More on the density zero ideal

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Outline

Basic definitions

Definition

An ideal $\mathcal{I}$ on ω is called a ***P-ideal*** *if $\mathcal{I}$ is countably directed mod finite. In other words, if $\{a_n : n \in \omega\} \subseteq \mathcal{I}$, then there exists $a \in \mathcal{I}$ such that $\forall n \in \omega\, [a_n \subseteq^* a]$.*

Remark

Ideals on ω are always assumed to be proper (i.e. $\omega \notin \mathcal{I}$) and non-principal (meaning every finite subset of ω belongs to $\mathcal{I}$).

- In this talk I am primarily interested in $\mathcal{I}$ that are definable.
- Especially analytic P-ideals.

Definition

When $\mathcal{I}$ is a tall P-ideal on ω you can define the following:

$$
\begin{aligned}
\mathrm{add}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F}\, [a \not\subseteq^* b]\},\\
\mathrm{cov}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall a \in [\omega]^{\omega} \exists b \in \mathcal{F}\, [|a \cap b| = \omega]\},\\
\mathrm{cof}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F}\, [b \subseteq^* a]\},\\
\mathrm{non}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F}\, [|a \cap b| < \omega]\}.
\end{aligned}
$$

- There are actually equal to the $\mathrm{add}, \mathrm{cov}, \mathrm{cof}$, and non of an associated σ-ideal.
- For each $a \in \mathcal{P}(\omega)$, let $\hat{a} = \{b \subseteq \omega : |a \cap b| = \omega\}$.
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- For a tall ideal $\mathcal{I}$, $\hat{\mathcal{I}} = \{X \subseteq \mathcal{P}(\omega) : \exists a \in \mathcal{I}\, [X \subseteq \hat{a}]\}$ is an ideal on $\mathcal{P}(\omega)$ generated by Borel sets.

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- $\mathcal{I}$ is a P-ideal iff $\hat{\mathcal{I}}$ is a σ-ideal.
- $\mathrm{add}(\hat{\mathcal{I}}) = \mathrm{add}^*(\mathcal{I})$, $\mathrm{cov}(\hat{\mathcal{I}}) = \mathrm{cov}^*(\mathcal{I})$, $\mathrm{cof}(\hat{\mathcal{I}}) = \mathrm{cof}^*(\mathcal{I})$, $\mathrm{non}(\hat{\mathcal{I}}) = \mathrm{non}^*(\mathcal{I})$ hold.

Definition

*A set $A \subseteq \omega$ is said to have **asymptotic density** 0 if* $\lim_{n\to\infty} \frac{|A \cap n|}{n} = 0$.

$$\mathcal{Z}_0 = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

- This an $F_{\sigma\delta}$ P-ideal.
- We are interested in the invariants $\mathrm{cov}^*(\mathcal{Z}_0)$ and $\mathrm{non}^*(\mathcal{Z}_0)$.

Four basic invariants

Definition

For $f, g \in \omega^{\omega}, f <^{*} g$ means that $|\{n \in \omega : g(n) \leq f(n)\}| < \omega$. A set $F \subseteq \omega^{\omega}$ is said to be ***unbounded*** *if there does not exist $g \in \omega^{\omega}$ such that $\forall f \in F\, [f <^{*} g]$. A set $F \subseteq \omega^{\omega}$ is said to be* ***dominating or cofinal*** *if $\forall f \in \omega^{\omega} \exists g \in F\, [f <^{*} g]$.*

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*For $f, g \in \omega^\omega$, $f <^* g$ means that $|\{n \in \omega : g(n) \leq f(n)\}| < \omega$. A set $F \subseteq \omega^\omega$ is said to be **unbounded** if there does not exist $g \in \omega^\omega$ such that $\forall f \in F\, [f <^* g]$. A set $F \subseteq \omega^\omega$ is said to be **dominating or cofinal** if $\forall f \in \omega^\omega \exists g \in F\, [f <^* g]$.*

Definition

*For $a, b \in \mathcal{P}(\omega)$ we say that a **splits** b if both $b \cap a$ and $b \cap (\omega \setminus a)$ are infinite. A family $F \subseteq \mathcal{P}(\omega)$ is called a **splitting family** if $\forall b \in [\omega]^\omega \exists a \in F\, [a$ splits $b]$.*

Definition

We define the cardinal invariants $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{s}$, *and* $\mathfrak{r}$ *as follows:*

$$\mathfrak{b} = \min\{|F| : F \subseteq \omega^{\omega} \wedge F \text{ is unbounded}\};$$
$$\mathfrak{d} = \min\{|F| : F \subseteq \omega^{\omega} \wedge F \text{ is dominating}\};$$
$$\mathfrak{s} = \min\{|F| : F \subseteq \mathcal{P}(\omega) \wedge F \text{ is a splitting family}\};$$
$$\mathfrak{r} = \min\{|F| : F \subseteq [\omega]^{\omega} \wedge \neg\exists a \in \mathcal{P}(\omega) \forall b \in F\, [a \text{ splits } b]\}.$$

Fact

$\aleph_1 \leq \max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{d} \leq \mathfrak{c}$.

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Theorem (Hernández-Hernández and Hrušák [1])

$\min\{\mathrm{cov}(\mathcal{N}), \mathfrak{b}\} \leq \mathrm{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$ *and*
$\min\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\} \leq \mathrm{non}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ *hold.*

Theorem (Hernández-Hernández and Hrušák [1])

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Theorem (R. and Shelah [3])

$\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{d}$ *and* $\mathfrak{b} \leq \mathrm{non}^*(\mathcal{Z}_0)$.

- This can be improved slightly.

- We adopt the convention that for a set $x \subseteq \omega$, $x^0 = x$ and $x^1 = \omega \setminus x$

Definition

Let $X = \langle x_i : i \in \omega \rangle$ be a sequence of elements of $\mathcal{P}(\omega)$. We say that X ***promptly splits*** *a if for each $n \in \omega$ and each $\sigma \in 2^{n+1}$, $\left(\bigcap_{i<n+1} x_i^{\sigma(i)} \right) \cap a$ is infinite. A family $\mathcal{F} \subseteq (\mathcal{P}(\omega))^{\omega}$ is said to be a* ***promptly splitting family*** *if for each $a \in [\omega]^{\omega}$, there exists $X \in \mathcal{F}$ which promptly splits a.*

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Definition

*Let $P = \langle x_i : i \in \omega \rangle$ be a partition of ω (that is, $\bigcup_{i\in\omega} x_i = \omega$ and for any $i < j < \omega$, $x_i \cap x_j = 0$). We say that P **splits** a if for each $i \in \omega$ $x_i \cap a$ is infinite. A family of partitions $\mathcal{F}$ is called a **splitting family of partitions** if for each $a \in [\omega]^{\omega}$, there exists $P \in \mathcal{F}$ which splits a.*

Definition

$$\mathfrak{s}^\omega = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a splitting family of partitions}\}.$$

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Lemma

$\mathfrak{s}^\omega = \min\{|\mathcal{F}| : \mathcal{F} \subseteq (\mathcal{P}(\omega))^\omega \wedge \mathcal{F}$ *is a promptly splitting family*$\}$.

- Next we will see that $\mathfrak{s}^\omega$ is also the least cardinal for which a certain type of strong coloring exists.

Definition

*Let κ be any cardinal. We say that a coloring $c : \kappa \times \omega \times \omega \to 2$ is **tortuous** if for each $A \in [\omega]^\omega$ and each partition of κ, $\langle K_n : n \in \omega \rangle$, there exists $n \in \omega$ such that*

$$\forall \sigma \in 2^{n+1} \exists \alpha \in K_n \exists k \in A \left[k > n \wedge \forall i < n+1 \left[\sigma(i) = c(\alpha, k, i) \right] \right].$$

Lemma

Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a promptly splitting family. There exists a tortuous coloring on κ.

Lemma

$\mathfrak{s}^\omega = \min\{\kappa :$ *there is a tortuous coloring on* $\kappa\}$.

Theorem ([2])

Let κ be a cardinal on which a tortuous coloring exists. Then $\mathrm{cov}^*(\mathcal{Z}_0) \leq \max\{\kappa, \mathfrak{b}\}$.

Corollary

$\mathrm{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{s}^{\omega}, \mathfrak{b}\}$.

Theorem ([2])

Let κ be a cardinal on which a tortuous coloring exists. Then $\text{cov}^*(\mathcal{Z}_0) \leq \max\{\kappa, \mathfrak{b}\}$.

Corollary

$\text{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{s}^\omega, \mathfrak{b}\}$.

Lemma

$\max\{\mathfrak{s}^\omega, \mathfrak{b}\} \leq \mathfrak{d}$.

Lemma

Suppose $\mathcal{F} \subseteq [\omega]^{\omega}$ *is a family of size less than* $\mathfrak{r}$. *Then there exists a sequence* $X = \langle x_k : k < \omega \rangle \in (\mathcal{P}(\omega))^{\omega}$ *such that* X *promptly splits* A, *for each* $A \in \mathcal{F}$.

Theorem ([2])

$\min\{\mathfrak{d}, \mathfrak{r}\} \leq \mathrm{non}^{*}(\mathcal{Z}_0)$.

- These results are all based on a general method for generating sets in $\mathcal{Z}_0$.

Definition

Let J be an interval partition where the size of J_n is some power of 2 (larger than n), for each $n \in \omega$. Let $\mathcal{F}_J$ be the family of all functions f in ω^{ω} such that for each $n, l \in \omega$:

1. $\dfrac{|\{k \in J_n : f(k) \geq l\}|}{|J_n|} \leq 2^{-l}$;
2. *for any $i, j \in \{k \in J_n : f(k) \geq l\}$, if $i \neq j$, then $|i - j| > 2^{l-1}$.*

Definition

Let J be an interval partition where the size of J_n is some power of 2 (larger than n), for each $n \in \omega$. For any interval partition I, function $f \in \mathcal{F}_J$, and $l \in \omega$, define $Z_{I,J,f,l} = \{m \in \omega : \exists k \in I_l \left[m \in J_k \wedge f(m) \geq l\right]\}$. Define $Z_{I,J,f} = \bigcup_{l \in \omega} Z_{I,J,f,l}$.

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Lemma

For any I, J, and f as above, $Z_{I,J,f}$ has density 0.

- In all cases the proof consists of identifying a "large enough" subclass $\mathcal{F} \subseteq \mathcal{F}_J$.
- Here "large enough" essentially means for every $A \in [\omega]^{\omega}$ there exists $f \in \mathcal{F}$ which is unbounded on A.

- To get $\text{cov}^*(\mathcal{Z}_0) \leq \kappa$, one needs to find an $\mathcal{F} \subseteq \mathcal{F}_J$ such that $|\mathcal{F}| \leq \kappa$ but still $\mathcal{F}$ is large enough in the above sense.
- To get $\kappa \geq \text{non}^*(\mathcal{Z}_0)$, one needs to find a single $f \in \mathcal{F}_J$ which is unbounded on κ many $A \in [\omega]^{\omega}$.

Is $\mathfrak{s}$ different from $\mathfrak{s}^\omega$?

Question

Is it true that $\mathfrak{s}^\omega \leq \max\{\mathfrak{s}, \mathfrak{b}\}$*? Is* $\mathfrak{s} = \mathfrak{s}^\omega$

Lemma

If $\mathbb{P}$ *is a Suslin c.c.c. forcing, then* $\mathbf{V} \cap (\mathcal{P}(\omega))^\omega$ *remains promptly splitting.*

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Definition

Let κ, λ, *and* θ *be cardinals. Then* $⫯(\kappa, \lambda, \theta)$ *is the following principle:*

- *There is a family* $\mathscr{C} \subseteq [\kappa]^{\aleph_0}$ *of size* λ *such that for any* $X \in [\kappa]^\theta$, *there exists* $A \in \mathscr{C}$ *such that* $A \subseteq X$.

Lemma ([2])

If $⫯(\mathfrak{s}, \mathfrak{s}, \aleph_1)$ *holds, then* $\mathfrak{s} = \mathfrak{s}^\omega$. *If* $⫯(\max\{\mathfrak{b}, \mathfrak{s}\}, \max\{\mathfrak{b}, \mathfrak{s}\}, \mathfrak{b})$ *holds, then* $\mathfrak{s}^\omega \leq \max\{\mathfrak{b}, \mathfrak{s}\}$.

Lemma ([2])

If $\Uparrow(\mathfrak{s}, \mathfrak{s}, \aleph_1)$ *holds, then* $\mathfrak{s} = \mathfrak{s}^\omega$. *If* $\Uparrow(\max\{\mathfrak{b}, \mathfrak{s}\}, \max\{\mathfrak{b}, \mathfrak{s}\}, \mathfrak{b})$ *holds, then* $\mathfrak{s}^\omega \leq \max\{\mathfrak{b}, \mathfrak{s}\}$.

Question

Is $\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{b}$*?*

Bibliography

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