Ramsey theory with and without pigeonhole principle

Noé de Rancourt

Université Paris VII, IMJ-PRG

Workshop “Set Theory and its Applications in Topology”

Casa Matemática Oaxaca

September 16, 2016

Theorem (Mathias-Silver)

Let $\mathcal{X}$ be an analytic set of infinite subsets of $\mathbb{N}$. Then there exists $M \subseteq \mathbb{N}$ infinite such that:

- *either for every infinite $A \subseteq M$, we have $A \in \mathcal{X}$;*
- *or for every infinite $A \subseteq M$, we have $A \notin \mathcal{X}$.*

The associated pigeonhole principle is the following:

Fact

For every $X \subseteq \mathbb{N}$, there exists an infinite $M \subseteq \mathbb{N}$ such that either $M \subseteq X$, or $M \subseteq X^c$.

Let $E = \mathbb{F}_2^{(\mathbb{N})}$ be the infinite-countable-dimensional vector space over $\mathbb{F}_2$. Recall that an (infinite-dimensional) block-subspace of E is a subspace having a basis $(f_i)_{i\in\mathbb{N}}$ with $\mathrm{supp}(f_0) < \mathrm{supp}(f_1) < \dots$

Theorem (Milliken)

Let $\mathcal{X}$ be an analytic set of block-subspaces of E. Then there exists an infinite-dimensional block-subspace F of E such that:

- *either every infinite-dimensional block-subspace of F belongs to $\mathcal{X}$;*
- *or every infinite-dimensional block-subspace of F belongs to $\mathcal{X}^c$.*

The associated (non-trivial!) pigeonhole principle is the following:

Theorem (Hindman)

For every $X \subseteq E \setminus \{0\}$, there exists an infinite-dimensional block-subspace F of E such that either $F \setminus \{0\} \subseteq X$, or $F \setminus \{0\} \subseteq X^c$.

The formalism of Gowers spaces

Let P be a set (the set of *subspaces*) and $\leqslant$ and $\leqslant^*$ be two quasi-orderings on P, satisfying:

1. for every $p, q \in P$, if $p \leqslant q$, then $p \leqslant^* q$;
2. for every $p, q \in P$, if $p \leqslant^* q$, then there exists $r \in P$ such that $r \leqslant p$, $r \leqslant q$ and $p \leqslant^* r$;
3. for every $\leqslant$-decreasing sequence $(p_i)_{i\in\mathbb{N}}$ of elements of P, there exists $p^* \in P$ such that for all $i \in \mathbb{N}$, we have $p^* \leqslant^* p_i$;

Write $p \lessapprox q$ for $p \leqslant q$ and $q \leqslant^* p$.

The formalism of Gowers spaces

Let P be a set (the set of *subspaces*) and $\leqslant$ and $\leqslant^*$ be two quasi-orderings on P, satisfying:

1. for every $p, q \in P$, if $p \leqslant q$, then $p \leqslant^* q$;
2. for every $p, q \in P$, if $p \leqslant^* q$, then there exists $r \in P$ such that $r \leqslant p$, $r \leqslant q$ and $p \leqslant^* r$;
3. for every $\leqslant$-decreasing sequence $(p_i)_{i \in \mathbb{N}}$ of elements of P, there exists $p^* \in P$ such that for all $i \in \mathbb{N}$, we have $p^* \leqslant^* p_i$;

Write $p \lessapprox q$ for $p \leqslant q$ and $q \leqslant^* p$.

Let X be an at most countable set (the set of *points*) and $\lhd \subseteq X \times P$ a binary relation, satisfying:

4. for every $p \in P$, there exists $x \in X$ such that $x \lhd p$.
5. for every $x \in X$ and every $p, q \in P$, if $x \lhd p$ and $p \leqslant q$, then $x \lhd q$.

The formalism of Gowers spaces

Let P be a set (the set of *subspaces*) and $\leqslant$ and $\leqslant^*$ be two quasi-orderings on P, satisfying:

1. for every $p, q \in P$, if $p \leqslant q$, then $p \leqslant^* q$;
2. for every $p, q \in P$, if $p \leqslant^* q$, then there exists $r \in P$ such that $r \leqslant p$, $r \leqslant q$ and $p \leqslant^* r$;
3. for every $\leqslant$-decreasing sequence $(p_i)_{i \in \mathbb{N}}$ of elements of P, there exists $p^* \in P$ such that for all $i \in \mathbb{N}$, we have $p^* \leqslant^* p_i$;

Write $p \lessapprox q$ for $p \leqslant q$ and $q \leqslant^* p$.

Let X be an at most countable set (the set of *points*) and $\lhd \subseteq X \times P$ a binary relation, satisfying:

4. for every $p \in P$, there exists $x \in X$ such that $x \lhd p$.
5. for every $x \in X$ and every $p, q \in P$, if $x \lhd p$ and $p \leqslant q$, then $x \lhd q$.

The quintuple $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ is called a *Gowers space*.

1. The Mathias-Silver space:
    - $X = \mathbb{N}$;
    - P is the set of infinite subsets of $\mathbb{N}$;
    - $\leqslant$ is the inclusion;
    - $\leqslant^*$ is the inclusion-by-finite;
    - $\lhd$ the membership relation.

1. The Mathias-Silver space:
   - $X = \mathbb{N}$;
   - P is the set of infinite subsets of $\mathbb{N}$;
   - $\leqslant$ is the inclusion;
   - $\leqslant^*$ is the inclusion-by-finite;
   - $\lhd$ the membership relation.

2. The Rosendal space over an at most countable field k:
   - $X = E$ is an infinite-countable-dimensional vector space over k;
   - P is the set of infinite-dimensional subspaces of E;
   - $\leqslant$ is the inclusion;
   - $\leqslant^*$ is the inclusion up to finite dimension ($F \leqslant^* G$ iff $F \cap G$ has finite codimension in F);
   - $\lhd$ is the membership relation.

Definition

The space $\mathcal{G}$ is said to satisfy the pigeonhole principle if for every $Y \subseteq X$ and every $p \in P$, there exists $q \leqslant p$ such that either for all $x \lhd q$, we have $x \in Y$, or for all $x \lhd q$, we have $x \in Y^c$.

Asymptotic games

Definition

Let $p \in P$. The asymptotic game below p, denoted by F_p, is the following two-players game:

$$
\begin{array}{lllllll}
\textbf{I} & p_0 \lessapprox p & & p_1 \lessapprox p & & \dots & \\
\textbf{II} & & x_0 \lhd p_0 & & x_1 \lhd p_1 & & \dots,
\end{array}
$$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

Asymptotic games

Definition

Let $p \in P$. The asymptotic game below p, denoted by F_p, is the following two-players game:

I	$p_0 \lessapprox p$		$p_1 \lessapprox p$		$\dots$
II		$x_0 \lhd p_0$		$x_1 \lhd p_1$	$\dots,$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

Saying that **I** has a strategy to reach $\mathcal{X} \subseteq X^{\mathbb{N}}$ in F_p means that "almost every" sequence below p belongs to $\mathcal{X}$.

Asymptotic games

Definition

Let $p \in P$. The asymptotic game below p, denoted by F_p, is the following two-players game:

$$\begin{array}{llllll} \textbf{I} & p_0 \lessapprox p & & p_1 \lessapprox p & & \dots \\ \textbf{II} & & x_0 \lhd p_0 & & x_1 \lhd p_1 & \dots, \end{array}$$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

Saying that **I** has a strategy to reach $\mathcal{X} \subseteq X^{\mathbb{N}}$ in F_p means that "almost every" sequence below p belongs to $\mathcal{X}$.

In the Mathias-Silver space, we have the following:

Proposition

If $\mathcal{X} \subset \mathbb{N}^{\mathbb{N}}$ is such that **I** *has a strategy to reach $\mathcal{X}$ in F_M, then there exists $N \subseteq M$ infinite such that every increasing sequence of elements of N belongs to $\mathcal{X}$.*

The abstract Mathias-Silver's theorem

So this is an equivalent formulation of Mathias-Silver's theorem:

Theorem

For every analytic $\mathcal{X} \subseteq \mathbb{N}^{\mathbb{N}}$, there exists $M \subseteq \mathbb{N}$ infinite such that:

- *either* **I** *has a strategy in F_M to reach $\mathcal{X}^c$;*
- *or* **I** *has a strategy in F_M to reach $\mathcal{X}$.*

The abstract Mathias-Silver's theorem

So this is an equivalent formulation of Mathias-Silver's theorem:

Theorem

For every analytic $\mathcal{X} \subseteq \mathbb{N}^{\mathbb{N}}$, there exists $M \subseteq \mathbb{N}$ infinite such that:

- *either* **I** *has a strategy in F_M to reach $\mathcal{X}^c$;*
- *or* **I** *has a strategy in F_M to reach $\mathcal{X}$.*

In general, we have:

Theorem (Abstract Mathias-Silver's)

Suppose that the space $\mathcal{G}$ satisfies the pigeonhole principle. Let $p \in P$ and $\mathcal{X} \subseteq X^{\mathbb{N}}$ be analytic. Then there exists $q \leqslant p$ such that:

- *either* **I** *has a strategy in F_q to reach $\mathcal{X}^c$;*
- *or* **I** *has a strategy in F_q to reach $\mathcal{X}$.*

Gowers' games and the abstract Rosendal's theorem

Definition

Let $p \in P$. The Gowers' game below p, denoted by G_p, is the following two-players game:

I	$p_0 \leqslant p$		$p_1 \leqslant p$		$\dots$	
II		$x_0 \lhd p_0$		$x_1 \lhd p_1$		$\dots,$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

Gowers' games and the abstract Rosendal's theorem

Definition

Let $p \in P$. The Gowers' game below p, denoted by G_p, is the following two-players game:

I	$p_0 \leqslant p$		$p_1 \leqslant p$		$\dots$	
II		$x_0 \lhd p_0$		$x_1 \lhd p_1$		$\dots,$

The outcome of the game is the sequence $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$.

We have the following implication : if **I** has a strategy to reach $\mathcal{X}$ in F_p, then **II** has a strategy to reach $\mathcal{X}$ in G_p.

Gowers' games and the abstract Rosendal's theorem

Definition

Let $p \in P$. The Gowers' game below p, denoted by G_p, is the following two-players game:

I	$p_0 \leqslant p$		$p_1 \leqslant p$		$\dots$	
II		$x_0 \lhd p_0$		$x_1 \lhd p_1$		$\dots,$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

We have the following implication : if **I** has a strategy to reach $\mathcal{X}$ in F_p, then **II** has a strategy to reach $\mathcal{X}$ in G_p. Under the pigeonhole principle, the converse is true up to taking a subspace.

Gowers' games and the abstract Rosendal's theorem

Definition

Let $p \in P$. The Gowers' game below p, denoted by G_p, is the following two-players game:

I	$p_0 \leqslant p$		$p_1 \leqslant p$		$\dots$	
II		$x_0 \lhd p_0$		$x_1 \lhd p_1$		$\dots,$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

We have the following implication : if **I** has a strategy to reach $\mathcal{X}$ in F_p, then **II** has a strategy to reach $\mathcal{X}$ in G_p. Under the pigeonhole principle, the converse is true up to taking a subspace.

Theorem (Abstract Rosendal's)

Let $p \in P$ and $\mathcal{X} \subseteq X^{\mathbb{N}}$ be analytic. Then there exists $q \leqslant p$ such that:

- *either **I** has a strategy in F_q to reach $\mathcal{X}^c$;*
- *or **II** has a strategy in G_q to reach $\mathcal{X}$.*

Strategically Ramsey sets

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ satisfying the conclusion of the abstract Rosendal's theorem is called *strategically Ramsey*.

Strategically Ramsey sets

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ satisfying the conclusion of the abstract Rosendal's theorem is called *strategically Ramsey*. If the pigeonhole principle holds, a strategically Ramsey set also satisfies the conclusion of the abstract Mathias-Silver's theorem.

Strategically Ramsey sets

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ satisfying the conclusion of the abstract Rosendal's theorem is called *strategically Ramsey*. If the pigeonhole principle holds, a strategically Ramsey set also satisfies the conclusion of the abstract Mathias-Silver's theorem.

Informally:

Proposition

- *If the pigeonhole principle holds, the class of strategically Ramsey sets is closed under complements;*

Strategically Ramsey sets

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ satisfying the conclusion of the abstract Rosendal's theorem is called *strategically Ramsey*. If the pigeonhole principle holds, a strategically Ramsey set also satisfies the conclusion of the abstract Mathias-Silver's theorem.

Informally:

Proposition

- *If the pigeonhole principle holds, the class of strategically Ramsey sets is closed under complements;*
- *If the pigeonhole principle doesn't hold in any subspace, and if, for a suitable class* Γ *of subsets of Polish spaces, every* Γ*-subset of* $X^{\mathbb{N}}$ *is strategically Ramsey, then so is every* $\exists^{2^{\mathbb{N}}}\Gamma$*-subset of* $X^{\mathbb{N}}$*.*

Strategically Ramsey sets

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ satisfying the conclusion of the abstract Rosendal's theorem is called *strategically Ramsey*. If the pigeonhole principle holds, a strategically Ramsey set also satisfies the conclusion of the abstract Mathias-Silver's theorem.

Informally:

Proposition

- *If the pigeonhole principle holds, the class of strategically Ramsey sets is closed under complements;*
- *If the pigeonhole principle doesn't hold in any subspace, and if, for a suitable class Γ of subsets of Polish spaces, every Γ-subset of $X^{\mathbb{N}}$ is strategically Ramsey, then so is every $\exists^{2^{\mathbb{N}}}\Gamma$-subset of $X^{\mathbb{N}}$.*

In particular, under the pigeonhole principle, every $\mathbf{\Pi}^1_1$ set is strategically Ramsey, whereas if $V = L$, then in spaces where it doesn't hold, there is always a non-strategically Ramsey $\mathbf{\Pi}^1_1$ set.

Adversarially Ramsey sets

The adversarial Gowers' games are obtained by mixing the asymptotic game and Gowers' game:

Definition (Rosendal)

Let $p \in P$. The adversarial Gowers' games below p, denoted by A_p and B_p, are the following:

- The game A_p:

I		$x_0 \lhd p_0,\ q_0 \lesssim p$		$x_1 \lhd p_1,\ \ldots$
II	$p_0 \leqslant p$		$y_0 \lhd q_0,\ p_1 \leqslant p$	

- The game B_p:

I		$x_0 \lhd p_0,\ q_0 \leqslant p$		$x_1 \lhd p_1,\ \ldots$
II	$p_0 \lesssim p$		$y_0 \lhd q_0,\ p_1 \lesssim p$	

The outcome of both games is the sequence $(x_0, y_0, x_1, y_1, \ldots) \in X^{\mathbb{N}}$.

Adversarially Ramsey sets

Definition

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ is *adversarially Ramsey* if for every $p \in P$, there exists $q \leqslant p$ such that:

- either **I** has a strategy in A_q to reach $\mathcal{X}^c$;
- or **II** as a strategy in B_q to reach $\mathcal{X}$.

Adversarially Ramsey sets

Definition

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ is *adversarially Ramsey* if for every $p \in P$, there exists $q \leqslant p$ such that:

- either **I** has a strategy in A_q to reach $\mathcal{X}^c$;
- or **II** as a strategy in B_q to reach $\mathcal{X}$.

Remark that when P has only one element, being adversarially Ramsey just means being determined.

Adversarially Ramsey sets

Definition

A set $\mathcal{X} \subseteq X^{\mathbb{N}}$ is *adversarially Ramsey* if for every $p \in P$, there exists $q \leqslant p$ such that:

- either **I** has a strategy in A_q to reach $\mathcal{X}^c$;
- or **II** as a strategy in B_q to reach $\mathcal{X}$.

Remark that when P has only one element, being adversarially Ramsey just means being determined.

Theorem (dR)

Every Borel subset of $X^{\mathbb{N}}$ is adversarially Ramsey.

Actually, if Γ is a suitable class of subsets of Polish spaces and if every Γ-subset of $\mathbb{R}^{\mathbb{N}}$ is determined, then every Γ-set is adversarially Ramsey.

Adversarially Ramsey sets

In spaces where the pigeonhole principle holds, being strategically Ramsey and being adversarially Ramsey are equivalent. So the notion of adversarially Ramsey sets is only useful in spaces without pigeonhole principle!

Adversarially Ramsey sets

In spaces where the pigeonhole principle holds, being strategically Ramsey and being adversarially Ramsey are equivalent. So the notion of adversarially Ramsey sets is only useful in spaces without pigeonhole principle!

Proposition

Suppose that the pigeonhole principle doesn't hold in the space $\mathcal{G}$. Let Γ be a suitable class of subsets of Polish spaces. If every Γ-subset of $X^{\mathbb{N}}$ is adversarially Ramsey, then every Γ-subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

Adversarially Ramsey sets

In spaces where the pigeonhole principle holds, being strategically Ramsey and being adversarially Ramsey are equivalent. So the notion of adversarially Ramsey sets is only useful in spaces without pigeonhole principle!

Proposition

Suppose that the pigeonhole principle doesn't hold in the space $\mathcal{G}$. Let Γ be a suitable class of subsets of Polish spaces. If every Γ-subset of $X^{\mathbb{N}}$ is adversarially Ramsey, then every Γ-subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

Question

Where does the adversarial Ramsey property lie between determinacy of games on integers and determinacy of games on reals?

The adversarial Ramsey property under large cardinal assumptions

Theorem (dR)

1. *Suppose that there are infinitely many Woodin cardinals and a measurable above them. Then in $L(\mathbb{R})$, every set is adversarially Ramsey.*
2. *Suppose that there are n Woodin cardinals and a measurable above them. Then every $\mathbf{\Pi}^1_{n+1}$ set is adversarially Ramsey.*

Thank you for your attention!