Chaotic dynamics of a classical string on AdS black hole spacetime

Waseda University
Kei-ichi Maeda

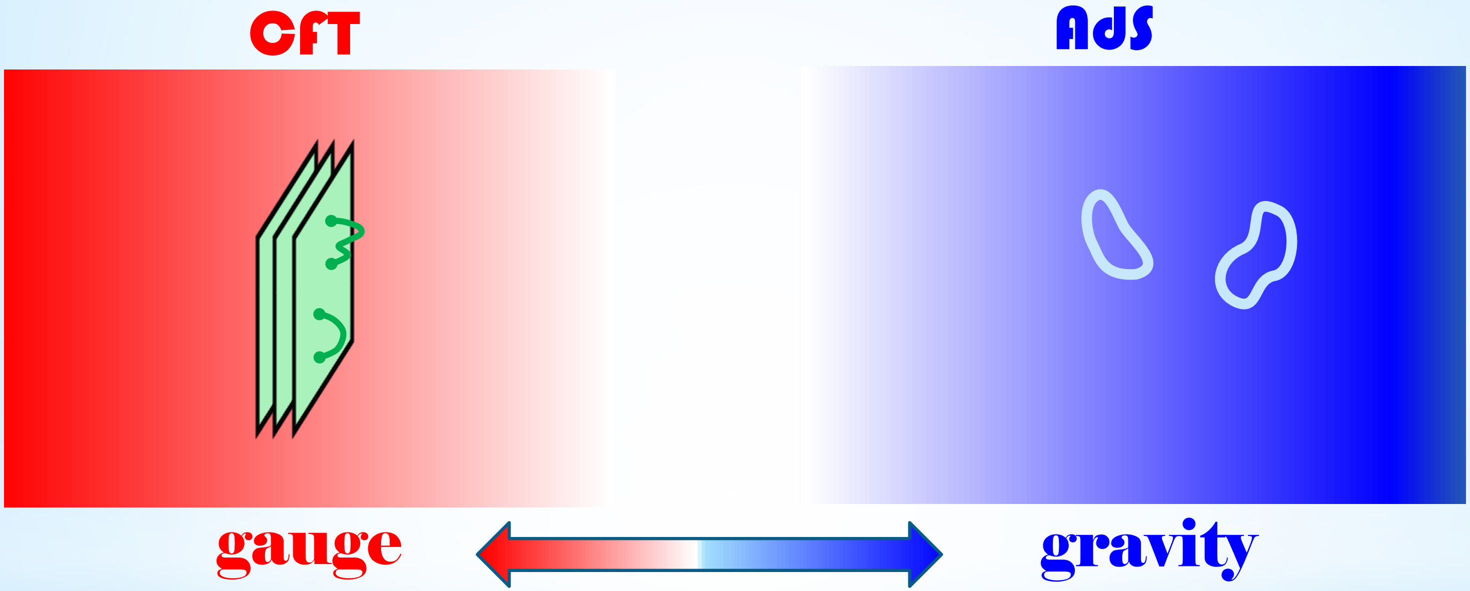
with Yuichiro Hoshino

AdS/CFT correspondence

Maldacena 1997

N = 4 supersymmetric Yang-Mills theory

type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$



Strong coupling phase

QCD
super conductor

Classical dynamics

5D AdS BH

Integrability

N = 4 supersymmetric Yang-Mills theory

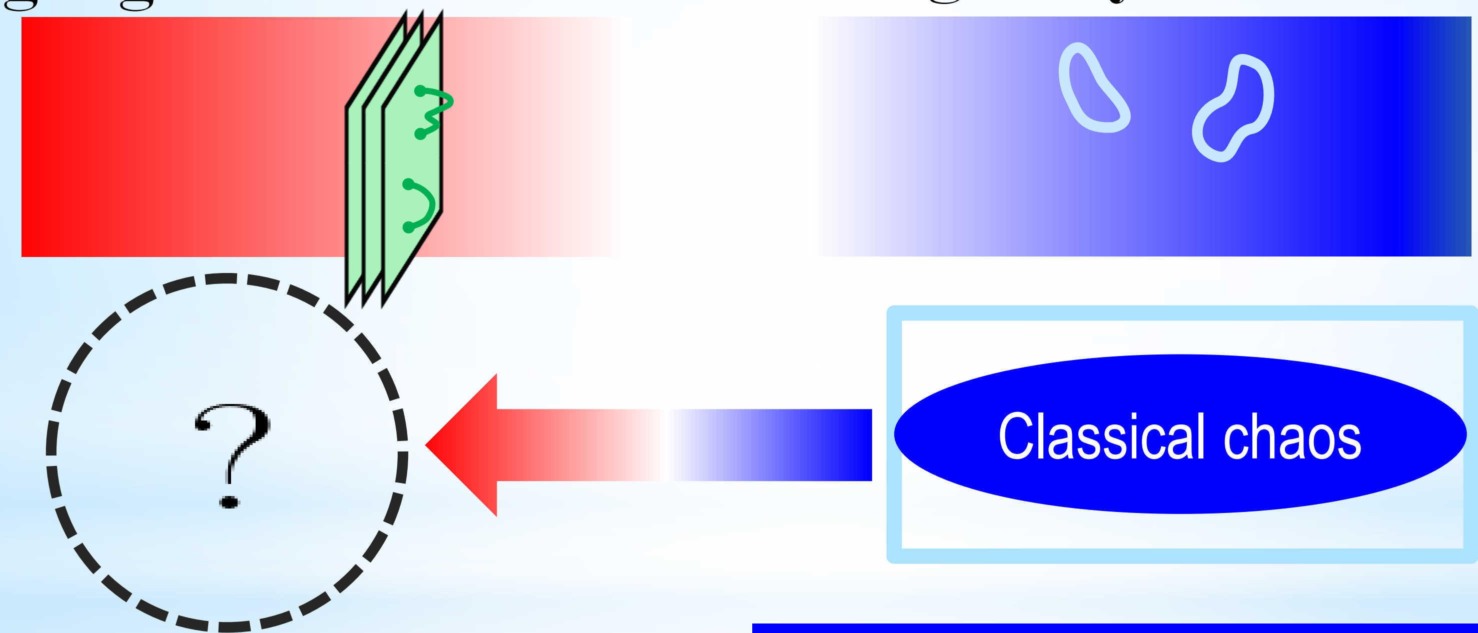
[Minahan, Zarembo 2003]

Classical string in $AdS_5 \times S^5$

[Bena, Polchinski, Roiban 2004]

gauge

gravity



Quantum chaos ?

Classical string in $AdS_5 \times T^{1,1}$

[Basu, Pando Zayas, 2011]

$\mathrm{AdS}_5\times \mathrm{T}^{1,1}$ [Basu, Pando Zayas, 2011]

metric
$$ds^2/b^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin\theta_i d\phi_i^2) + \frac{1}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2$$

Ansatz

$$t = t(\tau),\quad \rho = 0,\quad \psi = \psi(\tau),\quad \theta_i = \theta_i(\tau),\quad \phi_i = \alpha_i\sigma,$$

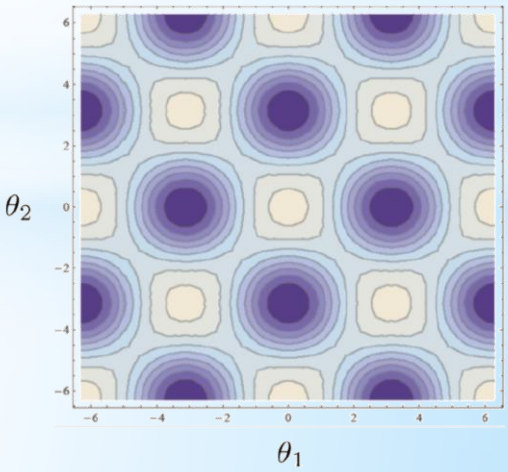
winding

Constant of motion

$$E = \dot{t} \qquad J = \dot{\psi} = 0$$

$$E^2 = \frac{1}{9}\sum_{i=1}^{2}\alpha_i^2 + \frac{1}{6}\sum_{i=1}^{2}\dot{\theta}_i^2 + V(\theta_1,\theta_2)$$

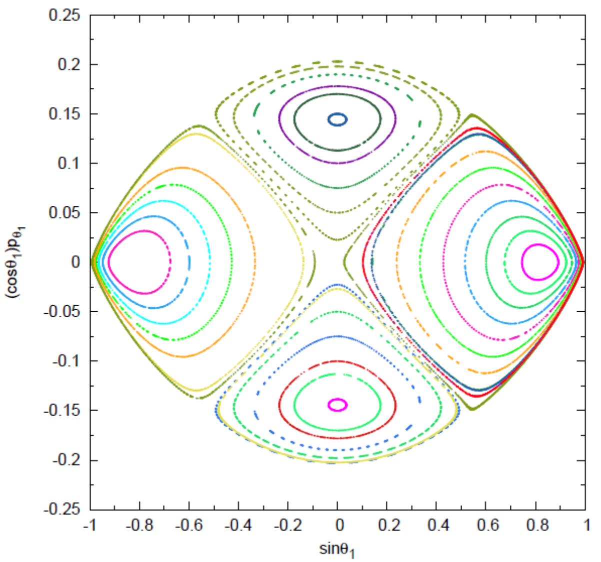
$$V(\theta_1,\theta_2) = \frac{1}{18}\sum_{i=1}^{2}\alpha_i^2\sin^2\theta_i + \frac{2}{9}\alpha_1\alpha_2\cos\theta_1\cos\theta_2$$



two pendula with the coupling

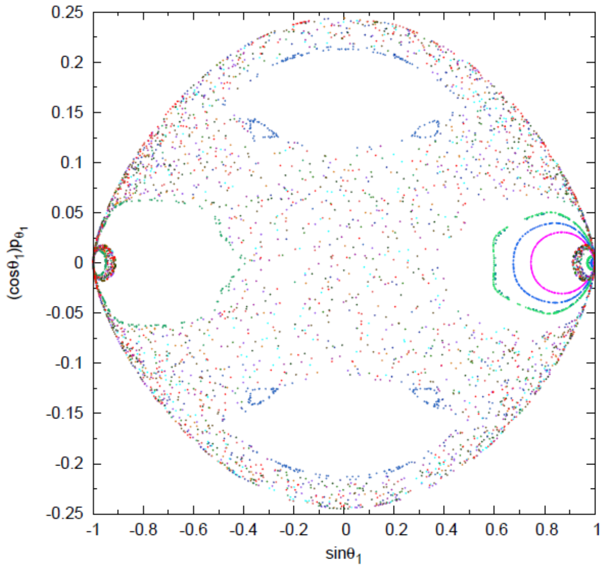
Poincaré map & Lyapunov exponent

$E = 0.5$



$\lambda_1 \approx 0.00$

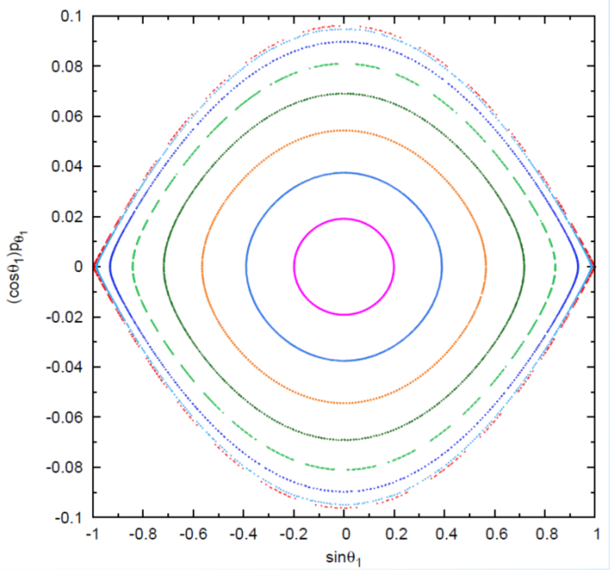
$E = 0.6$



$\lambda_1 \approx 0.18$

chaotic

$E = 10$



$\lambda_1 \approx 0.00$

Non-linear dynamics of string in AdS BH background

Important but complicate

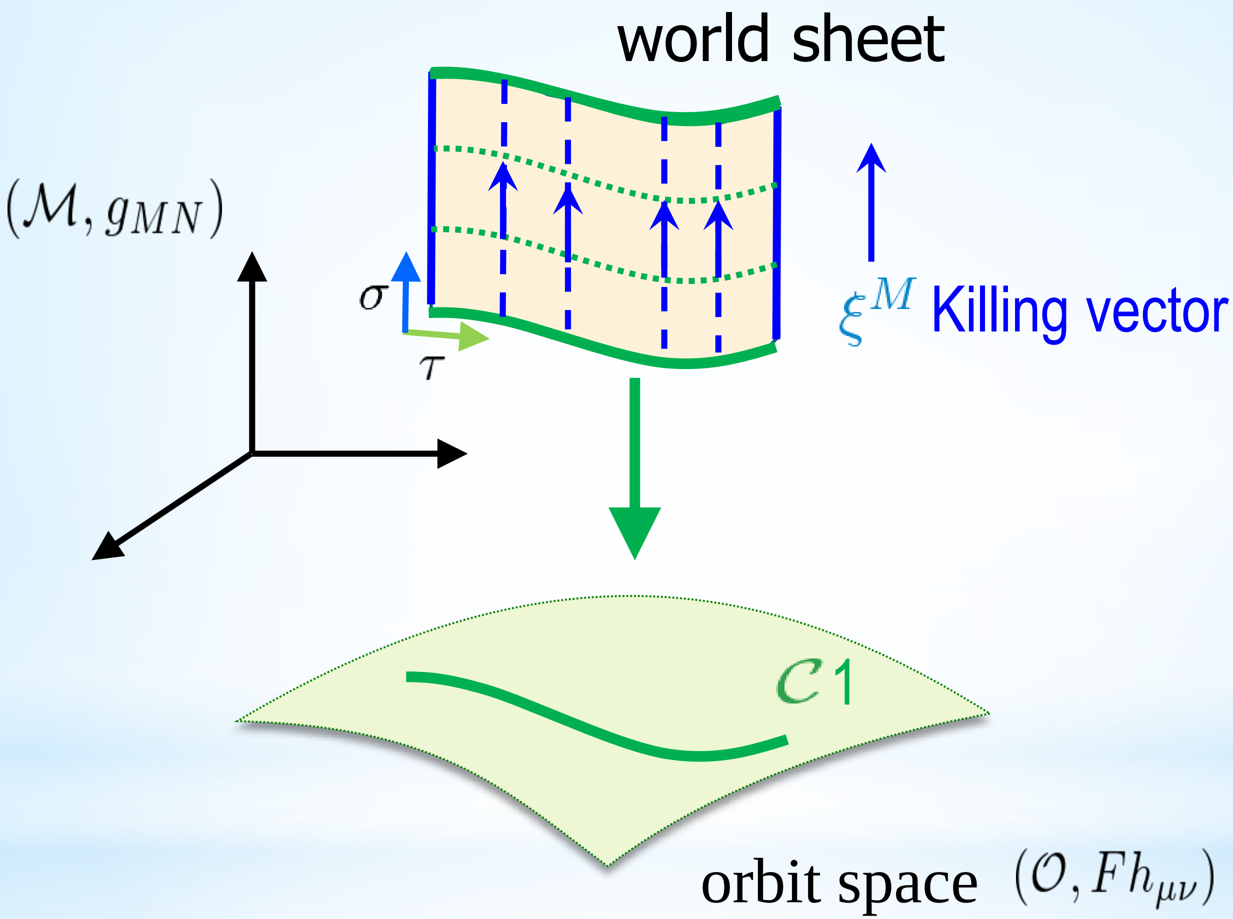
special configuration of string

cohomogeneity-one string

geodesics in lower dimensions

mathematically simple

Cohomogeneity-one string in 5D AdS BH spacetime



String Motion

[Frolov 1989, T. Koike 2005]

Nambu-Goto action
$$S_{\rm NG} = -T_0 \int d\tau d\sigma \sqrt{-\gamma}$$

projection

$$h_{\rm MN} = g_{\rm MN} - \frac{\xi_{\rm M}\xi_{\rm N}}{F}\,, \quad F = \xi_{\rm M}\xi^{\rm M}$$

ξ^M : Killing vector

geodesic in the orbit space ($\dim\mathcal{M} - 1$)

$$S_{C1} = -T_0 \Delta\sigma \int_C d\tau \sqrt{-F h_{\mu\nu}(x) \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

➢ 5D Sch-AdS BH

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\Phi^2 + \cos^2\theta d\Psi^2\right),$$

$$f(r) = 1 + \frac{r^2}{b^2} - \frac{M_s}{r^2}$$

b : AdS radius

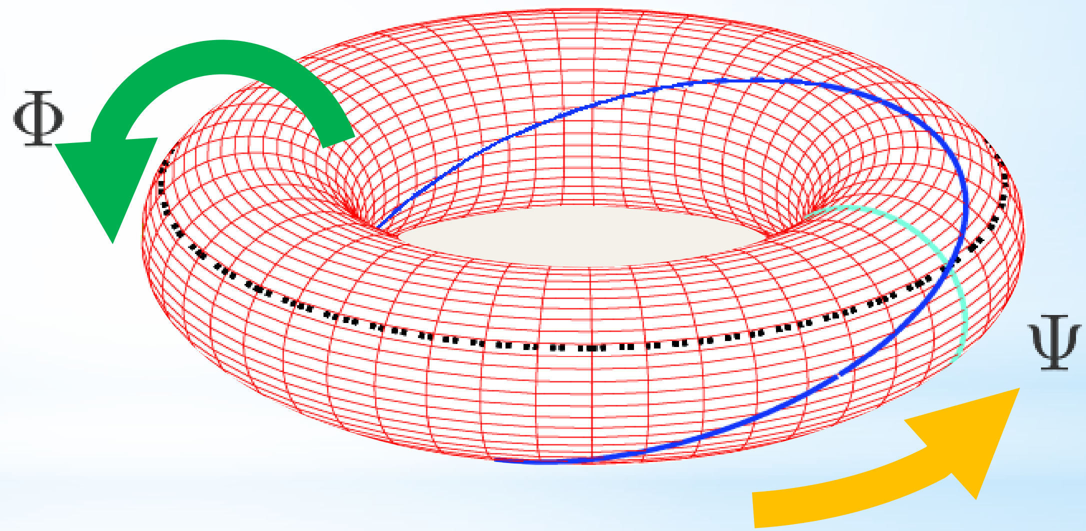
M_s : mass parameter

Killing vector

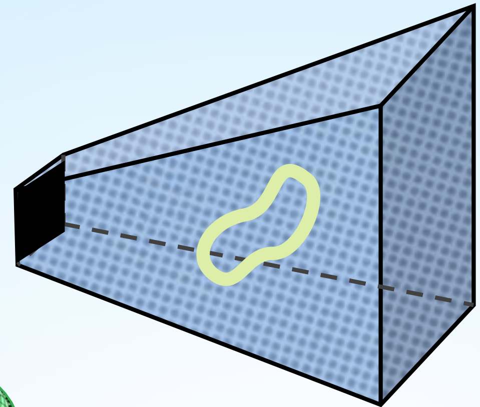
$$\xi = m\partial_\Phi + n\partial_\Psi$$

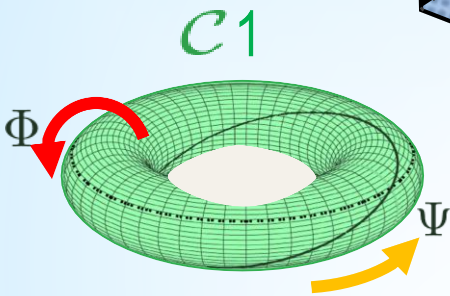
winding numbers :

$$m, n \in \mathbb{N}$$



String motion

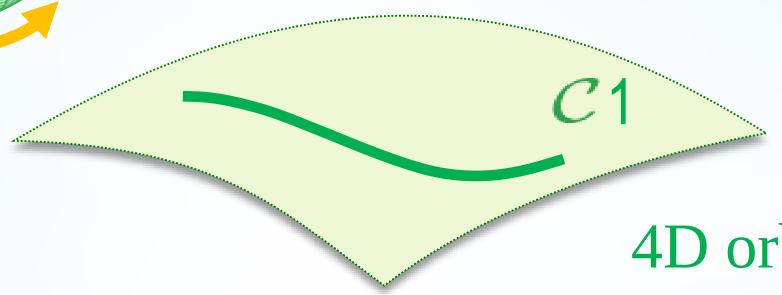




5D Sch-AdS BH

# of K.V. 3

“Particle” motion



4D orbit space $(\mathcal{O}, Fh_{\mu\nu})$

# of K.V. 2

System with 2 degrees of freedom

$$H(r, \theta, p_r, p_\theta; M, L, E; m, n)$$

$m = n$ integrable

[Igata , Ishihara 2010]

➢ Hamiltonian

$$f(r) = 1 + \frac{r^2}{b^2} - \frac{M_s}{r^2}$$

$$2H = fp_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{L^2(m^2\sin^2\theta + n^2\cos^2\theta)}{r^2\cos^2\theta\sin^2\theta} - \frac{E^2}{f} + r^2(m^2\sin^2\theta + n^2\cos^2\theta) = 0$$

➢ Equation of Motion

$$\dot{r} = fp_r \qquad \dot{\theta} = \frac{1}{r^2}p_\theta$$

$$\dot{p}_r = -\frac{f'}{2f^2}E^2 - \frac{1}{2}f'p_r^2 + \frac{1}{r^3}p_\theta^2 + \frac{L^2(m^2\sin^2\theta + n^2\cos^2\theta)}{r^3\cos^2\theta\sin^2\theta} - r(m^2\sin^2\theta + n^2\cos^2\theta)$$

$$\dot{p}_\theta = -\frac{L^2}{r^2}\left(\frac{m^2\sin\theta}{\cos^3\theta} - \frac{n^2\cos\theta}{\sin^3\theta}\right) - r^2\sin\theta\cos\theta(m^2 - n^2)$$

"Effective potential"

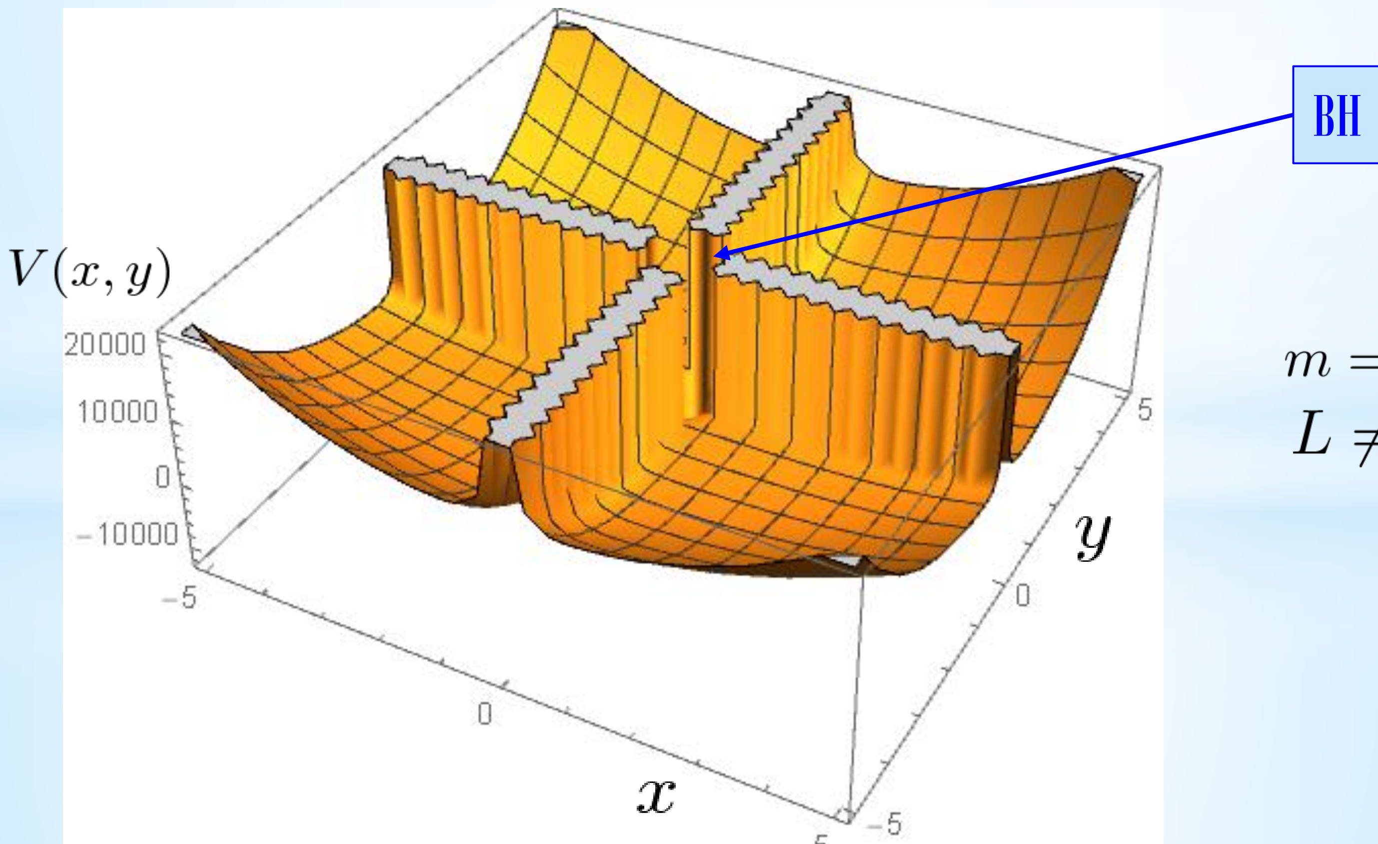
AdS radius $b = 1$

$$E^2 = f\left[f p_r^2 + \frac{1}{r^2} p_\theta^2\right] + V(x, y)$$

$$x = r\sin\theta, \quad y = r\cos\theta$$

$$f = \left(1 + x^2 + y^2 - \frac{M_s}{x^2 + y^2}\right)$$

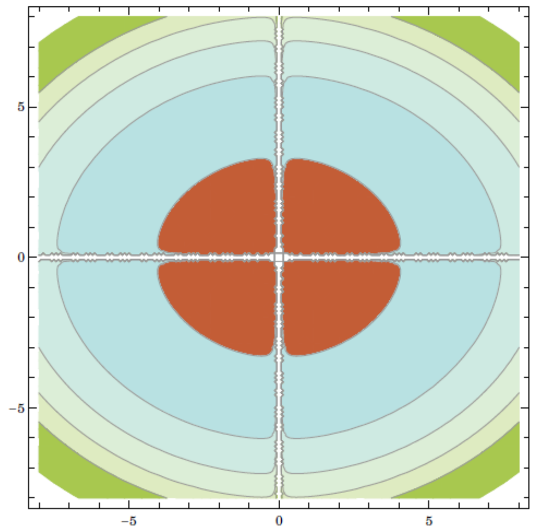
$$V(x, y) = f\left(1 + \frac{L^2}{x^2 y^2}\right)(m^2 y^2 + n^2 x^2)$$



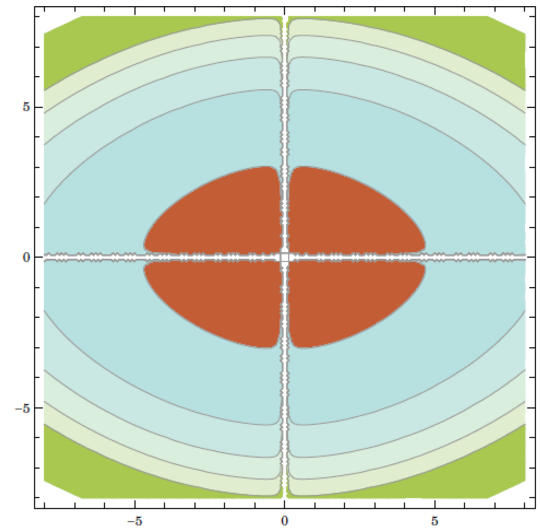
$m = 4\,, n = 2$

$L \neq 0$

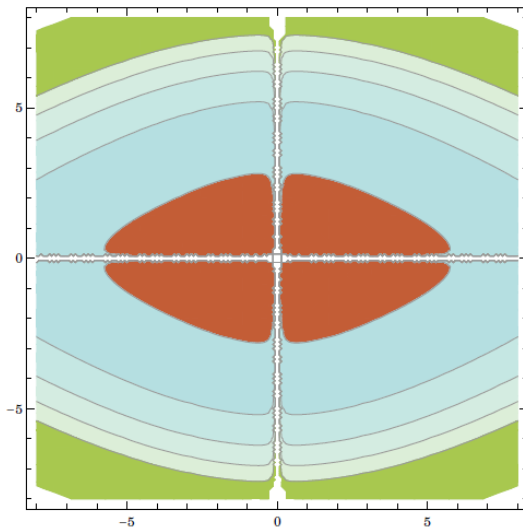
Contour



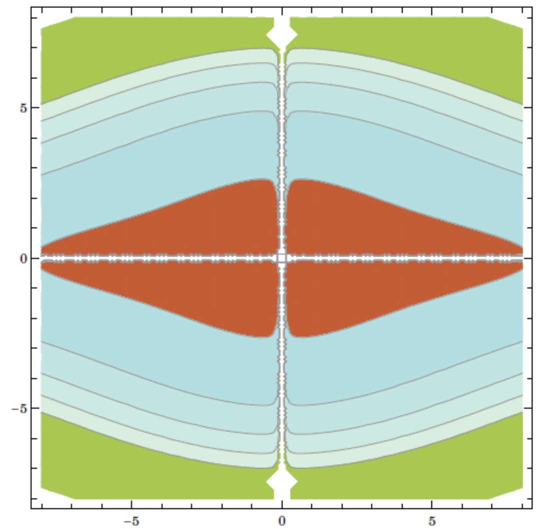
(m, n)=(6,4)



(m, n)=(7,3)



(m, n)=(8,2)



(m, n)=(9,1)

Two ways to analyze chaos

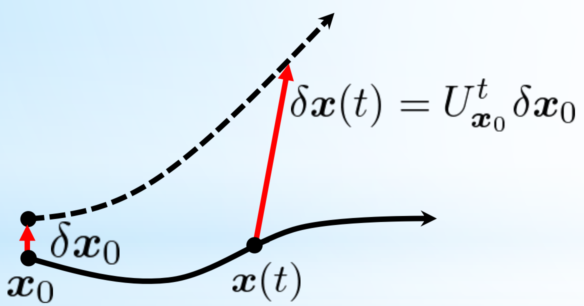
- **Poincare map**

The intersection of trajectories with a certain lower-dimensional subspace in the phase space

KAM tori —destroy→ chaos

- **Lyapunov exponent**

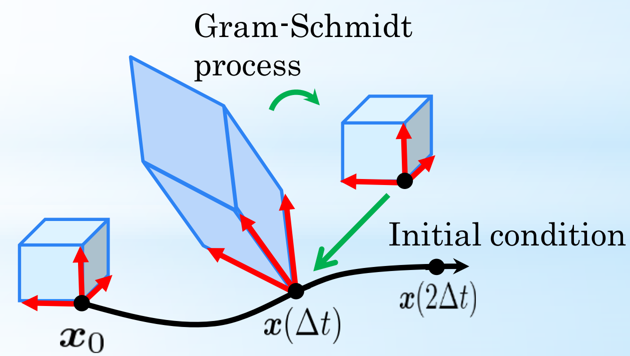
sensitive dependence on initial data



$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \frac{||\delta \boldsymbol{x}||}{||\delta \boldsymbol{x}_0||}$$

Lyapunov spectrum

Gram-Schmidt process



$$\lambda_i = \lim_{N \to \infty} \frac{1}{N \Delta t} \sum_{k=1}^{N} \| \boldsymbol{a}_i^k \|$$

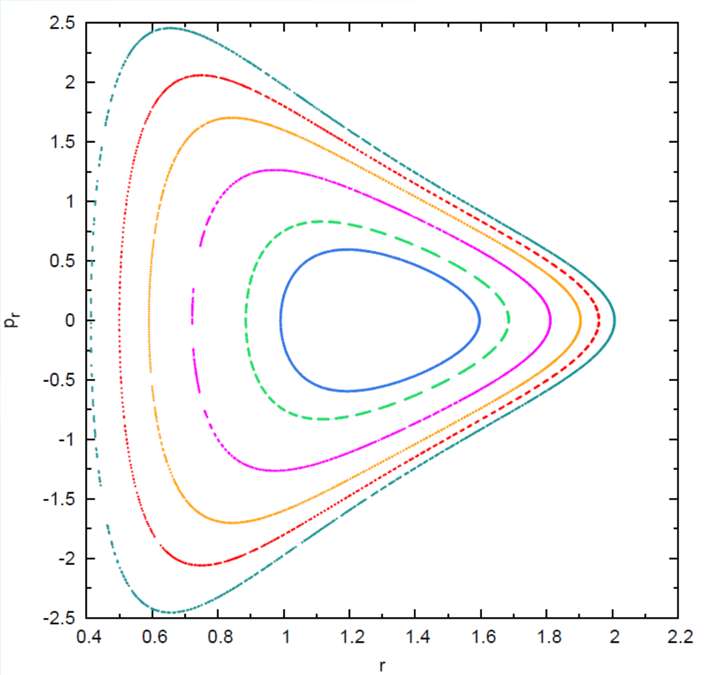
Integrable case

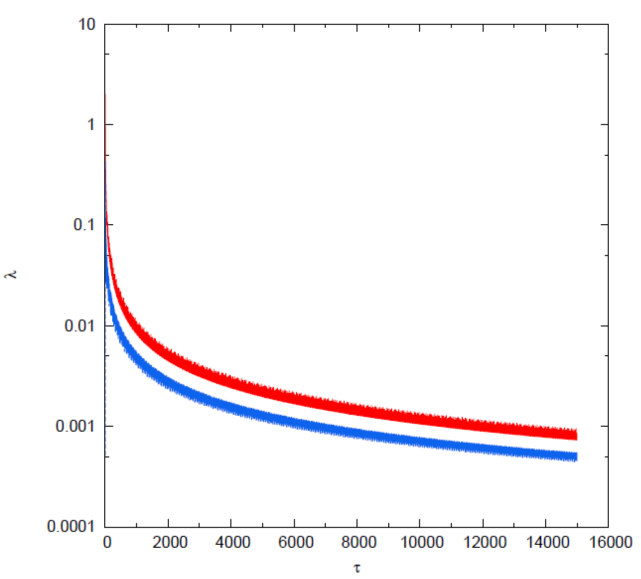
$$(m,n)=(1,1)$$
$$E=5$$

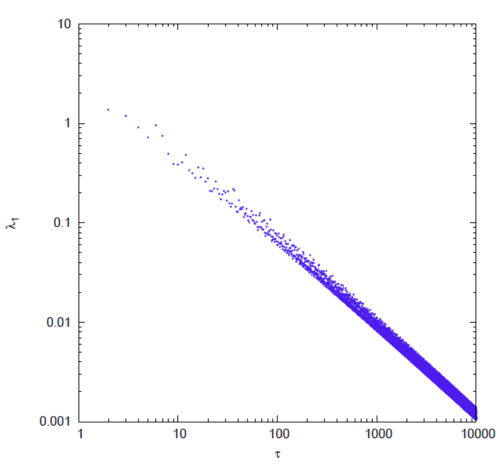
Lyapunov exponent

$$\lambda_1 \approx 0.00$$

Poincare map



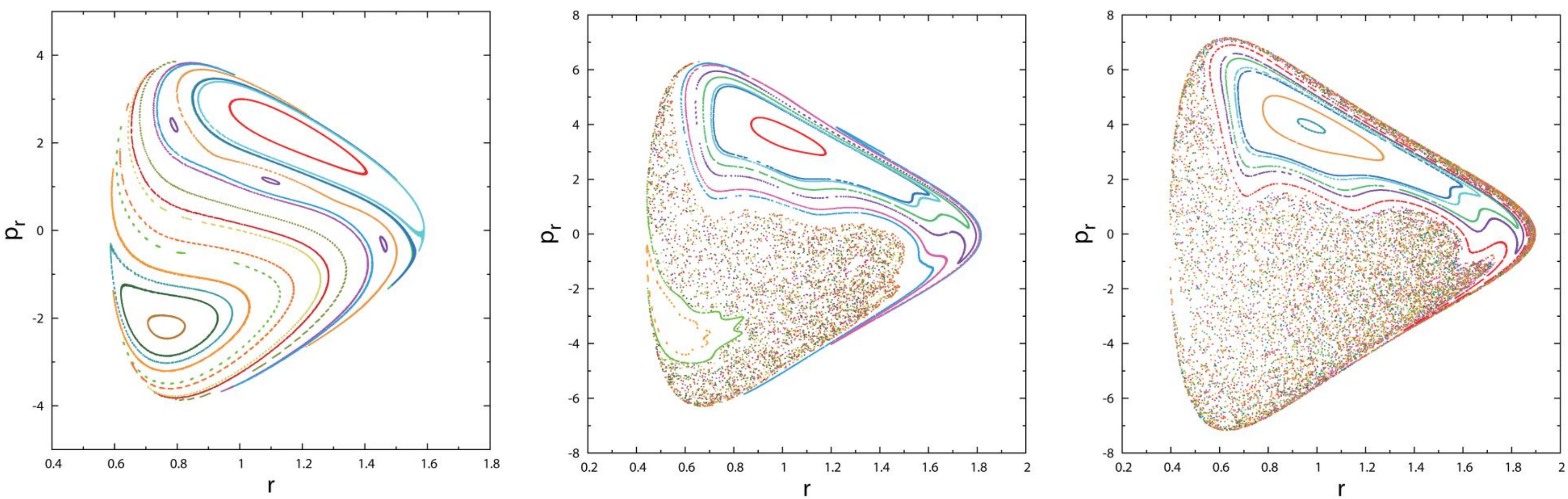




Energy dependence

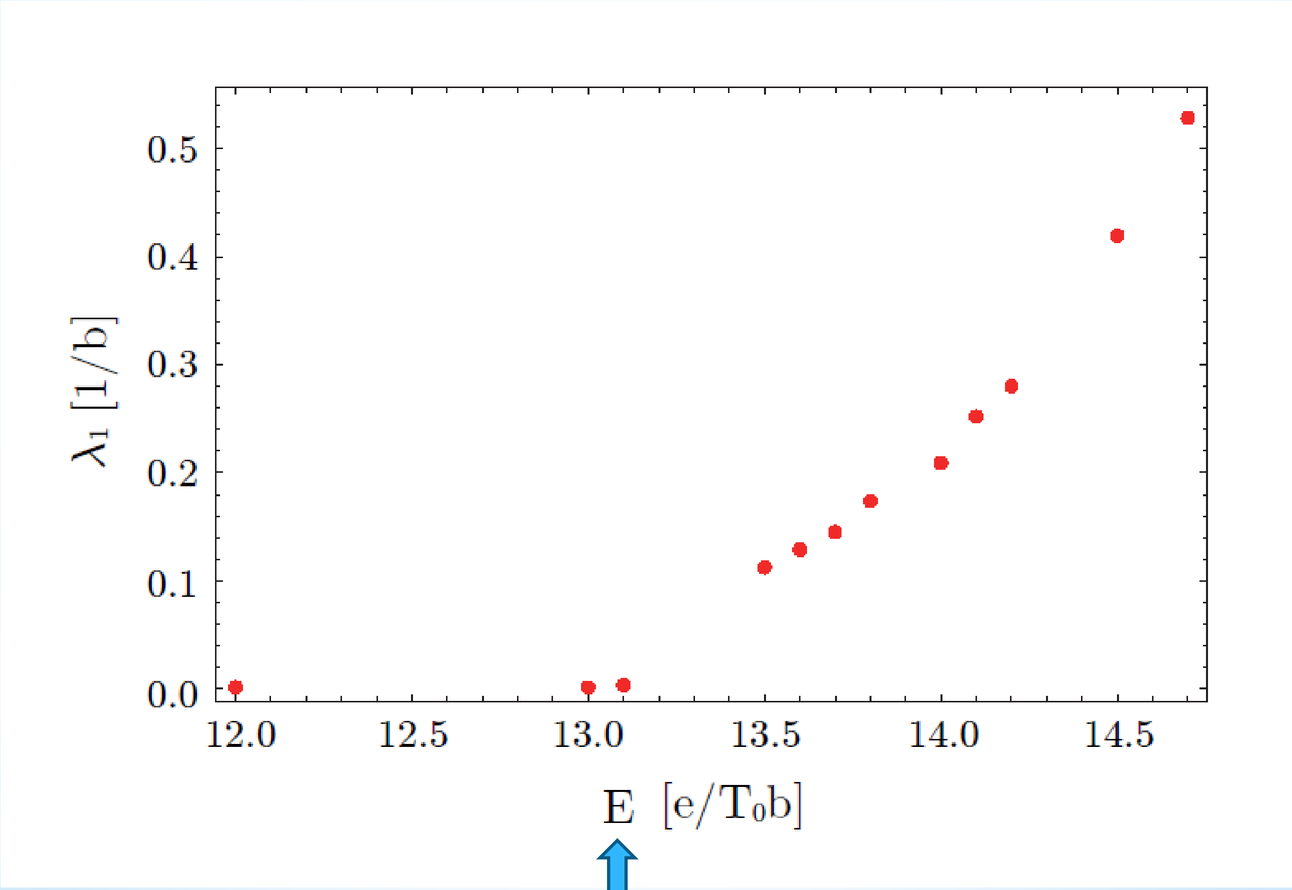
$(m,n) = (4,2),\ L = 1,\ M = 0.05$

$\lambda_1 = 0.00$ $\lambda_1 = 0.20$ $\lambda_1 = 0.52$



(i) E = 12 (ii) E = 14 (iii) E = 14.7

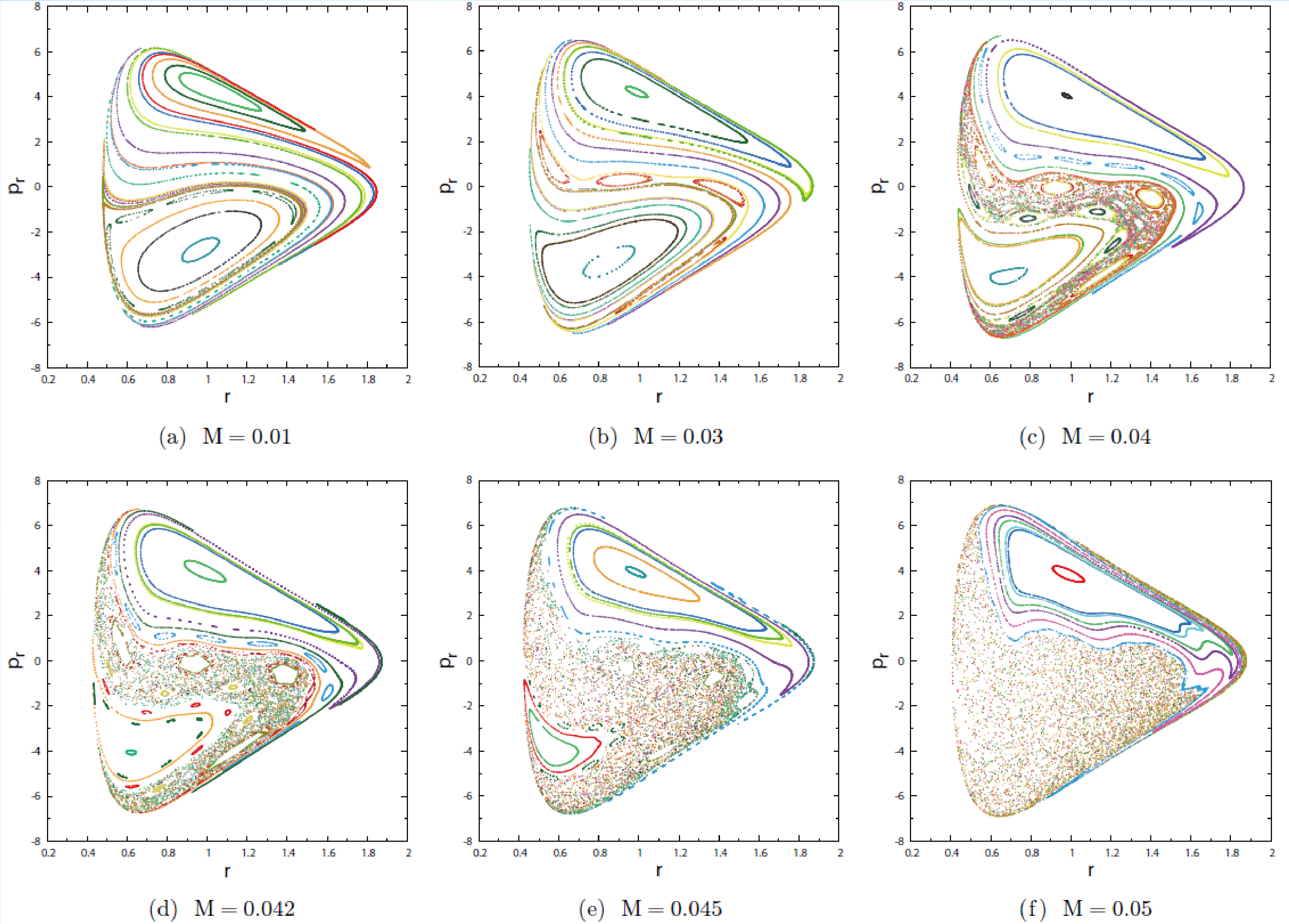
$$(m,n) = (4,2),\ L = 1,\ M = 0.05$$



$$E_{\rm cr} \approx 13.1$$

mass dependence

$(m,n) = (4,2),\ L = 1,\ E = 14.5$



(a) M = 0.01

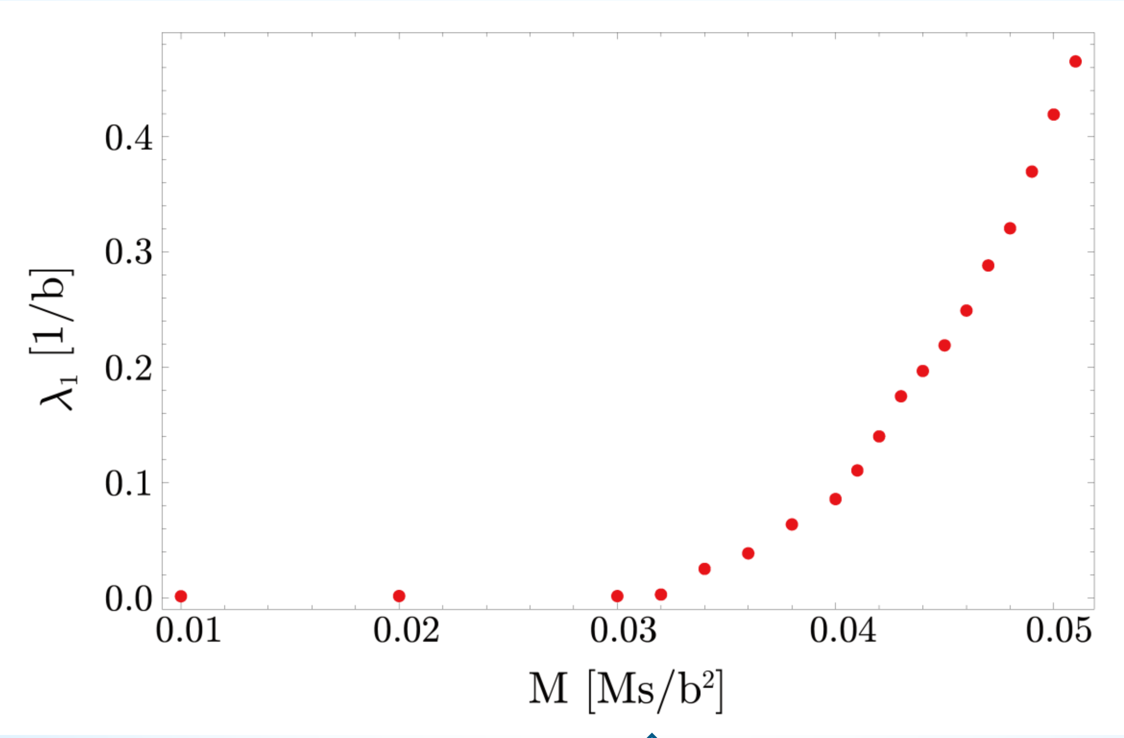
(b) M = 0.03

(c) M = 0.04

(d) M = 0.042

(e) M = 0.045

(f) M = 0.05



$M_{cr} \approx 0.32$

➤ Lyapunov exponent

$$M = 0.01,\ L = 0.4,\ E = 18$$

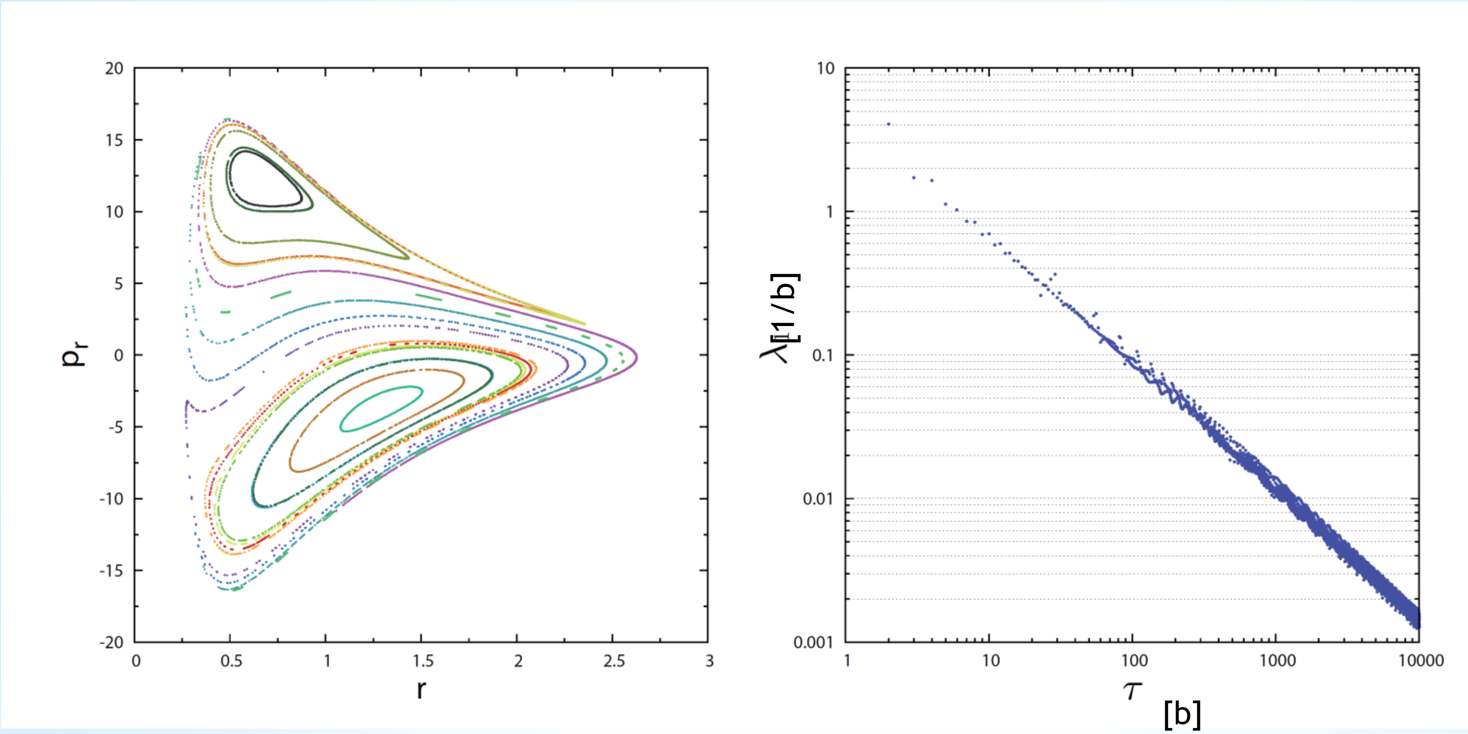
n \ m	1	2	3	4	5	6	7	8	9
1	0	*	*	*	*	*	*	0.49	0.33
2	-	0	*	*	*	*	0.22	0.00	0.12
3	-	-	0	*	*	0.55	0.34	0.15	0.00
4	-	-	-	0	0.61	0.22	0.13	0.00	0.00
5	-	-	-	-	0	0.00	0.00	0.00	0.00
6	-	-	-	-	-	0	0.00	0.00	0.00
7	-	-	-	-	-	-	0	0.00	0.00
8	-	-	-	-	-	-	-	0	0.00
9	-	-	-	-	-	-	-	-	0

becomes positive if E increases (rows 3–8)

AdS spacetime (M=0)

$M = 0$

$(m, n) = (4, 2),\ L = 1,\ E = 25$



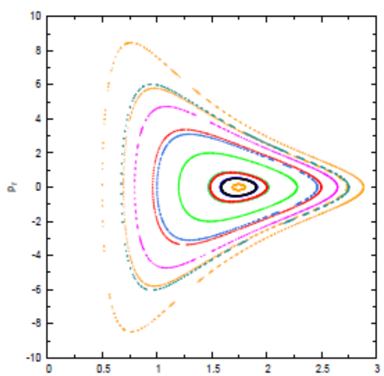


図 6.3.1 (m,n)=(2,2), E=20 ($\lambda_1 \approx 0.00$)

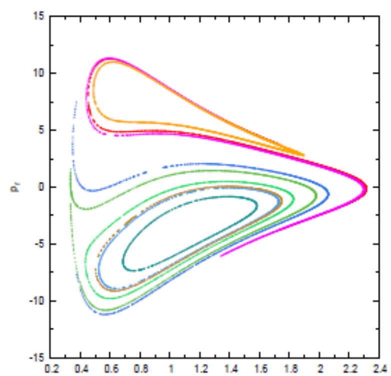


図 6.3.2 (m,n)=(4,2), E=20 ($\lambda_1 \approx 0.00$)

Integrable ?

Summary and Remarks

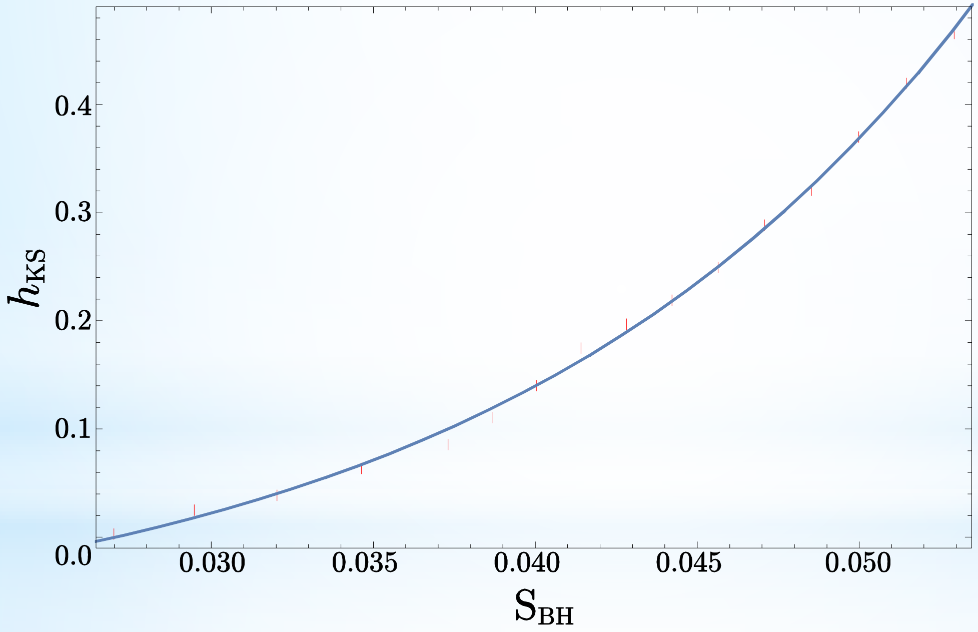
- We analyze the dynamics of **cohomogeneity one string** in 5D Sch-AdS BH
- **Chaotic behavior appears** when the winding numbers are different
- Lyapunov exponent increases when M and E increase
- Cohomogeneity one string may be **integrable in AdS spacetime**

  even if the winding numbers are different

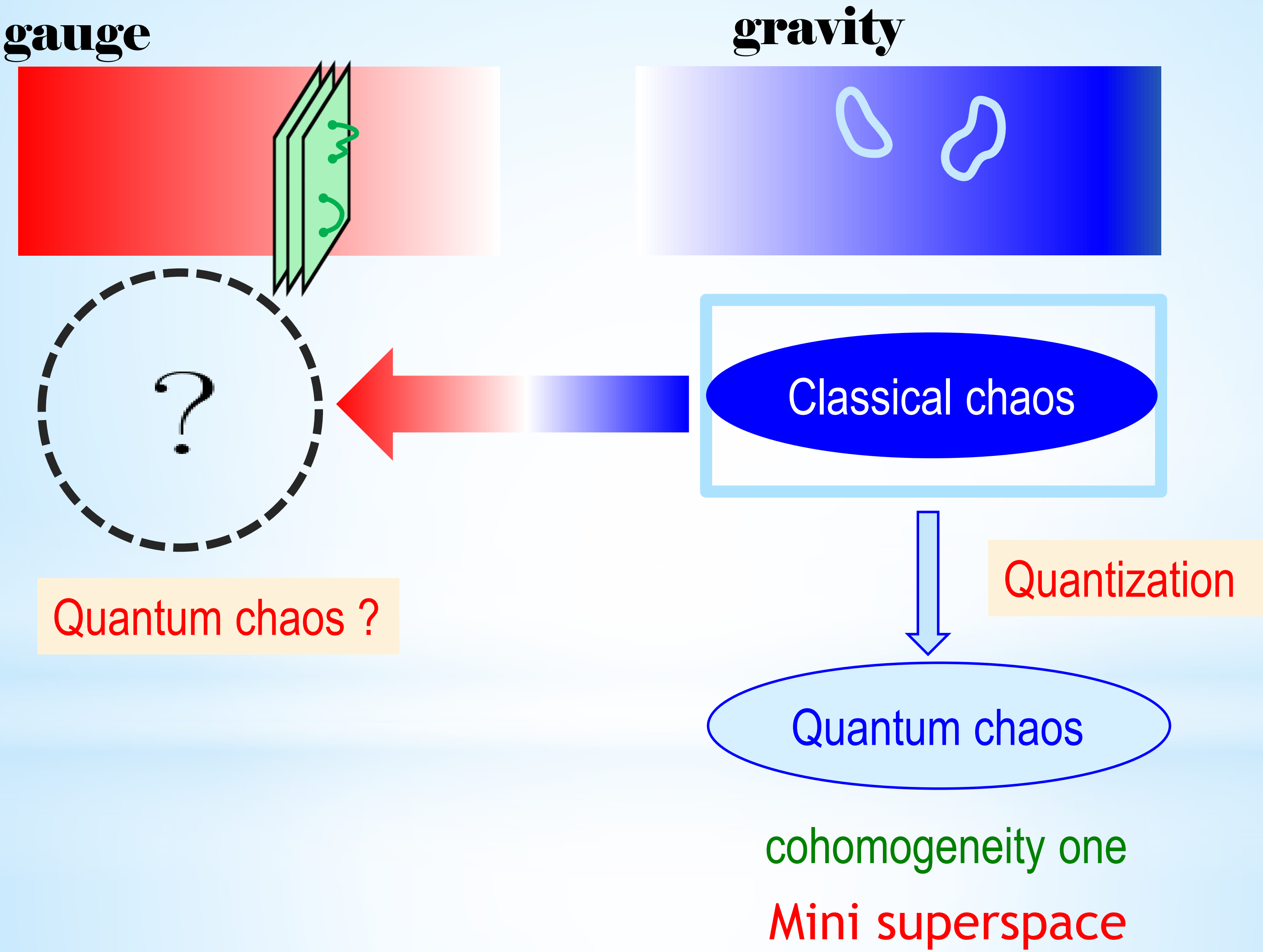
Is there any relation between KS entropy and BH entropy ?

Kolmogorov-Sinai (KS) entropy vs BH entropy

$(m, n) = (4, 2), \; L = 1, \; E = 14.5$



Quantization of string



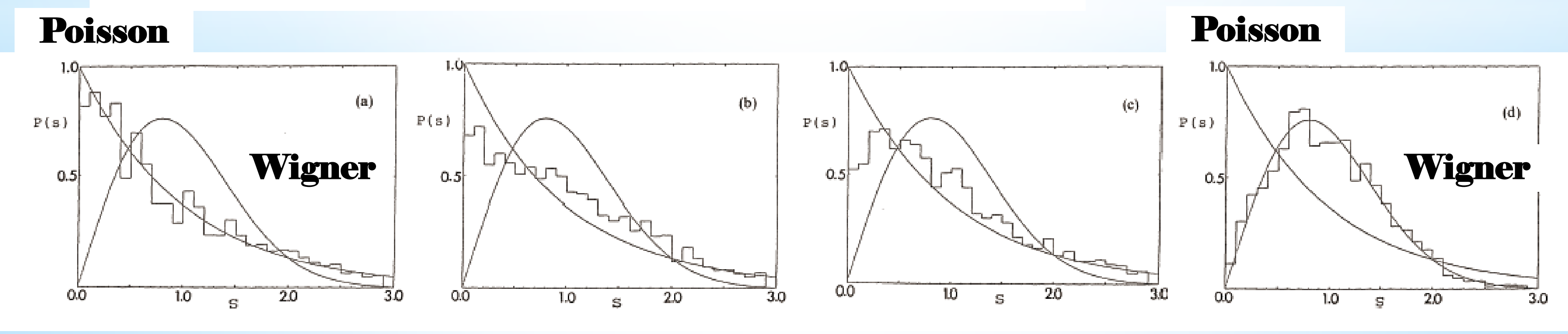
Quantum chaos

Lebel statistics

Integrable system — Poisson distribution $P(s) = \exp(-s)$

Completely chaotic — Wigner distribution $P(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4} s\right)$

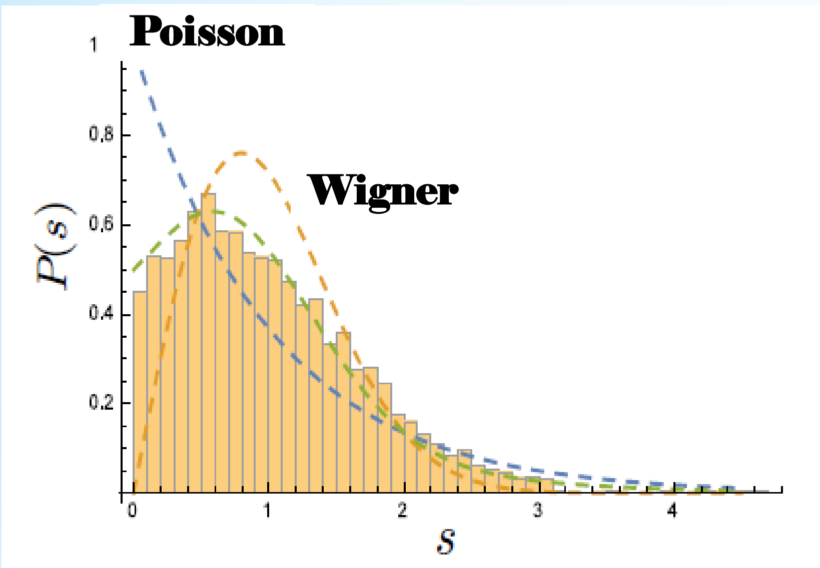
$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha}{4}(a q_1^4 + 2C q_1^2 q_2^2 + b q_2^4) \qquad \alpha = 0.088 \qquad a = 2\,,\ b = 1$$



$C = 0$ $\qquad$ $C = 1$ $\qquad$ $C = 4$ $\qquad$ $C = 10$

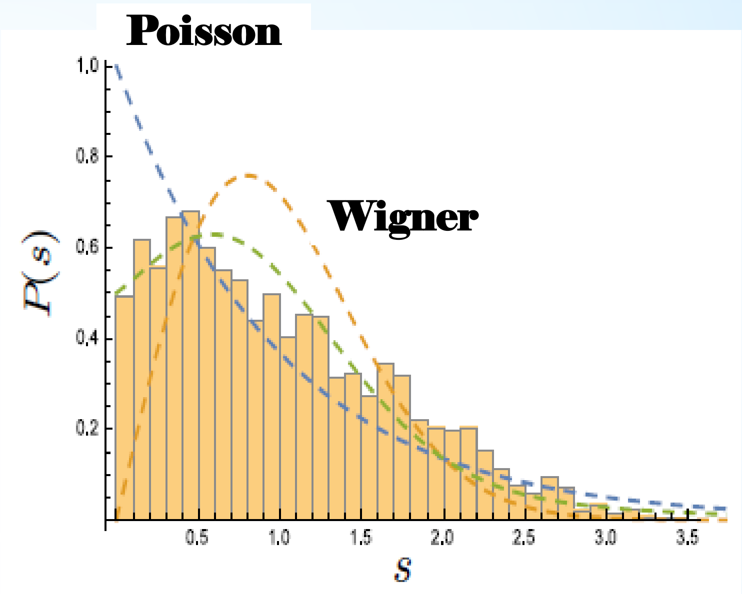
5D AdS

preliminary



$(m = n = 2,\ L = 1)$

integrable



$(m = 4,\ n = 2,\ L = 1)$

Integrable ?

Thank you for your attention