. The idea is that given an invertible integer matrix L, we pair L with some M-matrix M. Then we define $N := LM^{-1}$ and $S^+ := \{Nx : Nx \in \mathbb{Z}^n, x \in \mathbb{R}^n_{\geq 0}\}$. Then we chip-fire using the dynamics of M, but treating S^+ as our set of "nonnegative configurations." There are still many interesting open problems for this invertible integral matrix chip-firing. It is interesting even to consider the special case, closely related to work of Duval-Klivans-Martin [27], where $L = AA^T$ and A is a boundary map of some simplicial complex. Here are some specific questions:

(a) Given some L, what is a "good" M-matrix M to pair it with? What is the "closest" M-matrix to a given L? Is the space of M-matrices nice enough (e.g., convex) to have a projection?

(b) Find an M-matrix to pair with L so that we have a nice notion of grading in S^+; e.g., we could ask for a version of Merino's theorem [48] [47] using the Tutte polynomial of the matroid of A.

(c) Does there exist a natural toppling ideal (see [57, §4]) in this general setting?

4.2 David Perkinson: "Burning algorithm for M-matrices"

Is there a burning (or script in the sense of Speer [61]) algorithm for M-matrices? Bond-Levine [12, §5] have such an algorithm for abelian networks. The Laplacians of abelian networks that halt on all inputs are indeed M-matrices (see [11, Corollary 6.4]).

4.3 Luis Garcia Puente: "Bijection between recurrents for an M-matrix and its transpose"

Of course $\mathrm{coker}(M) = \mathrm{coker}(M^T)$ for M an M-matrix. The recurrent elements are certain representatives for $\mathrm{coker}(M)$. In the appendix of [60], Postnikov-Shapiro put forward the following natural question: can we find a bijection between the recurrents of M and of M^T? Note that this question is closely related to the question Spencer Backman asked above about a bijection between superstables and spanning trees for directed graphs because the superstables of M are the parking functions of M^T and vice-versa; and as mentioned, there is a spanning tree-parking function bijection for directed graphs due to Chebikin-Pylyavskyy [15].

4.4 Shaked Koplewitz: "Cohen-Lenstra heuristics for Jacobians of random regular graphs"

Building on work of Clancy et al. [16], Wood [64] has recently determined the distribution of $\mathrm{Jac}(G(n,p))$ as $n \to \infty$, where $G(n,p)$ is the Erdős-Rényi random graph. The distribution is closely related to the "Cohen-Lenstra heuristics" [17] that (conjecturally) govern the distribution of random class groups. Are there similar Cohen-Lenstra heuristics for the Jacobians of random regular graphs? Here our model can be $G_{n,d}$, the random d-regular graph on n vertices. Work of Van Vu and collaborators [43] [42] shows that it is not so unreasonable to expect random regular graphs to behave similarly to random graphs in many respects.

4.5 Nikita Kalinin: "Degree of tropical curves appearing in limits of sandpile stabilizations on a two-dimensional grid"

Consider the sandpile dynamics on an $n \times n$ two-dimensional grid (so every vertex has 4 neighbors and the sink is the "boundary" of this grid). If we start with the maximal stable configuration that assigns 3 chips to every vertex, and then add some finite number d more chips to various sites, and then stabilize, most of the vertices will return to having a value of 3. Physicists [13] [14] [55] observed experimentally that as we send n to infinity and rescale properly, the points that do not have a value of 3 form an interesting one-dimensional (in fact, piece-wise linear) set. Very recently, Kalinin-Shkolnikov [41] established rigorously that indeed in the limit the points which have a value different from 3 form a tropical curve (at least away from the boundary of the domain). But what is the degree of this tropical curve? Nikita conjectured that it should have degree $c\sqrt{d}$ for some absolute constant c asymptotically almost surely if the d extra chips are generically distributed. Note that some special assignments of d extra chips can produce curves of much higher degree.

4.6 Sam Hopkins: "Symmetric chip-firing and harmonic dihedral actions on graphs"

Dave Perkinson in his demonstration of the Sage sandpile package also sketched a kind of symmetric chip-firing dynamics: we start with a graph G that has some symmetry; we consider symmetric configurations of chips on this graph; when we topple a vertex we topple all vertices in its orbit under the automorphism group of the graph and so the configuration remains symmetric. On the other hand, Darren Glass [34] [33] has investigated the sandpile groups of graphs that come with the action of a dihedral group; in particular establishing a relationship between the sandpile group of the graph and the sandpile group of the quotient by this dihedral action. There are obvious differences between these approaches: on one hand, in Dave's setup the dynamics of symmetric chip-firing may not correspond to the Laplacian of any graph (they at least correspond to an M-matrix, however); on the other hand, in Darren's setup the dihedral group has to act *harmonically* on the graph. Still, could there be some relationship between these two versions of sandpile groups with symmetries?

4.7 Dustin Cartwright: "Harmonic dihedral actions on tropical curves"

Does the work of Glass [34] [33] on the Jacobians of graphs admitting a harmonic action of a dihedral group extend to tropical curves (i.e., metric graphs)?

4.8 Avi Levy and Farbod Shokrieh: "Alternate description of electrical network cohomology'

Avi Levy in his talk described a graph cohomology theory defined by students at the University of Washington REU studying inverse problems for electrical networks. Briefly, the story goes as follows. G is a graph, and $\partial G \subseteq V$ is a set of *boundary vertices*. We call $\Gamma = (G, \partial G)$ an *electrical network*. (Levy also allows edge-weights, but we will ignore this here.) A function $\varphi\colon V \to M$ is Γ-*harmonic* if it is harmonic (i.e. $\sum_{u\sim v} \varphi(u) = 0$) at all vertices v not in the boundary ∂G. Fix R a commutative ring. Define a functor $\mathcal{U}(\Gamma, -)\colon R - \mathrm{mod} \to R - \mathrm{mod}$ by

$$\mathcal{U}(\Gamma, M) := \{M\text{-valued } \Gamma\text{-harmonic functions}\}.$$

Then $\mathcal{U}(\Gamma, -)$ is left-exact and has a right derived functor. So set $\mathcal{U}^i(\Gamma, -) := R^i\mathcal{U}(\Gamma, -)$. $\mathcal{U}^i(\Gamma, M)$ is the *ith (electrical network) cohomology module* of Γ with coefficients in M.

A theorem of Levy is that $\mathcal{U}^1(\Gamma, \mathbb{Z}) = \mathrm{Jac}(G)$ when Γ is the network obtained from G by taking a single vertex q (the sink) to be the boundary ∂G. It is thus interesting to consider $\mathcal{U}^1(\Gamma, R)$ for R an arbitrary commutative ring in this case where Γ has a single boundary vertex. Farbod suggested an alternate description of $\mathcal{U}^1(\Gamma, R)$. Namely, let L be the R-module generated by cycles of G. And define

$$L^{\#} := \{\vec{x} \in L \otimes K\colon K = \text{ fraction field of } R,\ \vec{x} \cdot v \in R \text{ for all } v \in L\}.$$

Is it then the case that $L^{\#}/L \simeq \mathcal{U}^1(\Gamma, R)$? Farbod also suggested that this construction is very similar to the flow graph cohomology of Wagner [62].

4.9 Avi Levy: "Electrical network cohomology with coefficients in a polynomial ring"

The cohomology modules defined above have $\mathcal{U}^i(\Gamma, R) = 0$ for $i > 1$ if R is any PID, just for dimension reasons. However, we can ask what is $\mathcal{U}^i(\Gamma, R)$ for $i > 1$ with $R = \mathbb{C}[x_1, \ldots, x_n]$ a polynomial ring for some $n > 1$.

4.10 Avi Levy: "Moving between a single boundary vertex and a general set of boundary vertices in electrical networks"

We have $\mathcal{U}^1(\Gamma, \mathbb{Z}) = \mathrm{Jac}(G)$ when ∂G is a single vertex. How does $\mathcal{U}^1(\Gamma, \mathbb{Z})$ change for more general sets of boundary vertices? Let $\Gamma = (G, \partial G)$ be a network with arbitrary boundary. For each partition Π of ∂G we can set Γ_Π to be the network we get by collapsing all vertices in the same part into one vertex. So in particular $\Gamma_{\{\partial G\}}$ has only a single boundary vertex and thus $\mathcal{U}^1(\Gamma_{\{\partial G\}}, \mathbb{Z})$ is the sandpile group of a graph (namely, the graph obtained from Γ by collapsing all boundary vertices into a single sink). Is there some Galois connection coming from the poset of partitions of ∂G that interpolates between sandpile groups and cohomology for electrical networks with arbitrary boundaries?

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